A family of flat Minkowski planes over convex functions

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Abstract. Using suitable convex functions, we construct a new family of flat Minkowski planes whose automorphism groups are at least 3-dimensional. These planes admit groups of automorphisms isomorphic to the direct product of $\mathbb{R}$ and the connected component of the affine group on $\mathbb{R}$. We also determine isomorphism classes, automorphisms and possible Klein–Kroll types for our examples.

Keywords. Circle plane, Minkowski plane, automorphism group, convex function.

1. Introduction

Flat Minkowski planes are generalization of the geometry of nontrivial plane sections of the standard nondegenerate ruled quadric in the real 3-dimensional projective space $\mathbb{P}_3(\mathbb{R})$. A flat Minkowski plane $\mathcal{M}$ can be classified based on the dimension n of its automorphism group. It is known that n is at most 6, and the plane $\mathcal{M}$ is determined when $n \geq 4$, cp. [11]. The current open case of interest is when $n = 3$, in which a list of possible connected groups of automorphisms of $\mathcal{M}$ was presented in [2].

The purpose of this paper is to describe a new family of flat Minkowski planes that admit the 3-dimensional connected group

$$\Phi_\infty = \{(x, y) \mapsto (x + b, ay + c) \mid a, b, c \in \mathbb{R}, a > 0\}$$

as their group of automorphisms. The construction presented here is based on Chapter 6 of the author's doctoral dissertation [4], which was motivated from a family of flat Laguerre planes of translation type described by Löwen and Pfüller [7,8]. These Laguerre planes were constructed as follows: first a suitable class of functions, called 'strongly parabolic functions' is defined. Then circles are defined as the images of the graph of one such function under the group

Φ_∞. In our construction, we introduce the notion of a "strongly hyperbolic function", which is adapted from that of strongly parabolic functions to satisfy the incidence axioms of flat Minkowski planes.

The paper is organized as follows. Section 2 contains the preliminary results and examples. In Sect. 3, we define strongly hyperbolic functions and derive useful properties. In Sect. 4, we describe a family of flat Minkowski planes using strongly hyperbolic functions. The paper ends with a discussion on isomorphism classes, automorphisms and Klein–Kroll types of these planes in Sect. 5.

2. Preliminaries

2.1. Flat Minkowski planes and two halves of the circle set

A *flat Minkowski plane* is a geometry $\mathcal{M} = (\mathcal{P}, \mathcal{C}, \mathcal{G}^+, \mathcal{G}^-)$, whose

> point set $\mathcal{P}$ is the torus $\mathbb{S}^1 \times \mathbb{S}^1$,
> circles (elements of $\mathcal{C}$) are graphs of homeomorphisms of $\mathbb{S}^1$,
> $(+)$-parallel classes (elements of $\mathcal{G}^+$) are the verticals $\{x_0\} \times \mathbb{S}^1$,
> $(-)$-parallel classes (elements of $\mathcal{G}^-$) are the horizontals $\mathbb{S}^1 \times \{y_0\}$,

where $x_0, y_0 \in \mathbb{S}^1$. We denote the $(\pm)$-parallel class containing a point p by $[p]_\pm$. When two points p and q are on the same $(\pm)$-parallel class, we say they are $(\pm)$-*parallel* and denote this by $p \parallel_\pm q$. Two points p, q are *parallel* if they are $(+)$-parallel or $(-)$-parallel, and we denote this by $p \parallel q$. Furthermore, a flat Minkowski plane satisfies the following two axioms:

> *Axiom of Joining*: three pairwise nonparallel points p, q, r can be joined by a unique circle.
> *Axiom of Touching*: for each circle C and any two nonparallel points p, q with $p \in C$ and $q \notin C$, there is exactly one circle D that contains both points p, q and intersects C only at the point p.

The *derived plane* $\mathcal{M}_p$ of $\mathcal{M}$ *at the point* p is the incidence geometry whose point set is $\mathcal{P} \backslash ([p]_+ \cup [p]_-)$, whose lines are all parallel classes not going through p and all circles of $\mathcal{M}$ going through p. For every point $p \in \mathcal{P}$, the derived plane $\mathcal{M}_p$ is a flat affine plane, cp. [9, Theorem 4.2.1].

