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On α-conformal equivalence of statistical submanifolds

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Abstract. In this paper, we show a procedure to realize a statistical manifold, which is α-conformally equivalent to a manifold with an α-transitively flat connection, as a statistical submanifold.

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1. Introduction

Statistical manifolds are studied in terms of information geometry. The theory of α-connections of statistical manifolds plays an important role especially on statistical inference. In addition, considering conformal transformation into α-connections, Okamoto, Amari and Takeuchi obtain asymptotic theory of sequential estimation [3]. Kurose defined α-conformal equivalence and α-conformal flatness of statistical manifolds [2]. In our previous paper, we gave an example for a 1-conformally flat statistical submanifold of a flat statistical manifold, using a Hessian domain [4] [5]. In this paper, we show a procedure to realize a statistical manifold, which is α-conformally equivalent to a manifold with an α-transitively flat connection, as a statistical submanifold. An α-transitively flat connection is one of α-connections.

2. α-transitively flat connections on statistical manifolds

For a torsion-free affine connection ∇ and a pseudo-Riemannian metric h on a manifold N, the triple (N, ∇, h) is called a statistical manifold if ∇h is symmetric. If the curvature tensor R of ∇ vanishes, (N, ∇, h) is said to be flat.

For a statistical manifold (N, ∇, h), let ∇' be an affine connection on N such that

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla'_X Z) \quad \text{for } X, Y, Z \in \mathcal{X}(N),$$

where $\mathcal{X}(N)$ is the set of all tangent vector fields on N. The affine connection ∇' is torsion free, and $\nabla' h$ symmetric. Then ∇' is called the dual connection of ∇, the triple (N, ∇', h) the dual statistical manifold of (N, ∇, h), and (∇, ∇', h) the dualistic structure on N. The curvature tensor of ∇' vanishes if and only if that of ∇ does, and then (∇, ∇', h) is called the dually flat structure.

Let N be a manifold with a dualistic structure (∇, ∇', h). For a real number α, an affine connection defined by

$$\nabla^{(\alpha)} := \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla'$$

is called an α-connection of (N, ∇, h). The triple $(N, \nabla^{(\alpha)}, h)$ is also a statistical manifold, and $\nabla^{(-\alpha)}$ the dual connection of $\nabla^{(\alpha)}$. The 1-connection, the (-1)-connection and the 0-connection coincide with ∇, ∇' and Levi-Civita connection of (N, h), respectively. An α-connection is not always flat [1].

If (N, ∇, h) is a flat statistical manifold, we call $\nabla^{(\alpha)}$ an α-transitively flat connection of (N, ∇, h). An α-transitively flat connection is not always flat.

For $\alpha \in \mathbf{R}$, statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ are said to be α-conformally equivalent if there exists a function ϕ on N such that

$$\begin{aligned} \bar{h}(X, Y) &= e^{\phi} h(X, Y), \\ h(\bar{\nabla}_X Y, Z) &= h(\nabla_X Y, Z) - \frac{1+\alpha}{2} d\phi(Z) h(X, Y) \\ &\quad + \frac{1-\alpha}{2}\{d\phi(X) h(Y, Z) + d\phi(Y) h(X, Z)\} \end{aligned}$$

for $X, Y, Z \in \mathcal{X}(N)$. Two statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ are α-conformally equivalent if and only if the dual statistical manifolds (N, ∇', h) and $(N, \bar{\nabla}', \bar{h})$ are $(-\alpha)$-conformally equivalent. A statistical manifold (N, ∇, h) is called α-conformally flat if (N, ∇, h) is locally α-conformally equivalent to a flat statistical manifold [2].

3. α-transitively flat connections and α-conformal equivalence

We relate an α-transitively flat connection of a flat statistical manifold with an α-conformal equivalence of its statistical submanifold. Statistical submanifolds are defined in [4] and [6].

LEMMA 3.1. *Let (N, ∇, h) be a flat statistical manifold, and (M, D, g) a 1-conformally flat statistical submanifold realized in (N, ∇, h). Let M_o be a simply connected open set of M. If (M_o, D, g) is 1-conformally equivalent to a flat statistical manifold $(M_o, \bar{D}, \bar{g})$, $(M_o, D^{(\alpha)}, g)$ is α-conformally equivalent to $(M_o, \bar{D}^{(\alpha)}, \bar{g})$, where $D^{(\alpha)}$ the induced connection on M_o by an α-transitively flat connection $\nabla^{(\alpha)}$ of (N, ∇, h), and $\bar{D}^{(\alpha)}$ an α-transitively flat connection of $(M_o, \bar{D}, \bar{g})$.*

