J. Geom. (2023) 114:33

0047-2468/23/030001-14
published online November 15, 2023
https://doi.org/10.1007/s00022-023-00697-z

Journal of Geometry

On toroidal circle planes with groups of automorphisms fixing exactly one point

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Abstract. We characterize the family of flat Minkowski planes constructed from strongly hyperbolic functions in terms of their automorphism groups. We also improve the current characterization in the literature for the family of flat Minkowski planes constructed by Artzy and Groh. As a corollary, we show that the automorphism groups of toroidal circle planes constructed by Polster are 2-dimensional.

Keywords. Circle plane, Minkowski plane, Automorphism group.

1. Introduction

The classical Minkowski plane is the geometry of plane sections of the standard nondegenerate ruled quadric in real 3-dimensional projective space $\mathbf{P}_3(\mathbb{R})$. It is an example of a flat Minkowski plane, which is an incidence structure defined on the torus satisfying certain geometric axioms. Besides the classical Minkowski plane, there are many known examples of flat Minkowski planes.

There are two families of flat Minkowsi planes that we will consider in this paper. The first family is the one constructed by Artzy and Groh [1] using the 3-dimensional group

$$\Phi_1 = \{(x, y) \mapsto (ax + b, ay + c) \mid a, b, c \in \mathbb{R}, a > 0\}.$$

The second family comes from a special type of convex functions called strongly hyperbolic functions. These planes were constructed in [6] and admit the 3-dimensional group

$$\Phi_\infty = \{(x, y) \mapsto (x + b, ay + c) \mid a, b, c \in \mathbb{R}, a > 0\}$$

as their group of automorphisms.

Toroidal circle planes are a generalization of flat Minkowski planes in the sense that they are incidence structures on the torus satisfying all but one geometric axiom for flat Minkowski planes. There are toroidal circle planes which are not flat Minkowski planes, as shown by Polster with an explicit construction in [7].

In this paper, we characterize strongly hyperbolic planes and Artzy–Groh planes among the class of toroidal circle planes in terms of their automorphism groups. We also obtain some information on automorphism groups of the toroidal circle planes constructed by Polster. The content of this paper is based on Chapter 7 of the author's doctoral dissertation [5].

The paper is organized as follows. Section 2 contains preliminary results. In Sect. 3, we describe the circle set of toroidal circle planes with 3-dimensional groups of automorphisms fixing exactly one point. Characterizations for strongly hyperbolic planes and Artzy–Groh planes are contained in Sects. 4 and 5, respectively. Automorphism groups of Polster planes are considered in Sect. 6.

2. Preliminaries

2.1. Toroidal circle planes, flat Minkowski planes and derived planes

A *toroidal circle plane* is a geometry $\mathbb{T} = (\mathcal{P}, \mathcal{C}, \mathcal{G}^+, \mathcal{G}^-)$, whose

point set $\mathcal{P}$ is the torus $\mathbb{S}^1 \times \mathbb{S}^1$,
circles (elements of $\mathcal{C}$) are graphs of homeomorphisms of $\mathbb{S}^1$,
$(+)$-parallel classes (elements of $\mathcal{G}^+$) are the verticals $\{x_0\} \times \mathbb{S}^1$,
$(-)$-parallel classes (elements of $\mathcal{G}^-$) are the horizontals $\mathbb{S}^1 \times \{y_0\}$,

where $x_0, y_0 \in \mathbb{S}^1$. We shall view $\mathbb{S}^1$ as the one-point compactification $\mathbb{R} \cup \{\infty\}$.

We denote the $(\pm)$-parallel class containing a point p by $[p]_\pm$. When two points p, q are on the same $(\pm)$-parallel class, we say they are $(\pm)$-*parallel* and denote this by $p \parallel_\pm q$. Two points p, q are *parallel* if they are $(+)$-parallel or $(-)$-parallel, and we denote this by $p \parallel q$.

Furthermore, a toroidal circle plane satisfies the following

Axiom of joining: three pairwise non-parallel points p, q, r can be joined by a unique circle $\alpha(p, q, r)$.

A toroidal circle plane is called a *flat Minkowski plane* if it also satisfies the following

Axiom of touching: for each circle C and any two nonparallel points p, q with $p \in C$ and $q \notin C$, there is exactly one circle D that contains both points p, q and intersects C only at the point p.

A toroidal circle plane is in *standard representation* if the set $\{(x, x) \mid x \in \mathbb{S}^1\}$ is one of its circles, cf. [8, Sect. 4.2.3]. Up to isomorphisms, every toroidal circle plane can be described in standard representation. In this case, we omit the two parallelisms and refer to $(\mathcal{P}, \mathcal{C})$ as a toroidal circle plane.

