J. Geom. (2022) 113:41

0047-2468/22/030001-16
published online August 29, 2022
https://doi.org/10.1007/s00022-022-00657-z

Journal of Geometry

Real hypersurfaces in a nonflat complex space form whose certain tensor is recurrent

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Abstract. Real hypersurfaces in a nonflat complex space form (namely, a complex projective space or a complex hyperbolic space) are interesting objects among submanifolds in Riemannian manifolds. It is known that a real hypersurface in a nonflat complex space form admits an almost contact metric structure (ϕ, ξ, η, g) induced from the ambient space. Hence we are interested in real hypersurfaces from the aspects of both submanifolds and almost contact metric manifolds. In this paper, we study real hypersurfaces in a nonflat complex space form from the viewpoint of a recurrence of the tensor field $h(= (1/2)\mathcal{L}_\xi \phi)$. We note that the tensor h plays an important role in contact Riemannian geometry. We give a new classification which includes a special class of 3-dimensional ruled real hypersurfaces in a complex hyperbolic plane $\mathbb{C}H^2(c)$.

Mathematics Subject Classification. 53B25, 53C15, 53D15.

Keywords. Nonflat complex space forms, real hypersurfaces, hopf hypersurfaces, ruled real hypersurfaces, the tensor field h, recurrent tensors.

1. Introduction

In this paper, we denote by $\widetilde{M}_n(c)$ $(n \geqq 2)$ a nonflat complex space form (namely, $\widetilde{M}_n(c)$ is congruent to either a complex projective space $\mathbb{C}P^n(c)$ of constant holomophic sectional curvature $c > 0$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ of holomophic sectional curvature $c < 0$). In particular, we are interested in *real hypersurfaces* in $\widetilde{M}_n(c)$. It is well-known that a real hypersurface in $\widetilde{M}_n(c)$ admits an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure J of the ambient space. Hence the theory of real hypersurfaces in $\widetilde{M}_n(c)$ has the aspect of not only submanifolds theory but also almost contact metric geometry.

In particular, we investigate the behavior of the tensor $h(= (1/2)\mathcal{L}_\xi\phi)$ on real hypersurfaces in $\widetilde{M}_n(c)$, where $\mathcal{L}$ is the Lie derivative. In contact Riemannian geometry, the tensor h plays an important role. Indeed, a contact Riemannian manifold M^{2n-1} satisfies the condition $h = 0$ if and only if M^{2n-1} is a *K-contact manifold* (namely, the characteristic vector field ξ is a Killing vector field). The author studied the parallelism of the tensor h on real hypersurfaces in $\widetilde{M}_n(c)$ ([19,20]). Then we proved the following result:

Theorem 1. ([19]) *Let M^{2n-1} be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then M^{2n-1} satisfies*

$$\nabla_X h = 0 \tag{1}$$

for any tangent vector field X orthogonal to the characteristic vector field ξ if and only if M^{2n-1} is locally congruent to a real hypersurface of type (A) *in $\widetilde{M}_n(c)$.*

This theorem gives the characterization of real hypersurfaces of type (A) in $\widetilde{M}_n(c)$ from the viewpoint of the parallelism of the tensor h. Real hypersurfaces of type (A) in $\widetilde{M}_n(c)$ are known as nice examples, because these real hypersurfaces appear in many classification theorems. Many geometers have found characterizations of real hypersurface of type (A) in $\widetilde{M}_n(c)$ by various other conditions (see [4]). These results tell us that these real hypersurfaces tend to admit the common properties of both a complex projective space $\mathbb{C}P^n(c)$ and a complex hyperbolic space $\mathbb{C}H^n(c)$.

On the other hand, we are also interested in the differences of these spaces. For example, there exists a homogeneous ruled real hypersurface in $\mathbb{C}H^n(c)$ but there exists no homogeneous one in $\mathbb{C}P^n(c)$. In this paper, we consider the following question:

Question 1. *Does there exist a nice condition which gives the difference between $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$ from the aspect of the tensor h?*

The purpose of this paper is to give an answer to this question. To execute this, we define the following two conditions (2) and (3). The tensor h is $\mathbb{D}$*-recurrent* if there exists a 1-form ω on M^{2n-1} such that

$$(\nabla_X h)Y = \omega(X)hY \tag{2}$$

for all vectors X orthogonal to the characteristic vector field ξ and $Y \in TM$, where TM is the tangent bundle of M^{2n-1}. This condition is a generalization of the condition (1). In addition, this condition also gives a characterization of real hypersurfaces of type (A) in $\widetilde{M}_n(c)$. However we can not get the answer to the above question by the condition (2). Hence we consider an improvement of the condition (2). The tensor h is ϕ *-recurrent* if there exists a 1-form ω on M^{2n-1} such that

$$(\nabla_X h)Y = \omega(X)h\phi Y \tag{3}$$

for all vectors X orthogonal to the characteristic vector field ξ and $Y \in TM$. This condition gives a new classification of real hypersurfaces in $\widetilde{M}_n(c)$ which includes a special class of ruled real hypersurfaces in $\mathbb{C}H^2(c)$. In this paper, we shall prove the following:

Theorem 2. *Let M^{2n-1} be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then we have the following two statements* (1) *and* (2) :

(1) *M^{2n-1} satisfies the condition (2) if and only if M^{2n-1} is locally congruent to a real hypersurface of type* (A) *in $\widetilde{M}_n(c)$.*
(2) *M^{2n-1} satisfies the condition (3) if and only if M^{2n-1} is locally congruent to one of the following*:
   (i) *A real hypersurface of type* (A) *in $\widetilde{M}_n(c)$;*
   (ii) *A 3-dimensional ruled real hypersurface in $\mathbb{C}H^2(c)$ satisfying the condition $\beta = \sqrt{|c|}/2$, where the functions $\beta = \|A\xi - \alpha\xi\|, \alpha = g(A\xi, \xi)$ and A is the shape operator of M^{2n-1}.*

The case (2) of the above main theorem yields a certain answer to the above question. A special class of ruled real hypersurfaces which appears in the case (ii) of the above main theorem is extremely interesting because this class includes *a homogeneous minimal ruled real hypersurface in* $\mathbb{C}H^2(c)$. In addition, by the construction of [10], this class is characterized a class of ruled real hypersurfaces having constant scalar curvature in $\mathbb{C}H^2(c)$. We here emphasize that the statement (2) in the above main theorem also tells us the following two differences:

(a) The difference between the case of $n = 2$ and the case of $n \geqq 3$;
(b) The difference between the tensor h and *the structure Jacobi operator* $\ell(= R(\cdot, \xi)\xi)$, where R is the curvature tensor of M^{2n-1}.

In particular, the difference (b) is interesting, because there exist relationships between the tensor h and the structure Jacobi operator ℓ on contact Riemannian manifolds (see [1,22]). Moreover, many geometers have investigated the behavior of the structure Jacobi operator ℓ on real hypersurfaces in $\widetilde{M}_n(c)$ (see [4]). On the other hand, for real hypersurfaces in $\widetilde{M}_n(c)$, the point of view from the tensor h has hardly been investigated. So, it is natural to investigate real hypersurfaces in $\widetilde{M}_n(c)$ from the viewpoint of the tensor h. We shall prove that ruled real hypersurfaces do not have the analogue of the condition (3) which correspond to the structure Jacobi operator ℓ (see Proposition 1).

2. Real hypersurfaces in a nonflat complex space form

Let M^{2n-1} be a real hypersurface with a unit normal local vector field $\mathcal{N}$ of a complex n-dimensional non-flat complex space form $\widetilde{M}_n(c)$ of constant holomorphic sectional curvature c. *The Levi-Civita connections* $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$

and ∇ of M^{2n-1} are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX,\, Y)\mathcal{N}, \tag{4}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{5}$$

for vector fields X and Y tangent to M^{2n-1}, where g denotes the induced metric from the standard Riemannian metric of $\widetilde{M}_n(c)$ and A is the shape operator of M^{2n-1} in $\widetilde{M}_n(c)$. (4) is called *Gauss's formula*, and (5) is called *Weingarten's formula*. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal vectors* of M^{2n-1} in $\widetilde{M}_n(c)$, respectively.

It is known that M^{2n-1} has *the almost contact metric structure* (ϕ, ξ, η, g) induced from the Kähler structure J of $\widetilde{M}_n(c)$. The structure tensor ϕ, the characteristic vector field ξ and the contact form η of M^{2n-1} are defined by $\phi X = JX - g(JX, \mathcal{N})\mathcal{N}, \xi = -J\mathcal{N}$ and $\eta(X) = g(X, \xi)$, respectively. Furthermore this structure satisfies

$$\begin{aligned}&\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(\phi X) = 0,\\ &g(\phi X, Y) = -g(X, \phi Y) \quad \text{and} \quad g(\phi X,\, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\end{aligned} \tag{6}$$

where I denotes the identity map of the tangent bundle TM of M^{2n-1}.