Proof. Let D' and $\bar{D}'$ be the dual connection of D and $\bar{D}$, respectively. Since $D^{(\alpha)}$ is induced by $\nabla^{(\alpha)}$,

$$D^{(\alpha)} = \frac{1+\alpha}{2} D + \frac{1-\alpha}{2} D' \quad \text{on } M_o \tag{1}$$

holds. For 1-(resp. (-1)-) conformal equivalence of (D, g) and $(\bar{D}, \bar{g})$ (resp. of (D', g) and $(\bar{D}', \bar{g})$), there exists a function ϕ on M_o such that

$$\bar{g}(X, Y) = e^{\phi} g(X, Y), \tag{2}$$

$$g(\bar{D}_X Y, Z) = g(D_X Y, Z) - d\phi(Z) g(X, Y), \quad \text{and} \tag{3}$$

$$g(\bar{D}'_X Y, Z) = g(D'_X Y, Z) + d\phi(X) g(Y, Z) + d\phi(Y) g(X, Z) \tag{4}$$

for $X, Y, Z \in \mathcal{X}(M_o)$. From (3) and (4), it follows that

$$\begin{aligned} &g\left(\left(\frac{1+\alpha}{2}\bar{D} + \frac{1-\alpha}{2}\bar{D}'\right)_X Y, Z\right) \\ &= g\left(\left(\frac{1+\alpha}{2}D + \frac{1-\alpha}{2}D'\right)_X Y, Z\right) - \frac{1+\alpha}{2} d\phi(Z) g(X, Y) \\ &\quad + \frac{1-\alpha}{2}\{d\phi(X) g(Y, Z) + d\phi(Y) g(X, Z)\}. \end{aligned}$$

By (1) and the definition of an α-transitively flat connection of $(M_o, \bar{D}, \bar{g})$,

$$\begin{aligned} g(\bar{D}^{(\alpha)}_X Y, Z) \ = \ &g(D^{(\alpha)}_X Y, Z) - \frac{1+\alpha}{2} d\phi(Z) g(X, Y) \\ &+ \frac{1-\alpha}{2}\{d\phi(X) g(Y, Z) + d\phi(Y) g(X, Z)\} \end{aligned}$$

holds. This implies Lemma 3.1. □

4. Realization as statistical submanifolds

We call $(N, \tilde{\nabla}, h)$ a statistical manifold with an α-transitively flat connection if there exists a flat statistical manifold (N, ∇, h) such that $\tilde{\nabla}$ coincides with an α-transitively flat connection of (N, ∇, h). In this section, we give a procedure to realize a statistical manifold, which is α-conformally equivalent to a statistical manifold with an α-transitively flat connection, in another statistical manifold as a statistical submanifold of codimension one.

On realization of a 1-conformally flat statistical manifold, the next theorem is known.

THEOREM 4.1. ([4]) *An arbitrary* 1*-conformally flat statistical manifold of* $\dim n \geq 2$ *with a Riemannian metric can be locally realized as a submanifold of a flat statistical manifold of* $\dim(n+1)$.

For a statistical manifold which is α-conformally equivalent to a statistical manifold with an α-transitively flat connection, we obtain the next theorem.

THEOREM 4.2. *A statistical manifold of* $\dim n \geq 2$ *with a Riemannian metric, which is* α*-conformally equivalent to a statistical manifold with an* α*-transitively flat connection for non-zero* $\alpha \in \mathbf{R}$*, can be locally realized as a submanifold of a statistical manifold of* $\dim(n+1)$ *with an* α*-transitively flat connection.*

For the proof of Theorem 4.2, we show the next lemma.

LEMMA 4.3. *For non-zero* $\alpha \in \mathbf{R}$*, let* $(M, D^{(\alpha)}, g)$ *be an* α*-conformally equivalent statistical manifold to* $(M, \bar{D}^{(\alpha)}, \bar{g})$*, where* $\bar{D}^{(\alpha)}$ *is an* α*-transitively flat connection of* $(M, \bar{D}, \bar{g})$*. Set* $D^{(\beta)} := D^{LC} + \frac{\beta}{\alpha}(D^{(\alpha)} - D^{LC})$ *for an arbitrary* $\beta \in \mathbf{R}$*, where* D^{LC} *the Levi-Civita connection of* (M, g)*. Then* $(M, D^{(\beta)}, g)$ *is* β*-conformally equivalent to* $(M, \bar{D}^{(\beta)}, \bar{g})$*.*

Proof. First, we show that (M, D^{LC}, g) is 0-conformally equivalent to $(M, \bar{D}^{(0)}, \bar{g})$. Recall that the 0-connection $\bar{D}^{(0)}$ is the Levi-Civita connection. Setting by $D^{(\alpha)\prime}$ the dual connection of $D^{(\alpha)}$, we have $(-\alpha)$-conformal equivalence of $(M, D^{(\alpha)\prime}, g)$ and $(M, \bar{D}^{(-\alpha)}, \bar{g})$ from a fact described in Section 1. Thus we obtain that

