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Harmonic and analytic functions on graphs

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Abstract. Harmonic and analytic functions have natural discrete analogues. Harmonic functions can be defined on every graph, while analytic functions (or, more precisely, holomorphic forms) can be defined on graphs embedded in orientable surfaces. Many important properties of the "true" harmonic and analytic functions can be carried over to the discrete setting. We prove that a nonzero analytic function can vanish only on a very small connected piece. As an application, we describe a simple local random process on embedded graphs, which have the property that observing them in a small neighborhood of a node through a polynomial time, we can infer the genus of the surface.

1. Introduction

Harmonic and analytic functions are fundamental in analysis. Using basic notions in graph theory (flows and potentials), one can define natural discrete analogues of these notions. Harmonic functions can be defined on every graph, and have been used quite extensively. In this paper we briefly survey some of their properties, concentrating on their connection with random walks, electrical networks and rubber band structures. Discrete analytic functions (or, more precisely, holomorphic forms) can be defined on graphs embedded in orientable surfaces. They can be defined as rotation-free circulations (which is the same as requiring that the circulation is also a circulation on the dual graph). These functions were introduced for the case of the square grid a long time ago [7, 6]. For the case of a general map, the notion is implicit in [3] (cf. Section 5) and other previous work; they were formally introduced very recently by Mercat [8]. Many important properties of the "true" harmonic and analytic functions can be carried over to the discrete setting: maximum principles, Cauchy integrals etc. In this paper we focus on properties of discrete analytic functions, and sketch the proof of a somewhat deeper property that a nonzero analytic function can vanish only on a very small connected piece. As an application, we describe a simple local random process on embedded graphs, which has the property that observing it in a small neighborhood of a node through a polynomial time, we can infer the genus of the surface. For a detailed analysis of this application, see [1].

2. Notation

Let G be a finite graph with a reference orientation. Each edge $e \in E$ has a *tail* $t(e) \in V$ and a *head* $h(e) \in V$. For each node v, let $\delta v \in \mathbb{R}^E$ denote the coboundary of v:

$$(\delta v)_e = \begin{cases} 1, & \text{if } t(e) = v, \\ -1, & \text{if } h(e) = v, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $|\delta v|^2 = d_v$ is the degree of v. Let $\phi : E \to \mathbb{R}$ be a weighting of the edges by real numbers (equivalently, a vector $\phi \in \mathbb{R}^E$). We say that a node $v \in V$ is a *source* of ϕ if

$$(\delta v)^\mathsf{T} \phi = \sum_{e:\, t(e)=v} \phi(e) - \sum_{e:\, h(e)=v} \phi(e) > 0;$$

it is a *sink* if $(\delta v)^\mathsf{T}\phi < 0$; and we say that ϕ *satisfies the flow condition at* v if $(\delta v)^\mathsf{T}\phi = 0$. We say that ϕ is a *circulation* if it satisfies the flow condition at each node v. Let S be a closed compact surface, and consider a *map* on S, i.e., a graph $G = (V, E)$ embedded in S so that each face is a disc. We can describe the map as a triple $G = (V, E, \mathcal{F})$, where V is the set of nodes, E is the set of edges, and $\mathcal{F}$ is the set of faces of G. We set $n = |V|$, $m = |E|$, and $f = |\mathcal{F}|$. We fix a reference orientation of G; then each edge $e \in E$ has a *tail* $t(e) \in V$, a *head* $h(e) \in V$, a *right shore* $r(e) \in \mathcal{F}$, and a *left shore* $l(e) \in \mathcal{F}$. The embedding of G defines a *dual map* G^*. Combinatorially, we can think of G^* as the triple $(\mathcal{F}, E, V)$, where "node" and "face" are interchanged, "tail" is replaced "right shore", and "head" is replaced by "left shore". Note "right shore" is replaced by "head" and "left shore" is replaced by "tail". So $(G^*)^*$ is not G, but G with all edges reversed. For every face $F \in \mathcal{F}$, we denote by $\partial F \in \mathbb{R}^E$ the boundary of F:

$$(\partial F)_e = \begin{cases} 1, & \text{if } r(e) = F, \\ -1, & \text{if } l(e) = F, \\ 0, & \text{otherwise.} \end{cases}$$

Then $d_F = |\partial F|^2$ is the length of the cycle bounding F.

