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Infinite Volume Limit and Spontaneous Replica Symmetry Breaking in Mean Field Spin Glass Models

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Mean field spin glasses [1] were introduced around 30 years ago as an approximation of realistic models for disordered magnetic alloys. In what follows, we make particular reference to the well known Sherrington-Kirkpatrick (SK) model [2], but most of the results we present can be proven in greater generality. The Hamiltonian of the SK model in a magnetic field h, for a given configuration of the N Ising spins $\sigma_i = \pm 1$, $i = 1, \dots, N$, is defined as

$$H_N(\sigma, h; J) = -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} J_{ij} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i, \tag{1}$$

where the couplings J_{ij} are quenched i.i.d. Gaussian random variables with zero mean. The model can be generalized in many ways, for instance allowing for more general disorder distributions, or introducing interactions among p-ples of spins, with $p > 2$ (p-spin model [3]). The main object of interest is the infinite volume quenched free energy density, defined as

$$F(\beta, h) = \lim_{N \to \infty} F_N(\beta, h) \equiv \lim_{N \to \infty} -\frac{1}{N\beta} E \ln Z_N(\beta, h; J). \tag{2}$$

Here, Z_N is the disorder dependent partition function, at inverse temperature β, and E denotes average with respect to the quenched disorder J. Due to their mean field character, these models are exactly solvable, at least in the framework of Parisi theory of replica symmetry breaking [1]. This predicts a solution for the quenched free energy, which we denote by $F^{Parisi}(\beta, h)$, expressed as the *supremum* of a suitable trial functional over the space of functional order parameters [1]. From the rigorous point of view, the situation is much more delicate, and even the problem of proving that the limit in (2) exists remained open for a very long time.

Our methods make use of very simple interpolation techniques. Broadly speaking, many interesting results can be expressed in the form of comparisons (inequalities) between the free energies of two different systems, as we clarify below in some concrete cases. In order to obtain this, the idea is to introduce an auxiliary partition function, depending on an interpolating parameter $0 \le t \le 1$, which reduces to the partition function of the two systems to be compared when $t = 0$ or 1. Then, if the derivative of the t-dependent free energy has a definite sign, the inequality simply follows by integration. Let us give some examples.

In [4], [5] we solved the problem of the existence of the infinite volume limit (2), for a wide class of mean field spin glass models. The strategy is based on a suitable interpolation between the N spin system and a system made of two non-interacting subsystems of size N_1 and N_2, respectively, with $N_1 + N_2 = N$. Then the definite sign of the derivative allows, as explained above, to prove subadditivity for the free energy,

$$NF_N(\beta, h) \leq N_1 F_{N_1}(\beta, h) + N_2 F_{N_2}(\beta, h), \tag{3}$$

from which the existence of the infinite volume for F_N follows from standard methods.

A much deeper result concerns the relation between the quenched free energy F_N and the solution F^{Parisi} proposed by Parisi, which is shown to be a rigorous lower bound for F_N for all N, β, h [6]:

$$F_N(\beta, h) \geq F^{Parisi}(\beta, h). \tag{4}$$

While the precise form of the interpolating partition function is quite involved in this case (see [6] for details), the underlying physical idea is very simple: the model being of mean field type, it must be possible to substitute the random two-body interaction J_{ij} with an effective random external field acting on every spin σ_i. When the two-body potential is completely removed (corresponding to $t = 0$ in the interpolation), the model is exactly solvable, and its free energy is essentially given by the Parisi solution. Moreover, the difference between the l.h.s. and the r.h.s. of Eq. (4) can be expressed as the sum of positive fluctuations of the overlaps, in suitably defined states [6]. As a simple byproduct of (4) we showed that, below the Almeida-Thouless (AT) critical line [1], replica symmetry is broken, in the sense that the overlap between two configurations fluctuates even in the infinite volume limit [10].

Of course, one would like to prove also the bound opposite to (4), *i.e.*,

$$F_N(\beta, h) \leq F^{Parisi}(\beta, h) + o(1), \tag{5}$$

where $o(1)$ vanishes for $N \to \infty$, thereby fully justifying Parisi's solution. Even if this task has not been accomplished in the general case yet, we were able to prove Eq. (5) in a region of high temperature or high magnetic field, where the system is in the so called replica-symmetric phase. (For previous results in this direction, see for instance [11], [12]). In this region, we proved [8] that

$$F_N(\beta, h) = F^{Parisi}(\beta, h) + O\left(N^{-1}\right). \tag{6}$$

The proof requires to take two copies (replicas) of the system and to couple them via an interaction term depending on their mutual overlap ("quadratic replica coupling"). Moreover, in the same region we were able [9] to give limit theorems for the fluctuations. In particular, we proved that the fluctuations of the disorder dependent free energy density $-1/(N\beta) \ln Z_N(\beta, h; J)$ around the limit value

$F(\beta, h)$ behave, on the scale $1/\sqrt{N}$, as a centered Gaussian variable in the infinite volume limit. A central limit theorem holds also for the rescaled fluctuations of the overlap between two configurations, around the value predicted by Parisi theory. Unfortunately, we were not able to control the whole expected replica-symmetric region, whose boundary is marked by the AT line, but this seems to be a common problem of all the rigorous approaches proposed in the literature so far.

It is important to stress that our methods proved to be quite robust, in the sense that they are not specific of the SK model we are considering here. For instance, we showed in [13] how the quadratic replica coupling method can be extended to study the Viana-Bray diluted spin glass model [7]. This is a generalization of the SK model, where each site interacts only with a random finite number of other sites, even in the thermodynamic limit. We obtained a complete control of the system, including the thermodynamic limit for the free energy and limit theorems for the fluctuations, in the whole replica symmetric region, which in this case corresponds to high temperature or high dilution.

References

[1] M. Mézard, G. Parisi and M. A. Virasoro, *Spin glass theory and beyond*, World Scientific, Singapore (1987).

[2] D. Sherrington, S. Kirkpatrick, *Phys. Rev. Lett.* **35**, 1792–1796 (1975).

[3] B. Derrida, *Phys. Rev. B* **24**, 2613–2626 (1981).

[4] F. Guerra, F. L. Toninelli, *Commun. Math. Phys.* **230:1**, 71–79 (2002).

[5] F. Guerra, F. L. Toninelli, *Markov Proc. Rel. Fields*, to appear, `cond-mat/0208579`.

[6] F. Guerra, *Commun. Math. Phys.* **233:1**, 1–12 (2003).

[7] L. Viana, A. J. Bray, *J. Phys. C* **18**, 3037–3051 (1985).

[8] F. Guerra, F. L. Toninelli, *J. Math. Phys.* **43**, 3704–3716 (2002).

[9] F. Guerra, F. L. Toninelli, *J. Math. Phys.* **43**, 6224–6237 (2002).

[10] F. L. Toninelli, *Europhys. Lett.* **60**, 764–767 (2002).

[11] M. Scherbina, Helv. Phys. Acta **70**, 838 (1997).

[12] M. Talagrand, Ann. Probab **30**, 364-381 (2002).

[13] F. Guerra, F. L. Toninelli, to appear, `cond-mat/0302401`.

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