Next we compute the tensor h and the covariant derivative of the tensor h on M^{2n-1}. It is well-known that the covariant derivative of the structure tensor ϕ of M^{2n-1} and that of the characteristic vector field ξ of M^{2n-1} are given by:

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{7}$$

and

$$\nabla_X \xi = \phi AX \tag{8}$$

for any X and Y tangent to M^{2n-1}. By using (6), (7) and (8), the tensor h of M^{2n-1} is given by:

$$hX = (1/2)(\mathcal{L}_\xi \phi)X = (1/2)(\eta(X)A\xi - \phi A\phi X - AX), \tag{9}$$

where, $\mathcal{L}$ is the Lie derivative. By (6), (7), (8) and (9), we have

$$\begin{aligned}(\nabla_X h)Y =&(1/2)(g(\phi AX, Y)A\xi + \eta(Y)(\nabla_X A)\xi + \eta(Y)A\phi AX\\ &- \eta(A\phi Y)AX + g(AX, A\phi Y)\xi - \phi(\nabla_X A)\phi Y\\ &- \eta(Y)\phi A^2X + g(AX, Y)\phi A\xi - (\nabla_X A)Y)\end{aligned} \tag{10}$$

for all vectors X and Y tangent to M^{2n-1}.

3. Hopf Hypersurfaces in a nonflat complex space form

In this section, we shall give some results with respect to Hopf hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$. A real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ is said to be a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M^{2n-1}. It is known that every tube of sufficiently small constant radius around each Kähler submanifold of $\widetilde{M}_n(c)$ is a Hopf hypersurface (See [3,15]). The following lemma gives a useful properties of Hopf hypersurfaces in $\widetilde{M}_n(c)$:

Lemma 1. ([7,14]) *Let M^{2n-1} be a Hopf hypersurface with the principal curvature α corresponding to the characteristic vector field ξ in $\widetilde{M}_n(c)$. Then we have the following*:

(1) *α is locally constant on M^{2n-1}*;
(2) *If X is a tangent vector of M^{2n-1} perpendicular to ξ with $AX = \lambda X$, then $(2\lambda - \alpha)A\phi X = (\alpha\lambda + (c/2))\phi X$.*

In the theory of real hypersurfaces in a nonflat complex space form, the classes of Hopf hypersurfaces with constant principal curvatures play an important role. Indeed, these classes appear in many classifications of real hypersurfaces in $\widetilde{M}_n(c)$. Among them, the class of real hypersurfaces of type (A) is significant. We collectively refer to the following real hypersurfaces as type (A) (cf.[4,17]):

- A geodesic sphere $G(r)$ of radius r in $\mathbb{C}P^n(c)$, where $0 < r < \pi/\sqrt{c}$;
- A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/\sqrt{c}$;
- A horosphere in $\mathbb{C}H^n(c)$;
- A geodesic sphere $G(r)$ of radius r in $\mathbb{C}H^n(c)$, where $0 < r < \infty$;
- A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$ in $\mathbb{C}H^n(c)$, where $0 < r < \infty$;
- A tube of radius r around a totally geodesic $\mathbb{C}H^\ell(c)$ $(1 \leqq \ell \leqq n-2)$ in $\mathbb{C}H^n(c)$, where $0 < r < \infty$.

The following lemma gives the characterization of real hypersurfaces of type (A) in $\widetilde{M}_n(c)$:

Lemma 2. ([5,16,18]) *Let M^{2n-1} be a real hypersurface in $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then the following three conditions are equivalent*:

(1) *M^{2n-1} is locally congruent to a real hypersurface of type* (A);
(2) *$\phi A = A\phi$ on M^{2n-1}*;
(3) *$h = 0$ on M^{2n-1}.*

Remark 1. Needless to say, this lemma implies that real hypersurfaces of type (A) in $\widetilde{M}_n(c)$ satisfy recurrence conditions (2) and (3).

4. Ruled real hypersurfaces in a nonflat complex space form

Next, we define ruled real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$. It is known that ruled real hypersurfaces are examples of non-Hopf hypersurfaces in $\widetilde{M}_n(c)$. A real hypersurface M^{2n-1} is called a *ruled real hypersurface* of a non-flat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$ if the holomorphic distribution $\mathbb{D}$ defined by $\mathbb{D} = \{X \in TM \mid \eta(X) = 0\}$ is integrable and each of its maximal integral manifolds is a totally geodesic complex hypersurface $\widetilde{M}_{n-1}(c)$ of $\widetilde{M}_n(c)$. A ruled real hypersurface is constructed in the following way: Given an arbitrary regular real smooth curve γ in $\widetilde{M}_n(c)$ which is defined on an interval I we have at each point $\gamma(t)$ $(t \in I)$ a totally geodesic complex hypersurface $\widetilde{M}^{(t)}_{n-1}(c)$ that is orthogonal to the plane spanned by $\{\dot{\gamma}(t), J\dot{\gamma}(t)\}$. Then we have a ruled real hypersurface $M^{2n-1} = \bigcup_{t \in I} \widetilde{M}^{(t)}_{n-1}(c)$ in $\widetilde{M}_n(c)$. The following lemma is a well-known characterization of ruled real hypersurfaces from the viewpoint of the shape operator A.