3. Discrete harmonic functions

3.1. *Definition*

Let $G = (V, E)$ be a connected graph and $S \subseteq V$ (the orientation is not relevant right now). A function $\pi : V \to \mathbb{R}$ is called a "harmonic function with set of poles S" if

$$\frac{1}{d_v} \sum_{u \in N(v)} \pi(u) = \pi(v) \qquad \forall v \notin S.$$

It is trivial that every non-constant harmonic function has at least two poles (its minimum and maximum). For any two nodes $a, b \in V$ there is are harmonic functions with these poles. Such a harmonic function is uniquely determined up to scaling by a real number and translating by a constant. There are various natural ways to normalize; we'll somewhat arbitrarily decide on the following one:

$$\sum_{u \in N(v)} (\pi(u) - \pi(v)) = \begin{cases} 1, & \text{if } v = b, \\ -1, & \text{if } v = a, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

and

$$\sum_u \pi(u) = 0. \tag{2}$$

We denote this function by $\pi_{a,b}$ (if we want to express that it depends on a and b); if there is an edge e with $h(e) = b$ and $t(e) = a$, then we also denote $\pi_{a,b}$ by π_e. Expression (1) is equivalent to saying that $f_e = \pi(h(e)) - \pi(t(e))$ is a flow from a to b.

3.2. *Harmonic functions from random walks, electrical networks, and rubber bands*

Harmonic functions play an important role in the study of random walks: after all, the averaging in the definition can be interpreted as expectation after one move. They also come up in the theory of electrical networks, and also in statics. This provides a connection between these fields, which can be exploited. In particular, various methods and results from the theory of electricity and statics, often motivated by physics, can be applied to provide results about random walks. We start with describing three constructions of harmonic functions, one in each field mentioned.

EXAMPLE 1. Let $\pi(v)$ denote the probability that a random walk starting at node v hits s before it hits t. Clearly, π is a harmonic function with poles s and t. We have $\pi(s) = 1$ and $\pi(t) = 0$. More generally, if we have a set $S \subseteq V$ and a function $\pi_0 : S \to \mathbb{R}$, then we define $\pi(v)$ for $v \in V \setminus S$ as the expectation of $\pi_0(s)$, where s is the (random) node where a random walk starting at v first hits S. Then $\pi(v)$ is a harmonic function with pole set S. Moreover, $\pi(s) = \pi_0(s)$ for all $s \in S$.

EXAMPLE 2. Consider the graph G as an electrical network, where each edge represents a unit resistance. Assume that an electric current is flowing through G, entering at s and leaving at t. Let $\pi(v)$ be the voltage of node v. Then π is a harmonic function with poles s and t.

EXAMPLE 3. Consider the edges of the graph G as ideal springs with unit Hooke constant (i.e., it takes h units of force to stretch them to length h). Let us nail down nodes s and

t to points 1 and 0 on the real line, and let the graph find its equilibrium. The energy is a positive definite quadratic form of the positions of the nodes, and so there is a unique minimizing position, which is the equilibrium. Clearly all nodes will lie on the segment between 0 and 1, and the positions of the nodes define a harmonic function with poles s and t. More generally, if we have a set $S \subseteq V$ and we fix the positions of the nodes in S (in any dimension), and let the remaining nodes find their equilibrium, then every coordinate function is harmonic with pole set S.