Let $\mathbb{T} = (\mathcal{P}, \mathcal{C})$ be a toroidal circle plane. Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the sets of all circles in $\mathcal{C}$ that are graphs of orientation-preserving and orientation-reversing homeomorphisms of $\mathbb{S}_1$, respectively. Then $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$. We call $\mathcal{C}^+$ and $\mathcal{C}^-$ the *positive half* and *negative half* of $\mathbb{T}$, respectively. These two halves are independent of each other, that is, we can exchange halves from different toroidal circle planes and obtain another toroidal circle plane, cf. [8, Sect. 4.3.1]:

Theorem 2.1. *For $i = 1, 2$, let $\mathbb{T}_i$ be two toroidal circle planes. Then $\mathbb{T} = (\mathcal{P}, \mathcal{C}_1^+ \cup \mathcal{C}_2^-)$ is a toroidal circle plane.*

The *derived plane* $\mathbb{T}_p$ of a toroidal circle plane $\mathbb{T}$ at a point p is the incidence geometry whose point set is $\mathcal{P}\backslash([p]_+ \cup [p]_-)$, whose lines are all parallel classes not going through p and all circles of $\mathbb{T}$ going through p. For every point $p \in \mathcal{P}$, the derived plane $\mathbb{T}_p$ is an $\mathbb{R}^2$-plane and even a flat affine plane when $\mathbb{T}$ is a flat Minkowski plane, cf. [8, Theorem 4.2.1]. For more on $\mathbb{R}^2$-planes, we refer the reader to Salzmann et al. [10, Chapter 31] and references therein.

2.2. Automorphisms and the automorphism group

An *isomorphism* between two toroidal circle planes is a bijection between the point sets that maps circles to circles, and induces a bijection between the circle sets. An *automorphism* of a toroidal circle plane $\mathbb{T}$ is an isomorphism from $\mathbb{T}$ to itself. Every automorphism of a toroidal circle plane is continuous and thus a homeomorphism of the torus, cf. [8, Theorem 4.4.1]. With respect to composition, the set of all automorphisms of a toroidal circle plane is a group called the *automorphism group* $\mathrm{Aut}(\mathbb{T})$ of $\mathbb{T}$. The group $\mathrm{Aut}(\mathbb{T})$ is a Lie group of dimension at most 6 with respect to the compact-open topology, cf. [2]. We say a toroidal circle plane has *group dimension* n if its automorphism group has dimension n. Toroidal circle planes with group dimension $n \geqslant 4$ were classified in [2].

When $n = 3$, a list of possible connected groups of automorphisms of toroidal circle planes was presented in [3]. We define the following subgroups of $\mathrm{AGL}_2(\mathbb{R})$:

$$\Phi_\infty := \{(x, y) \mapsto (x + b, ay + c) \mid a, b, c \in \mathbb{R}, a > 0\},$$

and

$$\Phi_d := \{(x, y) \mapsto (ax + b, a^d y + c) \mid a, b, c \in \mathbb{R}, a > 0\},$$

for $d \in \mathbb{R}$. In our situation, we will need the following result from Creutz et al. [3, Lemma 4.3] (compare also Creutz et al. [5, Lemma 5.3.2]). For the convenience of the reader, a sketch of proof is provided.

Theorem 2.2. *Let $\mathbb{T}$ be a toroidal circle plane that admits a 3-dimensional connected group of automorphisms Σ fixing exactly one point p. Under a suitable coordinate system, let $p = (\infty, \infty)$ be the fixed point. Then the derived plane $\mathbb{T}_p$ is Desarguesian and Σ is isomorphic to Φ_d, for some $d \in \mathbb{R} \cup \{\infty\}$. The coordinates may be chosen such that the action of Σ is described by the definition of Φ_d on $\mathbb{R}^2$.*

Sketch of Proof. From Brouwer's Theorem (cf. [8, A2.3.8]) and the classification of $\mathbb{R}^2$-planes with point transitive 3-dimensional collineation group by Groh (cf. [4, Main Theorem 2.6]), one obtains that $\mathbb{T}_p$ is Desarguesian and that Σ is isomorphic to Φ_d for some $d \in \mathbb{R}$. Let the coordinates be chosen such that $\mathbb{T}_p$ equals the real affine plane and the parallel classes are the lines parallel to the x-axis and y-axis. The group Σ acts on $\mathbb{T}_p$ by automorphisms by its definition. It contains all translations, hence it is generated by the translations and the 1-dimensional stabilizer Σ_o fixing the origin. It follows that Σ_o is isomorphic to $\mathbb{R}$ and its action is described by the maps $\{(x, y) \mapsto (x, ay) \mid a > 0\}$, or $\{(x, y) \mapsto (ax, a^d y) \mid a > 0\}$. □

In fact, if the coordinates of $\mathbb{T}$ are chosen such that the conditions in Theorem 2.2 are satisfied, then $\mathbb{T}$ is in standard representation. From the uniqueness of the standard representation, the converse is also true. We then have the following.