$$\begin{aligned}
g(\bar{D}^{(0)}_X Y, Z) &= g\left(\left(\frac{1}{2}\bar{D}^{(\alpha)} + \frac{1}{2}\bar{D}^{(-\alpha)}\right)_X Y, Z\right) \\
&= \frac{1}{2}\Bigg\{ g(D^{(\alpha)}_X Y, Z) - \frac{1+\alpha}{2} d\phi(Z) g(X, Y) \\
&\quad + \frac{1-\alpha}{2}\{d\phi(X) g(Y, Z) + d\phi(Y) g(X, Z)\} \\
&\quad + g(D^{(\alpha)\prime}_X Y, Z) - \frac{1-\alpha}{2} d\phi(Z) g(X, Y) \\
&\quad + \frac{1+\alpha}{2}\{d\phi(X) g(Y, Z) + d\phi(Y) g(X, Z)\}\} \\
&= g\left(\left(\frac{1}{2}D^{(\alpha)} + \frac{1}{2}D^{(\alpha)\prime}\right)_X Y, Z\right) - \frac{1}{2} d\phi(Z) g(X, Y) \\
&\quad + \frac{1}{2}\{d\phi(X) g(Y, Z) + d\phi(Y) g(X, Z)\Bigg\} \\
&= g(D^{LC}_X Y, Z) - \frac{1}{2} d\phi(Z) g(X, Y) \\
&\quad + \frac{1}{2}\{d\phi(X) g(Y, Z) + d\phi(Y) g(X, Z)\}
\end{aligned}$$

for a certain function ϕ on $M_o \subset M$. This implies 0-conformal equivalence of (M, D^{LC}, g) and $(M, \bar{D}^{(0)}, \bar{g})$.

By definitions of $\bar{D}^{(\alpha)}$ and $\bar{D}^{(\beta)}$, $\bar{D}^{(\beta)} = \bar{D}^{(0)} + \frac{\beta}{\alpha}(\bar{D}^{(\alpha)} - \bar{D}^{(0)})$ holds. Hence it follows that

$$
\begin{aligned}
g(D_X^{(\beta)}Y, Z) &= g\left(\left(D^{LC} + \frac{\beta}{\alpha}(D^{(\alpha)} - D^{LC})\right)_X Y, Z\right) \\
&= \frac{\alpha-\beta}{\alpha} g(D_X^{LC}Y, Z) + \frac{\beta}{\alpha} g(D_X^{(\alpha)}Y, Z) \\
&= \frac{\alpha-\beta}{\alpha}\Big\{ g(\bar{D}_X^{(0)}Y, Z) + \frac{1}{2} d\phi(Z) g(X, Y) \\
&\quad - \frac{1}{2}\{d\phi(X)g(Y, Z) + d\phi(Y)g(X, Z)\}\Big\} \\
&\quad + \frac{\beta}{\alpha}\Big\{ g(\bar{D}_X^{(\alpha)}Y, Z) + \frac{1+\alpha}{2} d\phi(Z) g(X, Y) \\
&\quad - \frac{1-\alpha}{2}\{d\phi(X)g(Y, Z) + d\phi(Y)g(X, Z)\}\Big\} \\
&= g\left(\left(\bar{D}^{(0)} + \frac{\beta}{\alpha}(\bar{D}^{(\alpha)} - \bar{D}^{(0)})\right)_X Y, Z\right) + \frac{1+\beta}{2} d\phi(Z) g(X, Y) \\
&\quad - \frac{1-\beta}{2}\{d\phi(X)g(Y, Z) + d\phi(Y)g(X, Z)\} \\
&= g(\bar{D}_X^{(\beta)}Y, Z) + \frac{1+\beta}{2} d\phi(Z) g(X, Y) \\
&\quad - \frac{1-\beta}{2}\{d\phi(X)g(Y, Z) + d\phi(Y)g(X, Z)\}.
\end{aligned}
$$

This implies Lemma 4.3. □

Finally, we shall prove Theorem 4.2.

Let M be a manifold of $\dim n \geq 2$, and g, $\bar{g}$ Riemannian metrics. By Lemma 4.3, $(M, D^{(1)}, g)$ is 1-conformally equivalent to a flat statistical manifold $(M, \bar{D}^{(1)}, \bar{g})$. By Theorem 4.1, $(M, D^{(1)}, g)$ can be locally realized as a submanifold of a flat statistical manifold of $\dim(n+1)$. Suppose that $(M_o, D^{(1)}, g)$ is realized in a flat statistical manifold (N, ∇, h) for a simply connected open set $M_o \subset M$. Let $D_{sub}^{(\alpha)}$ be the induced connection on M_o by an α-connection $\nabla^{(\alpha)}$ of (N, ∇, h). By Lemma 3.1, $(M, D_{sub}^{(\alpha)}, g)$ is α-conformally equivalent to $(M, \bar{D}^{(\alpha)}, \bar{g})$. Moreover,

$$D_{sub}^{(\alpha)} = D^{LC} + \alpha(D^{(1)} - D^{LC})$$

holds by (1). Considering the definition of $D^{(1)}$, we have

$$D^{(\alpha)} = D^{LC} + \alpha(D^{(1)} - D^{LC}).$$

Thus $D^{(\alpha)}_{sub}$ coincides with $D^{(\alpha)}$. Hence $(M, D^{(\alpha)}, g)$, which is α-conformally equivalent to a statistical manifold with an α-transitively flat connection, can be realized in $(N, \nabla^{(\alpha)}, h)$ as a submanifold of codimension one.

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