A consequence of the uniqueness property is that the harmonic functions constructed (for the case $|S| = 2$) in examples 1, 2 and 3 are the same. As an application of this idea, we show the following interesting connections (see Nash-Williams [10], Chandra at al. [5]). Considering the graph G as an electrical network, let R_{st} denote the effective resistance between nodes s and t. Considering the graph G as a spring structure in equilibrium, with two nodes s and t nailed down at 1 and 0, let F_{ab} denote the force pulling the nails. Doing a random walk on the graph, let $\kappa(a, b)$ denote the commute time between nodes a and b (i.e., the expected time it takes to start at a, walk until you first hit b, and then walk until you first hit a again).

THEOREM 1. $\pi_{ab}(b) - \pi_{ab}(a) = R_{ab} = \dfrac{1}{F_{ab}} = \dfrac{\kappa(a,b)}{2m}.$

Using the "topological formulas" from the theory of electrical networks for the resistance, we get a further well-known characterization of these quantities:

COROLLARY 2. *Let G' denote the graph obtained from G by identifying a and b, and let $\mathcal{T}(G)$ denote the number of spanning trees of G. Then*

$$R_{ab} = \frac{\mathcal{T}(G)}{\mathcal{T}(G')}.$$

3.3. *Flows from harmonic functions*

Let $\pi \in \mathbb{R}^V$ be a function on a graph $G = (V, E)$ (with a reference orientation). The π gives rise to a vector $\delta\pi \in \mathbb{R}^E$, where

$$(\delta\pi)(uv) = \pi(u) - \pi(v) \tag{3}$$

In other words,

$$\delta\pi = \sum_v \pi(v)\delta v. \tag{4}$$

The function $\delta\pi$ satisfies the flow condition at node i if and only if π is harmonic at i. Indeed, if π is harmonic at i, then

$$\sum_j \delta\pi(ij) = \sum_j (\pi(j) - \pi(i)) = \sum_j \pi(j) - d_i\pi(i) = 0;$$

the same computation also gives the converse. Not every flow can be obtained from a harmonic function: for example, a non-zero circulation (a flow without sources and sinks) would correspond to a non-constant harmonic function with no poles, which cannot exist. In fact, the the flow obtained by (3) satisfies, for every cycle C, the following condition:

$$\sum_{e \in C^+} f_\pi(e) - \sum_{e \in C^-} f_\pi(e) = 0,$$

where C^+ and C^- denote the set of forward and backward edges of C, walking around it in an arbitrary direction.

4. Discrete analytic functions

4.1. *Circulations and homology*

If G is a map on a surface S, then the space of circulations on G has an important additional structure: for each face F, the vector ∂F is circulation. Circulations that are linear combinations of these special circulations ∂F are called *null-homologous*. Two circulations ϕ and ϕ' are *homologous* if $\phi - \phi'$ is null-homologous. Let ϕ be a circulation on G. We say that ϕ is *rotation-free*, if for every face $F \in \mathcal{F}$, we have

$$(\partial F)^{\mathsf{T}} \phi = \sum_{e:\, r(e)=F} \phi(e) - \sum_{e:\, l(e)=F} \phi(e) = 0.$$

This is equivalent to saying that ϕ is a circulation on the dual map G^*. To explain the connection between rotation-free circulations and analytic functions, let ϕ be a rotation-free circulation on a directed graph G embedded in a surface S. Consider a cycle C in G that bounds a planar piece S' of S. Let $G' = (V', E')$ be the subgraph of G contained in S' (including C). Then on the set $\mathcal{F}'$ of faces contained in S', we have a function $\sigma : \mathcal{F}' \to \mathbb{R}$ such that $\phi(e) = \sigma(r(e)) - \sigma(l(e))$ for every edge e. Similarly, we have a function $\pi : V' \to \mathbb{R}$ (where V' is the set of nodes in this planar piece), such that $\phi(e) = \pi(t(e)) - \pi(h(e))$ for every edge e. It is easy to see that π is harmonic in all nodes of G' in the interior of C and σ is harmonic in all nodes of the dual map in the interior of C. We can think of π and σ as the real and imaginary parts of a (discrete) analytic function. The relation $\delta\pi = \rho\phi$ is then a discrete analogue of the Cauchy-Riemann equations. Figure 1 shows a rotation-free circulation on a piece of a map. The first figure shows how to obtain it (locally) from a potential on the nodes, the second, how to obtain it from a potential on the faces.