A flat Minkowski plane is in *standard representation* if the set $\{(x, x) \mid x \in \mathbb{S}^1\}$ is one of its circles (cp. [9, Subsection 4.2.3]). Up to isomorphisms, every flat Minkowski plane can be described in standard representation. In this case, we omit the two parallelisms and refer to $(\mathcal{P}, \mathcal{C})$ as a flat Minkowski plane.

Let $\mathcal{M} = (\mathcal{P}, \mathcal{C})$ be a flat Minkowski plane in standard representation. Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the sets of all circles in $\mathcal{C}$ that are graphs of orientation-preserving and orientation-reversing homeomorphisms of $\mathbb{S}_1$, respectively. Clearly, $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$. We call $\mathcal{C}^+$ and $\mathcal{C}^-$ the *positive half* and *negative half* of $\mathcal{M}$, respectively. It turns out that these two halves are completely independent of each

other, that is, we can exchange halves from different flat Minkowski planes and obtain another flat Minkowski plane, see [9, Subsection 4.3.1]:

Theorem 2.1. *For $i = 1, 2$, let $\mathcal{M}_i = (\mathcal{P}, \mathcal{C}_i)$ be two flat Minkowski planes. Then $\mathcal{M} = (\mathcal{P}, \mathcal{C}_1^+ \cup \mathcal{C}_2^-)$ is a flat Minkowski plane.*

2.2. Examples

We identify $\mathbb{S}^1$ with $\mathbb{R} \cup \{\infty\}$ in the usual way. There are various known examples of flat Minkowski planes, cp. [9, Chapter 4]. For our purposes, we recall three particular examples.

Example 2.2. The circle set of the *classical flat Minkowski plane* $\mathcal{M}_C$ consists of sets of the form $\{(x, sx + t) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\}$, where $s, t \in \mathbb{R}$, $s \neq 0$, and $\{(x, y) \in \mathbb{R}^2 \mid (x - b)(y - c) = a\} \cup \{(\infty, c), (b, \infty)\}$, where $a, b, c \in \mathbb{R}$, $a \neq 0$.

Example 2.3. For $r, s > 0$, let $f_{r,s}$ be defined by

$$f_{r,s}(x) = \begin{cases} x^r & \text{for } x \geq 0, \\ -s|x|^r & \text{for } x < 0, \\ \infty & \text{for } x = \infty. \end{cases}$$

The circle set of a *generalised Hartmann plane* $\mathcal{M}_{GH}(r_1, s_1; r_2, s_2)$ consists of sets of the form

$$\{(x, sx + t) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\},$$

where $s, t \in \mathbb{R}$, $s \neq 0$, sets of the form

$$\left\{ \left(x, \frac{a}{f_{r_1,s_1}(x - b)} + c \right) \,\middle|\, x \in \mathbb{R} \right\} \cup \{(b, \infty), (\infty, c)\},$$

where $a, b, c \in \mathbb{R}$, $a > 0$, and sets of the form

$$\left\{ \left(x, \frac{a}{f_{r_2,s_2}(x - b)} + c \right) \,\middle|\, x \in \mathbb{R} \right\} \cup \{(b, \infty), (\infty, c)\},$$

where $a, b, c \in \mathbb{R}$, $a < 0$.

Example 2.4. Let f and g be two orientation-preserving homeomorphisms of $\mathbb{S}^1$. Denote $\mathrm{PGL}(2, \mathbb{R})$ by Ξ and $\mathrm{PSL}(2, \mathbb{R})$ by Λ. The circle set $\mathcal{C}(f, g)$ of a *half-classical plane* $\mathcal{M}_{HC}(f, g)$ consists of sets of the form $\{(x, \gamma(x)) \mid x \in \mathbb{S}^1\}$, where $\gamma \in \Lambda \cup g^{-1}(\Xi \backslash \Lambda) f$.