Theorem 2.3. *Let $\mathbb{T}$ be a toroidal circle plane in standard representation. Then $\mathbb{T}$ admits a 3-dimensional connected group of automorphisms Σ fixing exactly one point $p = (\infty, \infty)$ if and only if $\Sigma = \Phi_d$, for some $d \in \mathbb{R} \cup \{\infty\}$.*

2.3. Hyperbolic functions and strongly hyperbolic functions

In this section, we will define hyperbolic functions and derive some of their properties. Let $\mathbb{R}^+$ denote the set of positive real numbers. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is *hyperbolic* if it satisfies the following conditions.

(i) $\lim_{x \to 0^+} f(x) = +\infty$ and $\lim_{x \to +\infty} f(x) = 0$.
(ii) f is strictly convex.

Remark 2.4. We note that if f is a hyperbolic function, then f is bijective, continuous and strictly decreasing. The left-derivative f'_- exists, is left-continuous and strictly increasing (cf. [9, Theorem B p. 5]).

Let f be a hyperbolic function. In the remainder of this section, we study the roots of the function $\tilde{f}_{abc} : (\max\{-b, 0\}, \infty) \to \mathbb{R}$ defined by

$$\tilde{f}_{abc}(x) = af(x + b) + c - f(x),$$

where $a > 0, b, c \in \mathbb{R}$. Let $f^\dagger : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f^\dagger(x) = \ln(|f'_-(x)|).$$

Lemma 2.5. *Let f be a hyperbolic function. Assume that there exist $u, v \in \mathbb{R}$ such that the function $\bar{f}_{uv} : (\max\{-u, 0\}, \infty) \to \mathbb{R}$ defined by*

$$\bar{f}_{uv}(x) = f^\dagger(x + u) + v - f^\dagger(x)$$

changes sign at least two times. Then there exist $a > 0$, and $b, c \in \mathbb{R}$, $(a, b, c) \neq (1, 0, 0)$, such that $\tilde{f}_{abc}$ has at least 3 roots.

Proof. Let $a = e^v > 0$ so that $v = \ln a$. Also, let $b = u$. We consider the continuous function $h : (\max\{-u, 0\}, \infty) \to \mathbb{R}$ defined by

$$h(x) = \tilde{f}_{ab0}(x) = af(x + b) - f(x).$$

According to Remark 2.4, the left derivative h'_- exists. Furthermore, $\bar{f}_{uv}(x) \lesseqgtr 0$ if and only if $h'_-(x) \gtreqless 0$. From our assumption, it follows that h'_- also changes sign at least two times. This implies that h has two local extrema.

Assume that $\max\{-b, 0\} = 0$ so that $\lim_{x\to\infty} = 0$ and $\lim_{x\to 0+} = -\infty$. Since h has at least 2 local extrema, it has a local minimum and a local maximum. By the intermediate value theorem, there exists $c \in \mathbb{R}$ such that $h(x) = c$ has at least 3 roots. The case $\max\{-b, 0\} = -b$ is similar and the proof now follows. □

Lemma 2.6. *Let f be a hyperbolic function. If f is not differentiable, then there exist $a > 0$, and $b, c \in \mathbb{R}$, $(a, b, c) \neq (1, 0, 0)$, such that the function $\tilde{f}_{abc}$ has at least 3 roots.*

Proof. If f is not differentiable, then there exists x_0 at which f'_- has a jump discontinuity. Then $f^\dagger$ also has a jump discontinuity at x_0. It follows that there exist $u, v \in \mathbb{R}$ such that the function $\bar{f}_{uv}$ changes sign at least two times. The proof now follows from Lemma 2.5. □

Lemma 2.7. *Let f be a hyperbolic function. Assume that f is differentiable. If the function $f^\dagger = \ln|f'|$ is not strictly convex, then there exist $a > 0$, and $b, c \in \mathbb{R}$, $(a, b, c) \neq (1, 0, 0)$, such that the function $\tilde{f}_{abc}$ has at least 3 roots.*

Proof. We have two cases depending on the convexity of $f^\dagger(x)$.

Case 1: If $f^\dagger(x)$ is convex but not strictly convex, then there exists an interval (x_1, x_2) on which $f^\dagger(x)$ is affine. In particular, there exist $r, s \in \mathbb{R}$ such that $f^\dagger(x) = rx + s$ for all $x \in (x_1, x_2)$. This implies that $f'(x) = -e^{rx+s}$. But then f is an exponential function, which contradicts the assumption f is hyperbolic. So this case cannot occur.

Case 2: If $f^\dagger(x)$ is not convex, then there exists $u > 0$ such that the function $\bar{f} : (\max\{-u, 0\}, \infty) \to \mathbb{R}$ defined by

$$\bar{f}(x) = f^\dagger(x) - f^\dagger(x + u),$$

is not decreasing, cf. [5, Lemma B.2.2]. Also, we have that $\liminf_{x\to 0^+} f'(x) = -\infty$ (cf. [5, Lemma B.1.1]), so that $\limsup_{x\to 0^+} f^\dagger(x) = +\infty$. It follows that $\limsup_{x\to 0^+} \bar{f}(x) = +\infty$, and so $\bar{f}$ cannot be increasing either. Hence $\bar{f}$ is not monotone. This implies that there exist $v \in \mathbb{R}$ such that the function $\bar{f}_{uv} : (\max\{-u, 0\}, \infty) \to \mathbb{R}$ defined by

$$\bar{f}_{uv}(x) = f^\dagger(x + u) + v - f^\dagger(x)$$

changes sign at least two times. The proof now follows from Lemma 2.5.