From the definition of rotation-free circulations, we get two orthogonal linear subspaces: $\mathcal{A} \subseteq \mathbb{R}^E$ generated by the vectors δv $(v \in V)$ and $\mathcal{B} \subseteq \mathbb{R}^E$ generated by the vectors ∂F

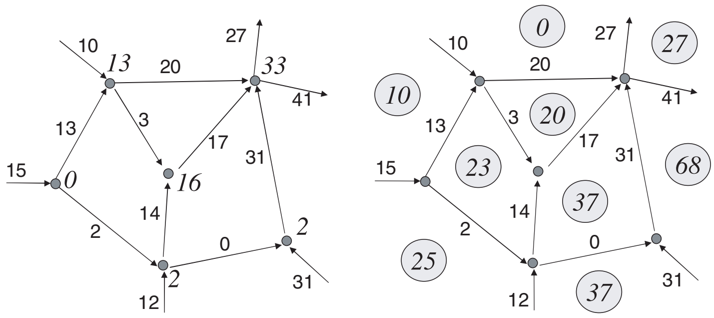

Figure 1 Rotation-free circulation

$(F \in \mathcal{F})$. Vectors in $\mathcal{B}$ are 0-homologous circulations. The orthogonal complement $\mathcal{A}^\perp$ is the space of all circulations, and $\mathcal{B}^\perp$ is the space of circulations on the dual graph. The intersection $\mathcal{C} = \mathcal{A}^\perp \cap \mathcal{B}^\perp$ is the space of rotation-free circulations. So $\mathbb{R}^E = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}$. From this picture we conclude the following.

LEMMA 3. *Every circulation is homologous to a unique rotation-free circulation.*

It also follows that $\mathcal{C}$ is isomorphic to the first homology group of S (over the reals), and hence we get the following [8]:

THEOREM 4. *The dimension of the space $\mathcal{C}$ of rotation-free circulations is $2g$.*

4.2. *Analytic functions from harmonic functions*

We can use harmonic functions to give a more explicit construction of rotation-free circulations. For $ab = e \in E$, consider the harmonic function π_e (as defined in Section 3.3). This is certainly rotation-free, but it is not a circulation: a is a source and b is a sink (all the other nodes satisfy the flow condition). We can try to repair this by sending a "blackflow" along the edge e; in other words, we consider $\pi_e - \chi_e$. This is now a circulation, but it is not rotation-free around the faces $r(ab)$ and $l(ab)$. The trick is to also consider the dual map, the dual edge $e^* = r(e)l(e)$, and the harmonic function π_{e^*}, and carry out the same construction as above, to get $\delta^*\pi_{e^*}$. Then we can combine these to really repair the flow condition and clear rotation: we define

$$\eta_e = \delta\pi_e - \delta^*\pi_{e^*} + \chi_e. \tag{5}$$

The considerations above show that η_e is a rotation-free circulation. In addition, it has the following description:

LEMMA 5. *The circulation η_e is the orthogonal projection of χ_e to the space $\mathcal{C}$ of rotation-free circulations.*

Proof. It suffices to show that

$$\chi_e - \eta_e = \delta\pi_e - \delta^*\pi_{e^*}$$

is orthogonal to every $\phi \in \mathcal{C}$. But $\delta\pi_e \in \mathcal{A}$ by (4), and similarly, $\delta^*\pi_{e^*} \in \mathcal{B}$. So both are orthogonal to $\mathcal{C}$. □

This lemma has some simple but interesting consequences. Since η_e is a projection of χ_e, we have

$$\eta_e(e) = \eta_e \cdot \chi_e = |\eta_e|^2 \geq 0. \tag{6}$$

Let R_e denote the effective resistance between the endpoints of e, and let R_e^* denote the effective resistance of the dual map between the endpoints of the edge dual to e. Then we get by Theorem 1 that

$$\eta_e(e) = 1 - R_e - R_e^*. \tag{7}$$

If we work with a map on the sphere, we must get 0 by Theorem 4. This fact has the following consequence (which is of course well known, and can be derived e.g. from Corollary 2): for every planar map, $R_e + R_e^* = 1$. For any other underlying surface, we get $R_e + R_e^* \leq 1$. It follows from Theorem 7 below that strict inequality holds here.