2.3. Automorphisms and Klein–Kroll types

An *isomorphism between two flat Minkowski planes* is a bijection between the point sets that maps circles to circles, and induces a bijection between the circle sets. An *automorphism of a flat Minkowski plane* $\mathcal{M}$ is an isomorphism from $\mathcal{M}$ to itself. Every automorphism of a flat Minkowski plane is continuous and thus a homeomorphism of the torus. With respect to composition, the set of all automorphisms of a flat Minkowski plane is a group called the automorphism

group $\text{Aut}(\mathcal{M})$ of $\mathcal{M}$. The group $\text{Aut}(\mathcal{M})$ is a Lie group of dimension at most 6 with respect to the compact-open topology. We say a flat Minkowski plane has *group dimension n* if its automorphism group has dimension n. For $n \geq 4$, we have the following classification, cp. [11].

Theorem 2.5. *Let $\mathcal{M}$ be a flat Minkowski plane with group dimension n. If $n \geq 4$, then exactly one of the following occurs.*

1. *$n = 6$ and $\mathcal{M}$ is isomorphic to the classical flat Minkowski plane.*
2. *$n = 4$ and $\mathcal{M}$ is isomorphic to a proper (nonclassical) generalised Hartmann plane $\mathcal{M}_{GH}(r_1, s_1; r_2, s_2)$, $r_1, s_1, r_2, s_2 \in \mathbb{R}^+$, $(r_1, s_1, r_2, s_2) \neq (1, 1, 1, 1)$.*
3. *$n = 4$ and $\mathcal{M}$ is isomorphic to a proper half-classical plane $\mathcal{M}_{HC}(f, id)$, where f is a homeomorphism of $\mathbb{S}_1$ of the form $f_{d,s}$, $(d, s) \neq (1, 1)$.*

A *central automorphism* of a Minkowski plane is an automorphism that fixes at least one point and induces a central collineation in the derived projective plane at each fixed point. Similar to the Lenz–Barlotti classification of projective planes with respect to central collineations, Minkowski planes have been classified by Klein and Kroll with respect to groups of central automorphisms that are 'linearly transitive', cp. [5,6]. In the case of flat Minkowski planes, possible Klein–Kroll types were determined by Steinke [12] as follows.

Theorem 2.6. *A flat Minkowski plane has Klein–Kroll type*

   I. *A.1, A.2, A.3, B.1, B.10, B.11, D.1,*
  II. *A.1, A.15,*
 III. *C.1, C.18, C.19,*
 IV. *A.1, or*
VII. *F.23.*

For each of these 14 types, except type II.A.15, examples are given in [12]. In the same paper, Steinke also characterised some families of flat Minkowski planes. The following result is adapted from [12, Proposition 5.9].

Theorem 2.7. *A flat Minkowski plane of Klein–Kroll type*

VII.F.23 *is isomorphic to the classical flat Minkowski plane;*
III.C.19 *is isomorphic to a proper generalised Hartmann plane $\mathcal{M}_{GH}(r, 1; r, 1)$, $r \neq 1$;*
III.C.18 *is isomorphic to an Artzy–Groh plane (cp. [1]) with group dimension 3.*

3. Strongly hyperbolic functions

In this section, we will define strongly hyperbolic functions and derive some properties. For convenience, we will abbreviate the mean value theorem by

MVT, and the intermediate value theorem by IVT. Let $\mathbb{R}^+$ denote the set of positive real numbers.

Definition 3.1. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *strongly hyperbolic function* if it satisfies the following conditions.

1. $\lim_{x \to 0+} f(x) = +\infty$ and $\lim_{x \to +\infty} f(x) = 0$.
2. f is strictly convex.
3. For each $b \in \mathbb{R}$,

$$\lim_{x \to +\infty} \frac{f(x + b)}{f(x)} = 1.$$

4. f is differentiable.
5. $\ln |f'(x)|$ is strictly convex.

3.1. Properties strongly hyperbolic functions

Let f be a strongly hyperbolic function. It is readily checked that f is strictly decreasing and consequently an orientation-reversing homeomorphism of $\mathbb{R}^+$. Since f is strictly convex and differentiable, it is continuously differentiable. We now consider some other properties of f.