□

3. On toroidal circle planes with 3-dimensional groups of automorphisms fixing exactly one point

In this section, we provide a general description for the circle set of a toroidal circle plane $\mathbb{T}$ with 3-dimensional group of automorphisms fixing exactly one point. Under a suitable coordinate system, we can assume that the fixed point is $p = (\infty, \infty)$. We will further assume that $\mathbb{T}$ is in standard representation. By Theorem 2.3, the connected component of $\mathrm{Aut}(\mathbb{T})$ is Φ_d, for some $d \in \mathbb{R} \cup \{\infty\}$.

For $s, t \in \mathbb{R}$, we define $\overline{l_{s,t}} := \{(x, sx + t) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\}$, and

$$L := \{\overline{l_{s,t}} \mid s, t \in \mathbb{R}, s < 0\}.$$

For $i = 1, 2$, let $f_i : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function. Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ be defined by

$$f(x) = \begin{cases} f_1(x) & \text{for } x > 0 \\ -f_2(-x) & \text{for } x < 0 \end{cases},$$

and let C_f be the graph of f extended by two points $(0, \infty)$ and $(\infty, 0)$, that is,

$$C_f := \{(x, f(x)) \mid x \in \mathbb{R}\backslash\{0\}\} \cup \{(0, \infty), (\infty, 0)\}.$$

For $d \in \mathbb{R} \cup \{\infty\}$, let F_d be the set of images of C_f under Φ_d and define

$$\mathcal{C}^-(f_1, f_2, \Phi_d) := F_d \cup L.$$

Lemma 3.1. *Let $\mathbb{T}$ be a toroidal circle plane in standard representation. If there exists $d \in \mathbb{R} \cup \{\infty\}$ such that Φ_d is a group of automorphisms of $\mathbb{T}$, then there exist two hyperbolic functions f_1, f_2 such that $\mathcal{C}^-(f_1, f_2, \Phi_d)$ is the negative half of $\mathbb{T}$.*

Proof. Let $\mathcal{C}^-$ be the negative half of $\mathbb{T}$. By Theorem 2.2, there exists a point p such that $\mathbb{T}_p$ is Desarguesian. Under suitable coordinatization, we can assume that $p = (\infty, \infty)$. This implies that $L \subset \mathcal{C}^-$. In the remainder of the proof, we describe circles in $\mathcal{C}^-$ not going through p.

(1) Let $C \in \mathcal{C}^-$ be a circle going through $(0, \infty)$ and $(\infty, 0)$ and let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ be a function such that C is the graph of f extended by two points $(0, \infty)$ and $(\infty, 0)$. Then f has the form

$$f(x) = \begin{cases} f_1(x) & \text{for } x > 0 \\ -f_2(-x) & \text{for } x < 0 \end{cases},$$

where f_1 and f_2 are continuous bijections of $\mathbb{R}^+$.

(2) We now show that f_1 is a hyperbolic function. Since C is the graph of an orientation-reversing homeomorphism of $\mathbb{S}_1$, it follows that $f_1(x)$ is a strictly decreasing. This also implies that $\lim_{x\to 0^+} f_1(x) = \infty$ and $\lim_{x\to\infty} f_1(x) = 0$.

Suppose f_1 is not strictly convex. Then there exists a line l with negative slope such that l intersects the graph of f_1 in at least three points. On the other hand, we have that $l \cup p \in L \subset \mathcal{C}^-$, which contradicts the Axiom of Joining. Therefore f_1 is strictly convex and consequently a hyperbolic function.

(3) Using arguments similar to the above, one can show that f_2 is also a hyperbolic function. Since Φ_d is a group of automorphism of $\mathbb{T}$, the remaining circles not going through p and different from C are images of C under Φ_d. This proof now follows. □

Let φ be the homeomorphism of the torus defined by $\varphi : (x, y) \mapsto (-x, y)$ and let $\mathcal{C}^+(f_1, f_2, \Phi_d) := \varphi(\mathcal{C}^-(f_1, f_2, \Phi_d))$. Our main result of this section the following.