4.3. *Nondegeneracy of rotation-free circulations*

Two useful non-degeneracy properties of rotation-free circulations were proved in [1]. We need a simple lemma about maps. For every face F, let a_F denote the number of times the orientation changes if we move along the the boundary of F. For every node v, let b_v denote the number of times the orientation changes in their cyclic order as they emanate from v.

LEMMA 6. *Let $G = (V, E, \mathcal{F})$ be any digraph embedded on an orientable surface S of genus g. Then*

$$\sum_{F\in\mathcal{F}}(a_F - 2) + \sum_{v\in V}(b_v - 2) = 4g - 4.$$

Proof. Clearly

$$\sum_F a_F = \sum_v (d_v - b_v),$$

and so by Euler's formula,

$$\sum_F a_F + \sum_v b_v = \sum_v d_v = 2m = 2n + 2f + 4g - 4.$$

Rearranging and dividing by 2, we get the equality in the lemma. □

THEOREM 7. *If $g > 0$, then $\eta_e \neq 0$ for every edge e.*

Proof. Suppose that $\eta_e = 0$ for some edge e. Then by the definition (5) of η_e, we have

$$\pi_e(h(f)) - \pi_e(t(f)) = \pi_{e^*}(r(f)) - \pi_{e^*}(l(f)) \tag{8}$$

for every edge $f \neq e$, but

$$\pi_e(h(e)) - \pi_e(t(e)) = \pi_{e^*}(r(e)) - \pi_{e^*}(l(e)) - 1. \tag{9}$$

We define a convenient orientation of G. Let $E(G) = E_1 \cup E_2$, where E_1 consists of edges a with $\phi(h(e)) \neq \phi(t(e))$, and E_2 is the rest. Every edge $a \in E_1$ is oriented so that $\pi(h(e)) > \pi(t(e))$. Consider any connected component C of the subgraph formed by edges in E_2. Let $u_1, \dots, u_k$ be the nodes of C that are incident with edges in E_1. Add a new node v to C and connect it to $u_1, \dots, u_k$ to get a graph C'. Clearly C' is 2-connected, so it has an acyclic orientation such that every node is contained in a path from v to u_1. The corresponding orientation of C is acyclic and every has the property that it has no source or sink other than possibly $u_1, \dots, u_k$. Carrying this out for every connected component of G', we get an orientation of G. We claim this orientation is acyclic. Indeed, if we had a directed cycle, then walking around it π would never decrease, so it would have to stay constant. But then all edges of the cycle would belong to E_2, contradicting the way these edges were oriented. We also claim this orientation has only one source and one sink. Indeed, if a node $v \neq h(e), t(e)$ is incident with an edge of E_1, then it has at least one edge of E_1 entering it and at least one leaving it, by (1). If v is not incident with any edge of E_1, then it is an internal node of a component C, and so it is not a source or sink by the construction of the orientation of C. Take the union of G and the dual graph G^*. This gives a graph H embedded in S. Clearly H inherits an orientation from G and from the corresponding orientation of G^*. We are going to apply Lemma 6. Every face of H will $a_F = 2$ (this just follows from the way how the orientation of G^* was defined). Those nodes of H which arise as the intersection of an edge of G with an edge of G^* will have $b_v = 2$. Consider a node v of G. If $v = h(e)$ then clearly all edges are directed toward v, so $b_{h(e)} = 0$. Similarly, we have $b_{t(v)} = 0$. We claim that $b_v = 2$ for every other node. Since obviously v is not a source or a sink, we have $b_v \geq 2$. Suppose that $b_v > 2$. Then we have for edges e_1, e_2, e_3, e_4 incident with v in this cyclic order, so that e_1 and e_2 form a corner of a face F, e_3 and e_4 form a corner of a face F', $h(e_1) = h(e_3) = v$ and $t(e_2) = t(e_3) = v$. Consider π_{e^*} of the faces incident with v. We may assume that $\pi_{e^*}(F) \leq \pi_{e^*}(F')$. From the orientation of the edges e_1 and e_2 it follows that $\pi_{e^*}(F)$ is larger than π_{e^*} of its neighbors. Let $\mathcal{F}$ be the union of all faces F'' with $\pi_{e^*}(F'') \geq \pi_{e^*}(F)$. The boundary of $\mathcal{F}$ is an eulerian subgraph, and so it can be decomposed into edge-disjoint cycles $D_1, \dots, D_t$. Since the boundary goes through v twice (once along e_1 and e_2, once along two other edges with the corner of F' on the left hand side), we have $t \geq 2$, and so one of these cycles, say D_1, does not contain e. But then by the definition of the orientation and by (8), D_1 is a directed cycle, which is a contradiction. A similar argument shows that if v is a node corresponding to a face not incident with e, then $b_v = 2$; while if v comes from $r(e)$ or from $l(e)$, then $b_v = 0$. So substituting in Lemma 6, only two terms on the left hand side will be non-zero, yielding $-4 = 4g - 4$, or $g = 0$. □