Lemma 3.2. *Let $b > 0$, $s, t \neq 0$. Then the following statements are true.*

1.

$$\lim_{x \to \infty} \frac{f'(x)}{f'(x + b)} = 1.$$

2.

$$\lim_{x \to +\infty} \frac{f'(x)}{f(x)} = 0.$$

3.

$$\lim_{x \to +\infty} \frac{f(x + s) - f(x)}{f'(x)} = s.$$

4.

$$\lim_{x \to +\infty} \frac{f(x + s) - f(x)}{f(x + t) - f(x)} = \frac{s}{t}.$$

5.

$$\liminf_{x \to 0+} \frac{f'(x)}{f(x)} = -\infty.$$

Proof.   1. Let $h(x) = \ln(|f'(x)|)$. Since f is strongly hyperbolic, h is strictly convex and strictly decreasing. We define $h_b : (0, \infty) \to \mathbb{R}$ by

$$h_b(x) = h(x) - h(x + b) = \ln \left(\left| \frac{f'(x)}{f'(x + b)} \right| \right).$$

Then h_b is strictly decreasing and is bounded below by 0. This implies that $\lim_{x\to\infty} h_b(x)$ exists and so does $\lim_{x\to\infty} \dfrac{f'(x)}{f'(x+b)}$. The claim now follows from L'Hospital's Rule.

2. Let $\delta > 0$. For $x > \delta$, by the MVT, there exists $r \in (x - \delta, x)$ such that

$$f'(r) = \frac{f(x) - f(x-\delta)}{\delta}.$$

Since f is strictly convex, f' is strictly increasing, so that $f'(r) < f'(x)$. Dividing both sides by $f(x)$, we get

$$\frac{f(x) - f(x-\delta)}{\delta f(x)} < \frac{f'(x)}{f(x)} < 0.$$

Since f is strongly hyperbolic,

$$\lim_{x\to+\infty} \frac{f(x) - f(x-\delta)}{f(x)} = 1 - \lim_{x\to+\infty} \frac{f(x-\delta)}{f(x)} = 1 - 1 = 0.$$

The claim now follows from the squeeze theorem.

3. We will assume $s > 0$, as the case $s < 0$ is similar. For $x > 0$, by the MVT, there exists $r \in (x, x+s)$ such that

$$f'(r) = \frac{f(x+s) - f(x)}{s}.$$

Since f' is strictly increasing and negative, $f'(x) < f'(r) < f'(x+s) < 0$. Hence,

$$sf'(x) < f(x+s) - f(x) < sf'(x+s) < 0,$$

and so

$$\frac{sf'(x+s)}{f'(x)} < \frac{f(x+s) - f(x)}{f'(x)} < s.$$

The claim now follows from the squeeze theorem and part 1.

4. We rewrite

$$\frac{f(x+s) - f(x)}{f(x+t) - f(x)} = \frac{f(x+s) - f(x)}{f'(x)} \cdot \frac{f'(x)}{f(x+t) - f(x)}.$$

The claim now follows from part 3.

5. This can be derived from the fact that $\lim_{x\to 0+} \ln f(x) = +\infty$. $\qquad\square$

We now study the roots of the function $\check{f} : (\max\{-b, 0\}, \infty) \to \mathbb{R}$ defined as

$$\check{f}(x) = af(x+b) + c - f(x),$$

where $a > 0, b, c \in \mathbb{R}$. We first consider the derivative $\check{f}'$.

Lemma 3.3. *The derivative $\check{f}'$ is continuous and has at most one root. Furthermore, if $\check{f}'$ has a root x_0, then $\check{f}'$ changes sign at x_0.*

Proof. Since $\ln|f'(x)|$ is strictly convex and strictly decreasing, the function $h : (\max\{-b, 0\}, \infty) \to \mathbb{R}$ defined by

$$h(x) = \ln|f'(x + b)| + \ln a - \ln|f'(x)|$$

is strictly monotonic. Hence, h has at most one root, and if h has a root x_0, then h changes sign at x_0. Note that $\breve{f}'$ has a root x_0 if and only if h has x_0 as a root, and $\breve{f}'(x) \lessgtr 0$ if and only if $h(x) \gtrless 0$. This completes the proof.   □

We note that if $a \neq 1$ and $b = c = 0$, then $\breve{f}$ has no roots. In the following lemma, we consider the special case when exactly one of b, c is zero. If $a = 1$, then $\breve{f}$ has no roots. We then further assume $a \neq 1$.