Theorem 3.2. *Let $\mathbb{T}$ be a toroidal circle plane in standard representation. Then there exists $d \in \mathbb{R} \cup \{\infty\}$ such that Φ_d is a group of automorphisms of $\mathbb{T}$ if and only if there exist four hyperbolic functions f_1, f_2, f_3, f_4 such that $\mathcal{C}^-(f_1, f_2, \Phi_d)$ is the negative half of $\mathbb{T}$ and $\mathcal{C}^+(f_3, f_4, \Phi_d)$ is the positive half of $\mathbb{T}$.*

Proof. Assume that there exists $d \in \mathbb{R} \cup \{\infty\}$ such that Φ_d is a group of automorphisms of $\mathbb{T}$. By Lemma 3.1, the negative half of $\mathbb{T}$ is $\mathcal{C}^-(f_1, f_2, \Phi_d)$ for two hyperbolic functions f_1 and f_2. On the other hand, the map φ defined above induces an isomorphism between the two toroidal circle planes $\mathbb{T}$ and $\varphi(\mathbb{T})$. Applying Lemma 3.1 to $\varphi(\mathbb{T})$, there exist two hyperbolic functions f_3, f_4 such that $\mathcal{C}^-(f_3, f_4, \Phi_d)$ is the negative half of $\varphi(\mathbb{T})$. This implies that $\mathcal{C}^+(f_3, f_4, \Phi_d) = \varphi(\mathcal{C}^-(f_3, f_4, \Phi_d))$ is the positive half of $\mathbb{T}$. The proof of the converse direction is straightforward. □

A natural problem is to determine the conditions on the functions f_i and the parameter d which guarantee that the construction described in Theorem 3.2 yields a toroidal circle plane. We do not know a general answer, but some special cases will be treated in the sequel.

4. Characterization of strongly hyperbolic planes

We recall that a strictly convex function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is hyperbolic if $\lim_{x \to 0^+} f(x) = +\infty$ and $\lim_{x \to +\infty} f(x) = 0$. A hyperbolic function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *strongly hyperbolic function* if it satisfies the following additional conditions.

1. For each $b \in \mathbb{R}$,

$$\lim_{x \to +\infty} \frac{f(x+b)}{f(x)} = 1.$$

2. f is differentiable.
3. $\ln |f'(x)|$ is strictly convex.

We also recall the following construction of flat Minkowski planes from Creutz et al. [6]. Let φ be the homeomorphism of the torus defined by $\varphi : (x, y) \mapsto (-x, y)$. For $i = 1..4$, let f_i be a strongly hyperbolic function. Let $\mathcal{C} := \mathcal{C}^-(f_1, f_2, \Phi_\infty) \cup \mathcal{C}^+(f_3, f_4, \Phi_\infty)$, where $\mathcal{C}^+(f_3, f_4, \Phi_\infty) = \varphi(\mathcal{C}^-(f_3, f_4, \Phi_\infty))$. Then $\mathcal{M}_f := (\mathcal{P}, \mathcal{C})$ is a flat Minkowski plane, which we will refer to as a *strongly hyperbolic (flat Minkowski) plane.*

In this section, we prove the following characterization of strongly hyperbolic planes.

Theorem 4.1. *Let $\mathbb{T}$ be a toroidal circle plane in standard representation. Then $\mathbb{T}$ is a flat Minkowski plane isomorphic to a strongly hyperbolic plane $\mathcal{M}_f$ if and only if $\mathbb{T}$ admits a group of automorphisms isomorphic to Φ_∞.*

Proof. The "only if" direction is straightforward from the construction of strongly hyperbolic planes. For the converse direction, we let $\mathbb{T}$ be a toroidal circle plane in standard representation admitting a group of automorphisms Σ isomorphic to Φ_∞. From [3, Theorem 1.2], Σ fixes exactly one point p. Under suitable coordinates, we can assume that $p = (\infty, \infty)$. By Theorem 2.3, $\Sigma = \Phi_\infty$. By Theorem 3.2, there exist two hyperbolic functions f_1, f_2 such that $\mathcal{C}^-(f_1, f_2, \Phi_\infty)$ is the negative half of $\mathbb{T}$. Similarly, there exist two hyperbolic functions f_3, f_4 such that $\mathcal{C}^+(f_3, f_4, \Phi_\infty)$ is the positive half of $\mathbb{T}$.

(1) We now show that f_1 is a strongly hyperbolic function. By Lemmas 2.6 and 2.7, f_1 is differentiable and $\ln |f_1'(x)|$ is strictly convex. It remains to show that, for each $b \in \mathbb{R}$,

$$\lim_{x\to+\infty} \frac{f_1(x+b)}{f_1(x)} = 1.$$

The case $b = 0$ is trivial. We consider the case $b > 0$. From the Axiom of Joining, there exists a unique circle going through the three points $(0, 1)$ and $(\infty, 0)$ and (x, y), where $x > 0, 0 < y < 1$. In particular, for each $x > 0$ and $0 < y < 1$, the equation $y = \dfrac{f_1(x+b)}{f_1(b)}$ has a unique solution $b > 0$. Define $g : \mathbb{R}^+ \to \mathbb{R}$ as

$$g(b) = \frac{f_1(x+b)}{f_1(b)}.$$

Then $g(b)$ is continuous, $g(b) \in (0, 1)$ for all $b > 0$, and $\lim_{b\to 0^+} g(b) = 0$. To satisfy the Axiom of Joining, $g(b)$ must be strictly increasing and $\lim_{b\to\infty} g(b) = 1$, that is,

$$\lim_{b\to\infty} \frac{f_1(x+b)}{f_1(b)} = 1.$$

Reverse the role of b and x gives the condition as stated. For the case $b < 0$, we can rewrite

$$\frac{f_1(x+b)}{f_1(x)} = \left(\frac{f_1(x'-b)}{f_1(x')}\right)^{-1}$$

and apply the argument as in the case $b > 0$.