COROLLARY 8. *If* $g > 0$*, then for every edge* e*,* $\eta_e(e) \geq n^{2-n} f^{2-f}$.

Indeed, combining Theorem 7 with (6), we see that $\eta_e(e) > 0$ if $g > 0$. But $\eta_e(e) = 1 - R_e - R_e^*$ is a rational number, and from Theorem 2 it follows that its denominator is not larger than $n^{n-2} f^{f-2}$.

COROLLARY 9. *If* $g > 0$*, then there exists a nowhere-0 rotation-free circulation.*

The second "non-degeneracy" result (also from [1]) is an analogue of the fact that an analytic function cannot vanish on a open set, unless it is identically 0. Here we are dealing with finite graphs, so instead of openness, we have to introduce a more complicated condition. Let H be a connected subgraph of G. We say H is *plane* if H is contained in a submanifold of S that is topologically a disc. We say H is *k-separable* in G, if G can be written as the union of two graphs G_1 and G_2 so that $|V(G_1) \cap V(G_2)| \leq k$, $V(H) \cap V(G_2) = \emptyset$, and G_2 is not plane. To illuminate this somewhat technical condition, assume that G is embedded sufficiently densely and uniformly in the sense that for every separating set X of k nodes, all but one of the components of $G - X$ are plane and have fewer than k' nodes. Then no connected subgraph with at least k' nodes is k-separable in G. To give another example, suppose that every non-contractible Jordan curve on the surface intersects the map in more than k points (this standard condition is called *representativity*). Then a non-plane connected subgraph is not k-separable in G.

THEOREM 10. *Let* G *be a graph embedded in an orientable surface* S *of genus* $g > 0$ *so that all faces are discs. Suppose that a non-zero rotation-free circulation* ϕ *vanishes on all edges incident with a connected subgraph* U *of* G (*including the edges of* U). *Then* U *is* $(4g - 3)$*-separable in* G.

The theorem is sharp, up to a factor of 2, in the following sense. Suppose X is a connected induced subgraph of G separated from the rest of G by at most $2g$ nodes, and suppose (for simplicity) that X is plane. Contract X to a single point x, and erase the resulting multiplicities of edges. We get a graph G' still embedded in S so that each face is a disc. Thus this graph has a $(2g)$-dimensional space of circulations, and hence there is a non-zero rotation-free circulation ψ vanishing on $2g - 1$ of the edges incident with x. Since this is a circulation, it must vanish on all the edges incident with x. Uncontracting X, and extending ψ with 0-s to the edges of X, it is not hard to check that we get a rotation-free circulation.