Lemma 3.4. *If $a \neq 1$ and either $b \neq 0, c = 0$ or $b = 0, c \neq 0$, then $\breve{f}$ has at most one root. Furthermore, exactly one of the following statements is true.*

1. *$\breve{f}$ has exactly one root x_0 at which it changes sign. The derivative $\breve{f}'$ is nonzero at x_0, and either*

$$a > 1, b > 0, c = 0 \text{ or } a > 1, b = 0, c < 0 \text{ or}$$
$$a < 1, b < 0, c = 0 \text{ or } a < 1, b = 0, c > 0.$$

2. *$\breve{f}$ has no root, and either*

$$a < 1, b > 0, c = 0 \text{ or } a < 1, b = 0, c < 0 \text{ or}$$
$$a > 1, b < 0, c = 0 \text{ or } a > 1, b = 0, c > 0.$$

Proof. There are four cases depending on the sign of b, c. We only prove the cases $b > 0, c = 0$ and $b = 0, c > 0$. The cases $b < 0, c = 0$ and $b = 0, c < 0$ are similar.

Case 1: $b > 0, c = 0$. We have $\breve{f}(x) = af(x + b) - f(x)$. If $a < 1$, then

$$\breve{f}(x) < f(x + b) - f(x) < 0,$$

and so $\breve{f}$ has no roots. We now claim that if $a > 1$, then $\breve{f}$ has exactly one root x_0 at which it changes sign. Let $g : (0, +\infty) \to \mathbb{R}$ be defined by

$$g(x) = \ln f(x + b) + \ln a - \ln f(x).$$

We note that $\breve{f}$ has a root if and only if g has a root, and the sign of $\breve{f}(x)$ is the same as the sign of $g(x)$. Also, g is continuous, $\lim_{x \to 0} g(x) = -\infty$, and $\lim_{x \to +\infty} g(x) = \ln a > 0$. It follows that g, and in particular $\breve{f}$, has at least one root.

By Lemma 3.3, $\breve{f}$ cannot have more than two roots. Suppose for a contradiction that $\breve{f}$ has exactly two roots $x_0 < x_1$. By Lemma 3.3 and Rolle's Theorem, $\breve{f}'$ is nonzero at these roots. Since $\breve{f}'$ is continuous, $\breve{f}$ is locally monotone at the roots. Hence $\breve{f}$ changes sign at the two roots. It follows that g has two roots at which it changes sign.

Since $\lim_{x \to 0} g(x) = -\infty$, $g(x) < 0$ for $x \in (0, x_0)$. Since g changes sign at x_0 and has no roots between x_0 and x_1, we have $g(x) > 0$ for $x \in (x_0, x_1)$. Since g changes sign at x_1 and has no roots larger than x_1, we have $g(x) < 0$ for $x > x_1$. This contradicts $\lim_{a \to +\infty} g(x) = \ln a > 0$.

Therefore $\check{f}$ has exactly one root x_0. Then g also has exactly one root at x_0. By the IVT, g changes sign at x_0. Then $\check{f}$ also changes sign at x_0. This proves the claim.

Case 2: $b = 0, c > 0$. Then $\check{f}(x) = (a-1)f(x) + c$. If $a > 1$, then $\check{f} > 0$ and has no roots. If $a < 1$, then from Definition 3.1, $\check{f}$ has exactly one root at which it changes sign. $\qquad\square$

We now consider the case when both $b, c \neq 0$.