(2) Let $\gamma : \mathcal{P} \to \mathcal{P}$ be the homeomorphism defined by $\gamma : (x, y) \mapsto (-x, -y)$. Then γ induces an isomorphism between the two toroidal circle planes $\mathbb{T}$ and $\gamma(\mathbb{T})$. In particular, $\gamma(\mathcal{C}^-(f_1, f_2, \Phi_\infty)) = \mathcal{C}^-(f_2, f_1, \Phi_\infty)$ is the negative half of $\gamma(\mathbb{T})$. Applying the argument in part 1) for $\gamma(\mathbb{T})$, it follows that f_2 is strongly hyperbolic. Similarly, one can show that f_3 and f_4 are also strongly hyperbolic functions. The proof now follows.

□

Remark 4.2. We note that the groups Φ_0 and Φ_∞ are isomorphic both as groups and as transformation groups of the real plane. Because of this isomorphism, Theorem 4.1 is equally true with Φ_0 in place of Φ_∞.

5. Characterization of Artzy–Groh planes

Perhaps the first group-theoretic construction of flat Minkowski planes is the one introduced in 1986 by Artzy and Groh, cf. [1]. Let $f, g : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ be two homeomorphisms satisfying the following conditions.

(i) The functions f and g are differentiable.
(ii) The restrictions of f' and g' to both $\mathbb{R}^+$ and $\mathbb{R}^-$ are strictly monotonic.
(iii) The restrictions of f and g to both $\mathbb{R}^+$ and $\mathbb{R}^-$ have the x-axis and the y-axis as asymptotes.
(iv) $f'(x) < 0$ and $g'(x) > 0$ for all $x \in \mathbb{R}\backslash\{0\}$.

For $a > 0, b, c \in \mathbb{R}$, let $f_{a,b,c} : \mathbb{R}\backslash\{-b\} \to \mathbb{R}\backslash\{c\}$ be defined by

$$f_{a,b,c}(x) = af\left(\frac{x+b}{a}\right) + c.$$

For $a < 0, b, c \in \mathbb{R}$, let $g_{a,b,c} : \mathbb{R}\backslash\{-b\} \to \mathbb{R}\backslash\{c\}$ be defined by

$$g_{a,b,c}(x) = ag\left(\frac{x+b}{|a|}\right) + c.$$

The circle set of an *Artzy–Groh plane* $\mathcal{M}'(f, g)$ consists of sets of the form

$$\{(x, sx + t) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\},$$

where $s, t \in \mathbb{R}$, $s \neq 0$, sets of the form

$$\{(x, f_{a,b,c}(x)) \mid x \in \mathbb{R}\backslash\{-b\}\} \cup \{(\infty, c), (-b, \infty)\},$$

where $a, b, c \in \mathbb{R}$, $a > 0$, and sets of the form

$$\{(x, g_{a,b,c}(x)) \mid x \in \mathbb{R}\backslash\{-b\}\} \cup \{(\infty, c), (-b, \infty)\},$$

where $a, b, c \in \mathbb{R}$, $a < 0$.

It is known that a flat Minkowski plane $\mathcal{M}$ is isomorphic to an Artzy–Groh plane if and only if $\mathcal{M}$ admits the group

$$\Phi_1 {:=} \{(x, y) \mapsto (ax + b, ay + c) \mid a, b, c \in \mathbb{R}, a > 0\}$$

as a group of automorphisms, cf. [8, Theorem 4.4.13]. In this section, we obtain the same result without assuming the Axiom of Touching. We start by defining a set of triangles $T(m_1, m_2, m_3)$.

A *triangle* abc is a triple $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2)$ of non-collinear points in $\mathbb{R}^2$. For $m_1, m_2, m_3 < 0$ such that $m_1 < m_2 < m_3$, let $T(m_1, m_2, m_3)$ be the set of triangles abc such that

$$\begin{cases} \dfrac{a_2 - b_2}{a_1 - b_1} = m_1 \\ \dfrac{a_2 - c_2}{a_1 - c_1} = m_2 \\ \dfrac{b_2 - c_2}{b_1 - c_1} = m_3 \\ a_1 < b_1 < c_1 \end{cases}$$

Each point a, b, c is called a *vertex* of the triangle abc. Given a triangle abc, we denote by $\phi(abc)$ the triangle formed by the images of a, b, c under a map $\phi \in \Phi_1$. The existence of such a triangle comes from the fact that Φ_1 is connected and preserves the slope of lines. Furthermore, Φ_1 is sharply transitive on $T(m_1, m_2, m_3)$.