Lemma 3. ([8,17]) *Let M^{2n-1} be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then the following three conditions are mutually equivalent*:

1. *M^{2n-1} is a ruled real hypersurface*;
2. *The shape operator A of M^{2n-1} satisfies the following equalities on the open dense subset $M_1 = \{x \in M^{2n-1} | \beta(x) \neq 0\}$ with a unit vector field U orthogonal to ξ :*

$$A\xi = \alpha\xi + \beta U, \;\; AU = \beta\xi, \;\; AX = 0 \tag{11}$$

   for an arbitrary tangent vector X orthogonal to ξ and U, where α, β are differentiable functions on M_1 by $\alpha = g(A\xi, \xi)$ and $\beta = \|A\xi - \alpha\xi\|$;
3. *The shape operator A of M^{2n-1} satisfies $g(AX, Y) = 0$ for arbitrary tangent vectors $X, Y \in \mathbb{D}$.*

We treat a ruled real hypersurface *locally*, because generally this hypersurface has singularities. When we investigate ruled real hypersurfaces, we usually omit points where ξ is principal and suppose that β does not vanish everywhere, namely a ruled real hypersurface M^{2n-1} is usually supposed $M_1 = M$.

The following lemma is given us a useful tool:

Lemma 4. ([6]) *Every ruled real hypersurface in $\widetilde{M}_n(c)$ $(n \geqq 2)$ satisfies the following properties*:

$$\beta \nabla_X U = \begin{cases} (\beta^2 - (c/4))\phi X & (X = U), \\ 0 & (X = \phi U), \\ -(c/4)\phi X & (X \in \mathbb{D}_U = \mathbb{D} \cap \operatorname{span}\{U, \phi U\}^{\perp}). \end{cases}$$

$$X\beta = \begin{cases} 0 & (X = U), \\ \beta^2 + (c/4) & (X = \phi U), \\ 0 & (X \in \mathbb{D}_U = \mathbb{D} \cap \operatorname{span}\{U, \phi U\}^{\perp}). \end{cases}$$

It is well known that there do not exist real hypersurfaces $\widetilde{M}_n(c)$ with parallel shape operator. However, ruled real hypersurfaces have the following property:

Lemma 5. ([9]) *Every ruled real hypersurface M^{2n-1} in $\widetilde{M}_n(c)\,(n \geqq 2)$ admits the η-parallelism with respect to the shape operator A. Namely, M^{2n-1} satisfies the following condition*:

$$g((\nabla_X A)Y, Z) = 0$$

for all vector fields $X, Y, Z \in \mathbb{D}$.

Remark 2. In general, a tensor field T of type $(1, 1)$ is η-parallel is equivalent to $(\nabla_X T)Y \in \mathrm{span}\,\{\xi\}$ for all vector fields X and Y in $\mathbb{D}$.

By virtue of Lemma 3, Lemma 4 and Lemma 5, we obtain the following three lemmas:

Lemma 6. *Every ruled real hyperusrface M^{2n-1} in $\widetilde{M}_n(c)$ $(n \geqq 2)$ does not satisfy the condition (2).*

Proof. We suppose that there exists a ruled real hypersurface M^{2n-1} in $\widetilde{M}_n$ (c) $(n \geqq 2)$ satisfying the condition (2). We put $X = U$ and $Y = \phi U$ in (2). By using (9), (10) and Lemma 4, then we have

$$\phi(\nabla_U A)U - (\nabla_U A)\phi U = 0.$$

From Lemma 5, we have $\phi(\nabla_U A)U = 0$. Hence we can see that

$$(\nabla_U A)\phi U = 0.$$

This equation implies that

$$\nabla_U(A\phi U) - A(\nabla_U \phi)U - A\phi\nabla_U U = 0.$$

Again, by using Lemma 4, we obtain $(\beta^2 - (c/4))\xi = 0$, namely,

$$\beta^2 = c/4. \tag{12}$$

Differentiating this equation with respect to ϕU, we can see that

$$2\beta(\phi U\beta) = 0.$$

Again, by using Lemma 4, we obtain

$$\beta(\beta^2 + (c/4)) = 0.$$

Since $\beta \neq 0$, we have $\beta^2 = -(c/4)$. This, combine with (12), yields $c = 0$, which is a contradiction. □

Lemma 7. *Every ruled real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ $(n \geqq 3)$ does not satisfy the condition (3).*

Proof. We suppose that there exists a ruled real hypersurface M^{2n-1} in $\widetilde{M}_n$ (c) $(n \geqq 3)$ satisfying the condition (3). We put $Y = V \in \mathbb{D}_U = \mathbb{D} \cap \mathrm{span}\{U, \phi U\}^{\perp}$ $(\|V\| = 1)$ in (3). Then we have

$$-\phi(\nabla_X A)\phi V - (\nabla_X A)V = 0$$

for any $X \in \mathbb{D}$. By using Lemma 5, we have $\phi(\nabla_X A)\phi V = 0$ for any $X \in \mathbb{D}$. Hence we have

$$(\nabla_X A)V = 0 \tag{13}$$

for any $X \in \mathbb{D}$. We set $X = \phi V$ in (13) and take the inner product with ξ.