Lemma 3.5. *If $b, c \neq 0$, then $\check{f}(x)$ has at most two roots. Furthermore, exactly one of the following statements is true.*

1. *$\check{f}$ has exactly two roots x_0 and x_1 at which it changes sign. The derivative $\check{f}'$ is nonzero at x_0 and x_1, and either*
$$a < 1, b < 0, c > 0 \text{ or } a > 1, b > 0, c < 0.$$

2. *$\check{f}$ has exactly one root x_0 at which it does not change sign. The derivative $\check{f}'$ also has a root at x_0, and either*
$$a < 1, b < 0, c > 0 \text{ or } a > 1, b > 0, c < 0.$$

3. *$\check{f}$ has exactly one root x_0 at which it changes sign. The derivative $\check{f}'$ is nonzero at x_0, and $bc > 0$.*
4. *$\check{f}$ has no roots, and $bc < 0$.*

Proof.   1. From Lemma 3.3 and Rolle's Theorem, $\check{f}$ cannot have more than two roots. Also, if $\check{f}$ has exactly two roots, then $\check{f}'$ is nonzero at these roots. Moreover, since $\check{f}$ is continuously differentiable, it must change sign at the two roots.

Assume $\check{f}$ has exactly one root x_0. If $\check{f}'(x_0) \neq 0$, then $\check{f}$ changes sign at x_0. In the case $\check{f}(x_0) = \check{f}'(x_0) = 0$, by Lemma 3.3, the derivative $\check{f}'$ changes sign at x_0. This implies $\check{f}$ has a local extremum at x_0 and so does not change sign.

If none of the above occurs, then it must be the case that $\check{f}$ has no roots. Therefore, excluding the conditions on the parameters a, b, c, exactly one of the statements in the lemma is true.

2. We claim that if $bc > 0$, then $\check{f}$ has exactly one root at which $\check{f}$ changes sign. We assume that $b > 0, c > 0$; the case $b < 0, c < 0$ is similar. Since $\lim_{x \to 0} \check{f}(x) = -\infty$, and $\lim_{x \to +\infty} \check{f}(x) = c$, by the IVT, $\check{f}$ has at least one root.

Suppose for a contradiction that $\check{f}$ has exactly two roots $x_0 < x_1$. Since $\lim_{x \to 0} \check{f}(x) = -\infty$, $\check{f}(x) < 0$ for $x \in (0, x_0)$. Since $\check{f}$ changes sign at x_0

and has no roots between x_0 and x_1, we have $\check{f}(x) > 0$ for $x \in (x_0, x_1)$. Since $\check{f}$ changes sign at x_1 and has no roots larger than x_1, we have $\check{f}(x) < 0$ for $x > x_1$. This contradicts $\lim_{x \to +\infty} \check{f}(x) = c > 0$. This proves the claim.

3. Assume $\check{f}$ has exactly two roots x_0 and x_1 at which it changes sign. From part 2 of the proof and Lemma 3.4, it must be the case that $bc < 0$. Assume $b > 0, c < 0$. We rewrite

$$\check{f}(x) = a\left(f(x+b) - f(x)\right) + c + (a-1)f(x).$$

Since f is strictly convex, $f(x+b) - f(x)$ is strictly increasing. If $0 < a \leq 1$, then $\check{f}$ is strictly increasing and thus has at most one root, which contradicts our assumption. Therefore $a > 1$. Similarly, when $b < 0, c > 0$, we have $a < 1$.

4. Assume $\check{f}$ has exactly one root at which it changes sign. We can say that there exists x^* such that $\check{f}(x^*) > 0$. Suppose for a contradiction that $b > 0, c < 0$. Then $\lim_{x \to 0} \check{f}(x) = -\infty$, and $\lim_{x \to +\infty} \check{f}(x) = c < 0$. By the IVT, there exist two roots $x_0 \in (0, x*)$ and $x_1 \in (x^*, +\infty)$, which contradicts our assumption. Similarly, it cannot be the case that $b < 0, c > 0$. Thus, $bc > 0$.