Lemma 5.1. *Let f be a hyperbolic function. If f is not differentiable, then there exist $m_1 < m_2 < m_3 < 0$ such that $T(m_1, m_2, m_3)$ has no elements on the graph of f.*

Proof. Assume that f is not differentiable. Then there exists a point x_0 such that

$$f'_-(x_0) < f'_+(x_0) < 0.$$

Let $m_1, m_2, m_3 \in (f'_-(x_0), f'_+(x_0))$ satisfying $m_1 < m_2 < m_3$. Suppose for a contradiction that there exists a triangle $abc \in T(m_1, m_2, m_3)$ that lies on the graph C of f. Let $C_1 = \{(x, f_1(x)) \mid 0 < x \leqslant x_0\}$ and $C_2 = \{(x, f_1(x)) \mid x > x_0\}$, so that $C = C_1 \cup C_2$. By the pigeonhole principle, either C_1 or C_2 contains at least two vertices of abc.Assume without loss of generality that a and b are on C_1. This implies that

$$m_1 < f'_{1-}(x_0) < m_1 < f'_{1+}(x_0),$$

which is a contradiction. □

Theorem 5.2. *Let $\mathbb{T}$ be a toroidal circle plane in standard representation. Then $\mathbb{T}$ is a flat Minkowski plane isomorphic to an Artzy–Groh plane if and only if $\mathbb{T}$ admits a group of automorphisms isomorphic to Φ_1.*

Proof. The "only if" is straightforward from the construction of Artzy–Groh planes. It remains to prove the converse direction. Similar to the proof of Theorem 4.1, without loss of generality we can assume that $\mathbb{T}$ is a toroidal circle plane in standard representation admitting Φ_1 as a group of automorphisms. By Theorem 3.2, there exist two hyperbolic functions f_1, f_2 such that $\mathcal{C}^-(f_1, f_2, \Phi_1)$ is the negative half of $\mathbb{T}$. Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ be defined by

$$f(x) = \begin{cases} f_1(x) & \text{for } x > 0 \\ -f_2(-x) & \text{for } x < 0 \end{cases},$$

and let C_f be the graph of f extended by two points $(0, \infty)$ and $(\infty, 0)$.

If f_1 is not differentiable, then by Lemma 5.1, there exists $m_1 < m_2 < m_3 < 0$ such that $T(m_1, m_2, m_3)$ has no elements on the graph of f_1. Let $abc \in T(m_1, m_2, m_3)$. From the Axiom of Joining, there exists a circle $D \in \mathcal{C}^-(f_1, f_2, \Phi_1)$ containing abc. Since a, b, c are non-collinear, D does not contain (∞, ∞). This implies that there exists $\phi \in \Phi_1$ such that $\phi(D) = C_f$. In particular, $\phi(abc) \subset C_f$. This is a contradiction, since $\phi(abc) \in T(m_1, m_2, m_3)$. Hence, f_1 is differentiable.

Similarly, f_2 is also differentiable. It follows that $\mathcal{C}^-(f_1, f_2, \Phi_1)$ is the negative half of an Artzy–Groh plane. The case of the positive half is analogous. The proof now follows. □

6. On automorphism groups of Polster planes

In this section, we consider a family of proper toroidal circle planes that are not flat Minkowski planes. This family was introduced by Polster [7] originally in the extended Cartesian coordinate system rotated by 45 degrees. We will refer to a plane in this family as a *Polster plane*. An intuitive explanation of this construction can be found in [8, Subsection 4.3.7]. In the standard coordinate system, a Polster plane can be described as follows.

Let f, g be functions satisfying the four conditions in the construction of Artzy–Groh planes, with f satisfying the additional property $f = f^{-1}$. For $a > 0, b, c \in \mathbb{R}$, let $f_{a,b,c} : \mathbb{R}\backslash\{-b\} \to \mathbb{R}\backslash\{c\}$ be defined by

$$f_{a,b,c}(x) = \begin{cases} af\left(\dfrac{x+b+1}{a}\right) + c & \text{for } x \geqslant x^*, \\[2ex] af\left(\dfrac{x+b}{a}\right) + c - 1 & \text{for } -b < x \leqslant x^*, \\[2ex] af\left(\dfrac{x+b}{a}\right) + c & \text{for } x < -b, \end{cases}$$

where $x^* \in (-b, +\infty)$ satisfies

$$af\left(\frac{x^*+b+1}{a}\right) = x^* + b.$$

For $a < 0, b, c \in \mathbb{R}$, let $g_{a,b,c} : \mathbb{R}\backslash\{-b\} \to \mathbb{R}\backslash\{c\}$ be defined by

$$g_{a,b,c}(x) = ag\left(\frac{x+b}{|a|}\right) + c.$$

The circle set of a *Polster plane* $\mathbb{T}(f,g)$ consists of sets of the form

$$\{(x, sx+t) \mid x \in \mathbb{R}\} \cup \{(\infty,\infty)\},$$

where $s,t \in \mathbb{R}$, $s \neq 0$, sets of the form

$$\{(x, f_{a,b,c}(x)) \mid x \in \mathbb{R}\backslash\{-b\}\} \cup \{(\infty,c),(-b,\infty)\},$$

where $a,b,c \in \mathbb{R}$, $a>0$, and sets of the form

$$\{(x, g_{a,b,c}(x)) \mid x \in \mathbb{R}\backslash\{-b\}\} \cup \{(\infty,c),(-b,\infty)\},$$

where $a,b,c \in \mathbb{R}$, $a<0$.