$$\begin{aligned} 0 &= g((\nabla_{\phi V} A)V, \xi) \\ &= g(\nabla_{\phi V}(AV) - A\nabla_{\phi V} V, \xi) \\ &= -\alpha g(\nabla_{\phi V} V, \xi) - \beta g(\nabla_{\phi V} V, U) \\ &= \alpha g(V, \phi A \phi V) + \beta g(V, \nabla_{\phi V} U) \quad \text{(from (8))} \\ &= g(V, (-c/4)\phi^2 V) \quad \text{(from Lemma 4)} \\ &= (c/4) \neq 0. \end{aligned}$$

This is a contradiction. □

However, a certain class of 3-dimensional ruled real hypersurfaces in $\mathbb{C}H^2(c)$ satisfies the condition (3). The following is a key lemma of our statements:

Lemma 8. *Let M^3 be a 3-dimensional ruled real hypersurface in $\widetilde{M_2}(c)$. Then M^3 satisfies the condition (3) if and only if M^3 is a ruled real hypersurface in $\mathbb{C}H^2(c)$ satisfying the condition $\beta = \sqrt{|c|}/2$.*

Proof. We suppose that M^3 satisfies the condition (3). Substituting $X = \phi U$ and $Y = U$ into (3). Then we have

$$-\phi(\nabla_{\phi U} A)\phi U - (\nabla_{\phi U} A)U = 0.$$

By using Lemma 5, this equation implies that

$$(\phi U \beta)\xi + \beta \phi A \phi U - A\nabla_{\phi U} U = 0.$$

By using Lemma 4, we can see that

$$\beta^2 + (c/4) = 0.$$

Hence, when $c < 0$, we have $\beta^2 = -(c/4)$. Since $\beta = \|A\xi - \alpha\xi\| > 0$, we obtain $\beta = \sqrt{|c|}/2$.

Next we shall check that the converse holds. We suppose that M^3 is a ruled real hypersurface in $\mathbb{C}H^2(c)$ which satisfies the condition $\beta = \sqrt{|c|}/2$. Now we define the 1-form ω as follows:

$$\omega(X) = \begin{cases} c/(2\beta) & (X = U), \\ 0 & (X = \phi U). \end{cases}$$

We put $Y = \xi$ in the left side of (3). Then we have

$$(\nabla_X h)\xi = (1/2)(A\phi A X - \phi A^2 X + \beta g(AX, \xi)\phi U)$$

for any $X \in \mathbb{D}$. Since $\omega(X) h \phi \xi = 0$, we shall check that

$$A\phi A X - \phi A^2 X + \beta g(AX, \xi)\phi U = 0 \tag{14}$$

for any $X \in \mathbb{D}$. We set $X = U$ in the left side of (14). Then we have

$$A\phi A\phi U - \phi A^2 U + \beta^2 \phi U = -\beta\phi A\xi + \beta^2\phi U = 0.$$

Similarly, when $X = \phi U$, the equation (14) holds trivially.

Next, we put $Y = U$ in the left side of (3). Then we obtain

$$(\nabla_X h)U = (1/2)(\phi A\phi\nabla_X U - (X\beta)\xi + A\nabla_X U)$$

for any $X \in \mathbb{D}$. Since $\omega(X)h\phi U = 0$, we shall show that

$$\phi A\phi\nabla_X U - (X\beta)\xi + A\nabla_X U = 0 \tag{15}$$

for any $X \in \mathbb{D}$. When $X = U$, Equation (15) holds obviously. When $X = \phi U$, we have

$$\phi A\phi\nabla_{\phi U} U - (\phi U\beta)\xi + A\nabla_{\phi U} U = -(\beta^2 + (c/4))\xi = (-(c/4) + (c/4))\xi = 0.$$

Finally, we put $Y = \phi U$ in the left side of (3). Then we get

$$(\nabla_X h)\phi U = (1/2)(\beta AX - \beta^2 g(U, X)\xi + \phi(\nabla_X A)U - (\nabla_X A)\phi U)$$

for any $X \in \mathbb{D}$. On the other hand, we have $\omega(X)h\phi^2 U = (1/2)\beta\omega(X)\xi$. Hence from Lemma 5, we shall prove that

$$\beta AX - \beta^2 g(U, X)\xi - (\nabla_X A)\phi U = \beta\omega(X)\xi$$

for any $X \in \mathbb{D}$. When $X = U$, we have

$$\begin{aligned} -(\nabla_U A)\phi U &= -(\nabla_U(A\phi U) - A(\nabla_U\phi)U - A\phi\nabla_U U) \\ &= A\phi\nabla_U U = -(\beta^2 - (c/4))\xi = (c/2)\xi. \end{aligned}$$