Proof. Let G' be the subgraph of G on which ϕ does not vanish. Let W be the connected component of $G \setminus V(G')$ containing U, and let Y denote the set of nodes in $V(G) \setminus V(W)$ adjacent to W. Consider an edge e with $\phi(e) = 0$. If e is not a loop, then we can contract e and get a map on the same surface with a rotation-free flow on it. If the two sides of e are different faces, then we can delete e and get a map on the same surface with a rotation-free flow on it. So we can eliminate edges with $\phi(e) = 0$ unless $h(e) = t(e)$ and $r(e) = l(e)$ (we call these edges strange loops). In this latter case, we can change $\phi(e)$ to any non-zero value and still have a rotation-free circulation. Applying this reduction procedure, we may assume that $W = \{w\}$ consists of a single node, and the only edges with $\phi = 0$ are the edges between w and Y, or between two nodes of Y. We cannot try to contract edges between nodes in Y (we don't want to reduce the size of Y), but we can try to delete them; if this does not work, then every such edge must have $r(e) = l(e)$. Also, if more than one edge remains between w and a node $y \in Y$, then each of them has $r(e) = l(e)$ (else, one of them could be deleted). Note that we may have some strange loops attached at w. Let D be the number of edges between w and Y. Re-orient each edge with $\phi \neq 0$ in the direction of the flow ϕ, and orient the edges between w and Y alternating in an out from w. Orient the edges with $\phi = 0$ between two nodes of Y arbitrarily. We get a digraph G_1. It is easy to check that G_1 has no sources or sinks, so $b_v \geq 2$ for every node v, and of course $b_w \geq |Y| - 1$. Furthermore, every face either has an edge with $\phi > 0$ on its boundary, or an edge with $r(e) = l(e)$. If a face has at least one edge with $\phi > 0$, then it cannot be bounded by a directed cycle, since ϕ would add up to a positive number on its boundary. If a face boundary goes through an edge with $r(e) = l(e)$, then it goes through it twice in different directions, so again it is not directed. So we have $a_F \geq 2$ for every face. Substituting in Lemma 6, we get that $|Y| - 1 \leq 4g - 4$, or $|Y| \leq d_w \leq 4g - 3$. Since Y separates U from G', this proves the theorem. □

5. A classical application: Dissection of rectangles into squares

Let us briefly sketch the classic paper of Brooks, Smith, Stone and Tutte [3]. Consider a rectangle R dissected into a finite number of squares. Assume that the rectangle is made of thin homogeneous conducting material, and let us send unit electrical current from the top edge to the bottom edge. Consider any maximal horizontal segment I composed of edges of the squares. Clearly all points of this segment will be at the same potential, and so we can represent them by a single node. Furthermore, the resistance of every square between its top and bottom edge is the same (since the resistance of a rectangle is proportional to its height but inversely proportional to its width); choosing appropriate units, we can assume that each square has resistance 1. Each square "connects" two horizontal segments, and we can represent it by an edge connecting the two corresponding nodes, directed top-down. We get a directed graph G (Figure 2). It is easy to see that G is planar, and the two special nodes