In this section, we determine the dimension of the automorphism group of a Polster plane and describe its connected component. For convenience, we denote the restriction of $f_{a,0,0}$ to $\mathbb{R}^+$ by f_a, that is, for $a>0$, let

$$f_a(x) = \begin{cases} af\left(\dfrac{x+1}{a}\right) & \text{for } x \geqslant x_a^*, \\ af\left(\dfrac{x}{a}\right)-1 & \text{for } 0 < x \leqslant x_a^*, \end{cases}$$

where x_a^* satisfies the equation

$$af\left(\frac{x_a^*+1}{a}\right) = x_a^*.$$

Lemma 6.1. *Let $\mathbb{T}(f,g)$ be a Polster plane in standard representation. For $d \in \mathbb{R}\cup\{\infty\}$, the group Φ_d is not a group of automorphims of $\mathbb{T}(f,g)$.*

Proof. A Polster plane $\mathbb{T}(f,g)$ is not a flat Minkowksi plane since it does not satisfy the Axiom of Touching. From Theorems 4.1, 5.2 and Remark 4.2, for $d \in \{0,1,\infty\}$, the group Φ_d is not a group of automorphims of $\mathbb{T}(f,g)$. For the remaining cases, fix $d \in \mathbb{R}\backslash\{0,1\}$ and suppose for a contradiction that the group Φ_d is a group of automorphims of $\mathbb{T}(f,g)$.

For $1 \neq r > 0$, let $\phi \in \Phi_d$ be the map defined by $\phi : (x,y) \mapsto (rx, r^d y)$. Let $a_0 > 0$. Under ϕ, the set of points$\{(x, f_{a_0}(x)) \mid x \in \mathbb{R}^+\}$ is mapped onto $\left\{\left(x, r^d f_{a_0}\left(\frac{x}{r}\right)\right) \mid x \in \mathbb{R}^+\right\}$. Since ϕ fixes $(0,\infty)$ and $(\infty,0)$, it fixes the set of circles going through these two points. This implies that there exists $a_1 > 0$ such that, for all $x>0$,

$$r^d f_{a_0}\left(\frac{x}{r}\right) = f_{a_1}(x).$$

It is necessary that $rx_{a_0}^* = x_{a_1}^*$ and $r^d f_{a_0}\left(\dfrac{rx_{a_0}^*}{r}\right) = f_{a_1}\left(x_{a_1}^*\right)$. In particular,

$$ra_0 f\left(\frac{x_{a_0}^*+1}{a_0}\right) = a_1 f\left(\frac{x_{a_1}^*+1}{a_1}\right),$$

and

$$r^d a_0 f\left(\frac{x_{a_0}^*+1}{a_0}\right) = a_1 f\left(\frac{x_{a_1}^*+1}{a_1}\right).$$

It follows that $r^d = r$, a contradiction. □

Theorem 6.2. *The full automorphism group of a Polster plane* $\mathbb{T}(f, g)$ *is* 2-*dimensional. Its connected component is the translation group* $\mathbb{R}^2$.

Proof. Let Σ be the connected component of the full automorphism group of $\mathbb{T}(f, g)$. Since Σ contains the group of translations $\mathbb{R}^2$, its dimension is at least 2. On the other hand, Σ cannot have dimension greater than 3, cf. [2, Theorem 1.2]. Suppose that $\dim \Sigma = 3$. From [3, Theorem 1.2], Σ fixes exactly one point p. Under suitable coordinates, we can assume that $p = (\infty, \infty)$. By Theorem 2.3, $\Sigma = \Phi_d$, for some $d \in \mathbb{R} \cup \{\infty\}$. But this contradicts Lemma 6.1. Therefore, $\dim \Sigma = 2$. From [10, 93.12], Σ is the translation group and the proof now follows. □

Acknowledgements

The author would like to thank Günter Steinke and Brendan Creutz for valuable discussions and suggestions. The author also wishes to thank the anonymous reviewer for the careful reading and for pointing out an issue with isomorphisms in the preliminary version of the paper. This work was supported by a University of Canterbury Doctoral Scholarship and partially by UAEU Grant G00003490.

Author contributions D.H. wrote the main manuscript text and reviewed the manuscript.

Data availability statement Not applicable.

Declarations

Conflict of interest The corresponding author states that there is no conflict of interest.

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Received: August 23, 2023.
Revised: October 20, 2023.
Accepted: October 27, 2023.