On the other hand, we can see that

$$\beta\omega(U)\xi = \beta(c/(2\beta))\xi = (c/2)\xi.$$

When $X = \phi U$, we have

$$-(\nabla_{\phi U} A)\phi U = -(\nabla_{\phi U}(A\phi U) - A(\nabla_{\phi U}\phi)U - A\phi\nabla_{\phi U} U) = A\phi\nabla_{\phi U} U = 0.$$

On the other hand, we obtain $\beta\omega(\phi U)\xi = 0$.

Hence M^3 satisfies the condition (3). □

5. Proof of Theorem 2

(1) First we suppose that there exists a non-Hopf hypersurface M^{2n-1} satisfying Condition (3). Since M^{2n-1} is a non-Hopf hypersurface, the shape operator A fulfills $A\xi = \alpha\xi + \beta U$, where the function β fulfills $\beta \neq 0$ and a unit vector field U orthogonal to the characteristic vector field ξ.

Putting $Y = U$ in (2). By using equations (9) and (10), then we get

$$\begin{aligned} g(\phi AX, U)A\xi + &g(AX, A\phi U)\xi - \phi(\nabla_X A)\phi U \\ &+ \beta g(AX, U)\phi U - (\nabla_X A)U = \omega(X)(-\phi A\phi U - AU) \end{aligned} \tag{16}$$

for any $X \in \mathbb{D}$. Taking the inner product of this equation with U and ϕU, respectively. Then we have

$$\begin{aligned} \beta g(\phi AX, U) - g(\phi(\nabla_X A)\phi U, U) - g((\nabla_X A)U, U) \\ = \omega(X)(g(A\phi U, \phi U) - g(AU, U)), \end{aligned} \tag{17}$$

$$\beta g(AX, U) - 2g((\nabla_X A)\phi U, U) = -2\omega(X)g(A\phi U, U) \tag{18}$$

for any $X \in \mathbb{D}$. Similarly, we put $Y = \phi U$ in (2). Then we obtain

$$\begin{aligned} g(AX, U)A\xi + \beta AX - g(AX, AU)\xi + \phi(\nabla_X A)U \\ + \beta g(AX, \phi U)\phi U - (\nabla_X A)\phi U = \omega(X)(\phi AU - A\phi U) \end{aligned} \tag{19}$$

for any vector $X \in \mathbb{D}$. We take the inner product of this equation with U and ϕU, respectively. Then we can see that

$$2\beta g(AX, U) - 2g((\nabla_X A)U, \phi U) = -2\omega(X)g(A\phi U, U), \tag{20}$$

$$\begin{aligned} 2\beta g(AX, \phi U) + g((\nabla_X A)U, U) - g((\nabla_X A)\phi U, \phi U) \\ = \omega(X)(g(AU, U) - g(A\phi U, \phi U)) \end{aligned} \tag{21}$$

for any $X \in \mathbb{D}$. By using (17) and (21), we get $g(A\phi U, X) = 0$ for any vector $X \in \mathbb{D}$. Since $\eta(A\phi U) = 0$, we have

$$A\phi U = 0. \tag{22}$$

From (18) and (20), we obtain $g(AU, X) = 0$ for any $X \in \mathbb{D}$. This implies that

$$AU = \beta\xi. \tag{23}$$

Next we put $Y = \xi$ in (2). Again, by using (9) and (10), we can see that

$$A\phi AX - \phi A^2 X + \beta^2 g(X, U)\phi U = 0 \tag{24}$$

for any tangent vector field $X \in \mathbb{D}$. Now we take a unit vector field $Z \in \mathbb{D}_U = \mathbb{D} \cap \operatorname{span}\{U, \phi U\}^\perp$ such that $AZ = \gamma Z$. We set $X = Z$ in (24). Then we obtain

$$\gamma(A\phi Z - \gamma\phi Z) = 0.$$

By this equation, our discussion divide into two cases.

Case (1):$\gamma = 0$.

This case means that $AX = 0$ for an arbitrary vector field $X \in \mathbb{D}_U$. This, combined with (22), yields

$$AX = 0 \tag{25}$$

for any tangent vector field X orthogonal to both ξ and U. This, together with relations (23) and $A\xi = \alpha\xi + \beta U$, implies that M^{2n-1} is locally congruent to a ruled real hypersurface in $\widetilde{M}_n(c)$ (Lemma 3). However, by Lemma 6, ruled real hypersurfaces do not fulfill the condition (2).

Case (2):$A\phi Z = \gamma\phi Z$ ($\gamma \neq 0$).