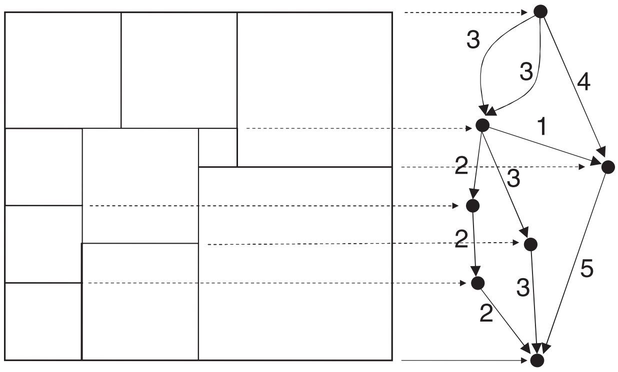

Figure 2 The Brooks-Smith-Stone-Tutte construction

a and b corresponding to the top and bottom edge of the rectangle R are on the boundary of the infinite face. Furthermore, if we assign the edge-length of the corresponding square to each edge, then we get a unit flow from a to b. Since the edge-length of a square is also the potential difference between the upper and lower edges, this flow is rotation-free. Conversely, if we take a 2-connected planar graph with two specified nodes a and b on the boundary of the infinite face, then we can send unit electrical current from a to b, to get a rotation-free flow. Brooks et al. show (it is not hard after all these preparations) that this current can be obtained from a tiling of a rectangle by squares, where the edge-length of a square is the electrical flow through the corresponding edge. (There is a little ambiguity when two nodes on the boundary of the same finite face are at the same potential: then in the square tiling we get four squares meeting at the same node, and according to the construction above, we should have represented the corresponding segments by the same node. In other words, these nodes would be identified.) Brooks et al. use this method to construct a tiling of a square with squares whose edge-lengths are all different.

6. A new application: Global information from local observation

Suppose that we live in a (finite) graph, embedded in an orientable surface with genus G (we assume the embedding is reasonably dense). On the graph, a random process is going on, with local transitions. Can we determine the genus g, by observing the process in a small neighborhood of our location? Discrete analytic functions motivate a reasonably natural and simple process, called *noisy circulator*, which allows such a conclusion. Informally, this can be described as follows. Each edge carries a weight. With some frequency, a node wakes up, and balances the weights on the edges incident with it, so that locally the flow condition is restored. With the same frequency, a face wakes up, and balances the weights on the edges incident with it, so that the rotation around the face is cancelled. Finally, with a much lower frequency, an edge wakes up, and increases or decreases its weight by 1. To be precise, we consider a finite graph G, embedded in an orientable surface S, so that each face is a disk bounded by a simple cycle. We fix a reference orientation of G, and a number $0 < p < 1$. We start with the vector $x = 0 \in \mathbb{R}^E$. At each step, the following two operations are carried out on the current vector $x \in \mathbb{R}^E$:

(a) [Node balancing.] We choose a random node v. Let $a = (\delta v)^\mathsf{T} x$ be the "imbalance" at node v (the value by which the flow condition at v is violated). We modify f by subtracting $(a/d_v)\delta v$ from x.

(b) [Face balancing.] We choose a random face F. Let $r = (\partial F)^\mathsf{T} x$ be the rotation around F. We modify f by subtracting $(r/d_F)\partial F$ from x. In addition, with some given probability $p > 0$, we do the following:

(c) [Excitation.] We choose a random edge e and a random number $X \in \{-1, 1\}$, and add X to x_e.

Discrete analytic functions are invariant under node and face balancing, and under repeated application of (a) and (b), any vector converges to a discrete analytic function.

Next we describe how we observe the process. Let U be a connected subgraph of G, which is not $(4g-3)$-separable in G. Let E_0 be the set of edges incident with U (including the edges of U). Let $x(t) \in \mathbb{R}^E$ be the vector of edge-weights after t steps, and let $y(t)$ be the restriction of $x(t)$ to the edges in E_0. So we can observe the sequence random vectors $y(0), y(1), \ldots$. The main result of [1] about the noisy circulator, somewhat simplified, is the following.

THEOREM 11. *Assume that we know in advance an upper bound N on $n + m + f$. Then there is a constant $c > 0$ such that if $p < N^{-c}$, then observing the Noisy Circulator for N^c/p steps, we can determine g with high probability.*

The idea behind the recovery of the genus g is that if the excitation frequency p is sufficiently small, then most of the time $x(t)$ will be essentially a rotation-free circulation. The random excitations guarantee that over sufficient time we get $2g$ linearly independent discrete analytic functions. Theorem 10 implies that even in our small window, we see $2g$ linearly independent weight assignments $y(t)$. The details are quite involved, however, and the reader is referred to [1] for details.

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