By the same discussion in the proof of Theorem 4.1 in [19], Case (2) does not occur.

Finally, we consider the case of Hopf hypersurfaces in $\widetilde{M}_n(c)$. We suppose that M^{2n-1} is a Hopf hypersurface (with $A\xi = \alpha\xi$) in $\widetilde{M}_n(c)$. We put $Y = \xi$ in (2). Then we get

$$A\phi AX - \phi A^2 X = 0$$

for any tangent vector field $X \in \mathbb{D}$. We take a vector $V \in \mathbb{D}$ with $AV = \lambda V$. By Lemma 1 and the above equation, we can see that

$$\lambda(2\lambda^2 - 2\alpha\lambda - (c/2)) = 0.$$

This equation implies that the function λ is locally constant, and $\lambda = 0$ or $2\lambda^2 - 2\alpha\lambda - (c/2) = 0$. The former does not occur. Indeed, there exist no Hopf hypersurfaces with constant principal curvatures in $\widetilde{M}_n(c)$ which satisfy $\lambda = 0$ (see [17]). The latter gives that M^{2n-1} is locally congruent to a real hypersurface of type (A) in $\widetilde{M}_n(c)$ (see the proof of Theorem 4.1 in [19]).

(2) We suppose that M^{2n-1} is a non-Hopf hypersurface in $\widetilde{M}_n(c)$ satisfying the condition (3). By the same discussion as in the proof of the statement (1), we obtain the following equations instead of Equations (17), (18), (20) and (21), respectively.

$$\beta g(\phi AX, U) - g(\phi(\nabla_X A)\phi U, U) - g((\nabla_X A)U, U) = -2\omega(X)g(A\phi U, U), \tag{26}$$

$$\beta g(AX, U) - 2g((\nabla_X A)\phi U, U) = \omega(X)(g(AU, U) - g(A\phi U, \phi U)), \tag{27}$$

$$2\beta g(AX, U) - 2g((\nabla_X A)U, \phi U) = \omega(X)(g(AU, U) - g(A\phi U, \phi U)), \tag{28}$$

$$2\beta g(AX, \phi U) + g((\nabla_X A)U, U) - g((\nabla_X A)\phi U, \phi U) = 2\omega(X)g(A\phi U, U) \tag{29}$$

for any $X \in \mathbb{D}$. Equations (26) and (29) imply the relation $A\phi U = 0$, and Equations (27) and (28) imply the relation $AU = \beta\xi$.

Putting $Y = \xi$ in (3). Then we have

$$A\phi AX - \phi A^2 X + \beta^2 g(X, U)\phi U = 0$$

for any $X \in \mathbb{D}$. We here take a unit vector field $Z \in \mathbb{D}_U = \mathbb{D} \cap \mathrm{span}\{U, \phi U\}^\perp$ such that $AZ = \gamma Z$, and set $X = Z$ in this equation. Then we obtain

$$\gamma(A\phi Z - \gamma\phi Z) = 0.$$

If $\gamma = 0$, then M^{2n-1} is locally congruent to a ruled real hypersurface in $\widetilde{M}_n(c)$. By virtue of Lemma 7 and Lemma 8, we have the case (ii) of our theorem. By the same discussion as in the proof of (1), the case of $A\phi Z = \gamma\phi Z\,(\gamma \neq 0)$ does not hold. In addition, the same is true for the case of Hopf hypersurfaces.

Therefore we obtain the desired conclusion.

6. Concluding remarks

6.1

The special class of ruled real hypersurfaces which appears in case (2) of our theorem includes a homogeneous minimal ruled real hypersurface in $\mathbb{C}H^2(c)$. Indeed, the following lemma tells us this fact:

Lemma 9. ([2,12,13]) *Let M^{2n-1} be a ruled real hypersurface in $\mathbb{C}H^n(c)$ $(n \geqq 2)$. Then M^{2n-1} is the homogeneous minimal real hypersurface in $\mathbb{C}H^n(c)$ if and only if M^{2n-1} fulfills the following condition*:

$$\begin{cases} A\xi &= (\sqrt{|c|}/2)U, \\ AU &= (\sqrt{|c|}/2)\xi, \\ AX &= 0 \quad \text{for any tangent vector field } X \perp \xi, U. \end{cases}$$

6.2

Recently, M. Kimura, S. Maeda and H. Tanabe found a new construction of ruled real hypersurfaces in $\mathbb{C}H^n(c)$ (see [10]).

Theorem 3. ([10]) *Ruled real hypersurfaces in complex hyperbolic space $\mathbb{C}H^n$ (-4) of constant holomorphic sectional curvature -4 are in one-to-one correspondence with real 1-dimensional curves in indefinite complex projective space $\mathbb{C}P^n_1(4)$ of constant holomorphic sectional curvature 4.*

By using this theorem, they also gave the following:

Theorem 4. *([10]) Let M^{2n-1} be a ruled real hypersurface in $\mathbb{C}H^n(-4)$. Then M^{2n-1} has constant scalar curvature if and only if the corresponding curve δ in $\mathbb{C}P^n_1(4)$ is lightlike.*

For ruled real hypersurfaces M^{2n-1} in $\widetilde{M}_n(c)$, we note that the scalar curvature of M^{2n-1} is constant if and only if the function β is constant. Hence the existence of the special class of ruled real hypersurfaces which appears in case (2) of our theorem is guaranteed by Theorem 3 and 4. Moreover this class is characterized a class of ruled real hypersurfaces having constant scalar curvature.

6.3

There exists a nice relationship between the tensor h and *the structure Jacobi operator* $\ell(= R(\cdot,\xi)\xi)$ in contact Riemannian geometry (see [1,22]). In addition, many geometers have investigated the behavior of the structure Jacobi operator ℓ on real hypersurfaces in $\widetilde{M}_n(c)$ (see [4]). It is well-known the following result which correspond to (1) of our theorem.

Theorem 5. ([11,21,23]) *There exist no real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ $(n \geqq 2)$ satisfying the following condition*:

$$(\nabla_X \ell)Y = \omega(X)\ell Y$$

for any $X \in \mathbb{D}$, $Y \in TM$ and ω is a 1-form on M^{2n-1}.

Hence it is natural to study the recurrent structure Jacobi operator correspond to the condition (3).

For the structure Jacobi operator ℓ, we consider the analogue of the condition (3), namely,

$$(\nabla_X \ell)Y = \omega(X)\ell\phi Y \tag{30}$$

for any $X \in \mathbb{D}$, $Y \in TM$ and ω is a 1-form on M^{2n-1}. Then we obtain the following proposition:

Proposition 1. *Every ruled real hypersurface M^{2n-1} in $\widetilde{M}_n(c)\,(n \geqq 2)$ does not satisfy the condition (30).*

Proof. First we prepare the fundamental relation of the structure Jacobi operator ℓ. Let R be the curvature tensor of M^{2n-1} in $\widetilde{M}_n(c)$. *The Gauss equation* is given by:

$$\begin{aligned} R(X,Y)Z = (c/4)\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X \\ &- g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} \\ &+ g(AY,Z)AX - g(AX,Z)AY \end{aligned} \tag{31}$$

for all vectors X, Y and Z tangent to M^{2n-1}. By using (31), we have

$$\ell X = (c/4)(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi \tag{32}$$

for any vector field $X \in TM$, where $\alpha = g(A\xi, \xi)$. The covariant derivative of the tensor ℓ is given by:

$$\begin{aligned} (\nabla_X \ell)Y = &-(c/4)g(\phi AX,Y)\xi - (c/4)\eta(Y)\phi AX + (X\alpha)AY \\ &+ \alpha(\nabla_X A)Y - g((\nabla_X A)Y,\xi)A\xi - g(A\phi AX,Y)A\xi \\ &- \eta(AY)(\nabla_X A)\xi - \eta(AY)A\phi AX \end{aligned} \tag{33}$$

for any $X, Y \in TM$.

We suppose that there exists a ruled real hypersurface in $\widetilde{M}_n(c)$ satisfying the condition (30). Putting $Y = U$ in (30). By using (32), (33) and Lemma 4, we have

$$-2\beta(X\beta)U - \beta^2\nabla_X U = (c/4)\omega(X)\phi U \tag{34}$$

for any $X \in \mathbb{D}$. Setting $X = \phi U$ in (34), from Lemma 4, we obtain

$$-2\beta(\phi U\beta)U = (c/4)\omega(\phi U)\phi U.$$

Taking the inner product of this equation with U, form Lemma 4, we get

$$\beta^2 = -(c/4). \tag{35}$$

Next we put $X = U$ in (34). Then we have

$$-\beta(\beta^2 - (c/4)) = (c/4)\omega(U).$$

This, together with (35), yields

$$\omega(U) = 2\beta. \tag{36}$$

Setting $X = U$ and $Y = \phi U$ in (30). Then we have

$$\beta(\beta^2 - (c/4)) = \omega(U)((c/4) - \beta^2).$$

This, combined with (35) and (36), gives $\beta c = 0$. This is a contradiction. □

At the end of this paper, we pose the following problem:

Problem 1. *Does there exist a real hypersurface in $\widetilde{M}_n(c)$ satisfying the condition (30)?*

Acknowledgements

The author would like to thank Professor Yasuhiko Furihata for his valuable comments.

Data Availability Statement This manuscript has no associate data.

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Received: April 27, 2022.
Accepted: August 18, 2022.