5. When $\check{f}$ has exactly one root x_0 at which it does not change sign, the conditions on a, b, c follow in a similar manner as in part 3. When $\check{f}$ has no roots, then by part 2 of the proof, $bc < 0$. This completes the proof. $\qquad \square$

3.2. Graphs of strongly hyperbolic functions under Φ_∞

Let $a_1, a_2 > 0$, $b_1, b_2, c_1, c_2 \in \mathbb{R}$. For two strongly hyperbolic functions f_1 and f_2, we define

$$\check{f} : (\max\{-b_1, -b_2\}, +\infty) \to \mathbb{R} : x \mapsto a_1 f_1(x + b_1) + c_1 - a_2 f_1(x + b_2) - c_2,$$
$$\hat{f} : (-\infty, \min\{-b_1, -b_2\}) \to \mathbb{R} : x \mapsto -a_1 f_2(-x - b_1) + c_1$$
$$+ a_2 f_2(-x - b_2) - c_2.$$

From the previous subsection, we obtain the following two lemmas.

Lemma 3.6. *Assume $b_1 \neq b_2, c_1 = c_2$ or $b_1 = b_2, c_1 \neq c_2$.*

1. *If $a_1 = a_2$, then both $\check{f}$ and $\hat{f}$ have no roots.*
2. *If $a_1 \neq a_2$, then either $\check{f}$ or $\hat{f}$, but not both, has exactly one root x_0 at which it changes sign. The corresponding derivative is nonzero at x_0.*

Lemma 3.7. *Assume $b_1 \neq b_2, c_1 \neq c_2$. If $\check{f}$ has at least one root, then exactly one of the following is true.*

1. *$\check{f}$ has exactly two roots x_0 and x_1 at which it changes sign, $\hat{f}$ has no roots. The derivative $\check{f}'$ is nonzero at x_0 and x_1.*

2. $\check{f}$ has exactly one root x_0 at which it does not change sign, $\widehat{f}$ has no roots. The derivative $\check{f}'$ also has a root at x_0.
3. $\check{f}$ has exactly one root x_0 at which it changes sign, $\widehat{f}$ also has exactly one root x_1 at which it changes sign. The derivatives $\widehat{f}'(x_0)$ and $\widehat{f}'(x_1)$ are nonzero.

3.3. Some examples and remarks

Example 3.8. The following functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ are strongly hyperbolic.

1. $f(x) = \dfrac{1}{x^i}$, where $i \in \mathbb{N}$.
2. $f(x) = \sum\limits_{i=1}^{n} \dfrac{1}{x^i}$, where $n \in \mathbb{N}$.
3. $f(x) = \dfrac{1}{x + \arctan(x)}$.
4. $f(x) = \ln\left(\dfrac{1}{x} + \sqrt{\dfrac{1}{x^2} + 1}\right)$.

Remark 3.9. The inverse of a strongly hyperbolic function is not necessarily a strongly hyperbolic function. For instance, the inverse of the last function f in Example 3.8 is given by

$$f^{-1}(x) = \frac{1}{\sinh(x)}.$$

As mentioned in [3], for $b \neq 0$,

$$\lim_{x \to \infty} \frac{\sinh(x)}{\sinh(x + b)} = e^{-b} \neq 1.$$

This implies that the inverse f^{-1} is not strongly hyperbolic.

Remark 3.10. The definition of strongly hyperbolic functions shares similarities with the definition of strongly parabolic functions in the construction of flat Laguerre planes of translation type by Löwen and Pfüller [7]. We note that strongly parabolic functions are assumed to be twice differentiable. As mentioned by the authors in [7, Remark 2.6], this condition was proved to be unnecessary by Schellhammer [10]. For the definition of strongly hyperbolic functions, we omit this condition.

4. Flat Minkowski planes from strongly hyperbolic functions

Let f_1, f_2 be strongly hyperbolic functions. For $a > 0, b, c \in \mathbb{R}$, let

$$f_{a,b,c} : \mathbb{R}\backslash\{-b\} \to \mathbb{R}\backslash\{c\} : x \mapsto \begin{cases} a f_1(x + b) + c & \text{for } x > -b, \\ -a f_2(-x - b) + c & \text{for } x < -b. \end{cases}$$

We also define the following sets:

$$\overline{f_{a,b,c}} := \{(x, f_{a,b,c}(x)) \mid x \in \mathbb{R}\backslash\{-b\}\} \cup \{(-b, \infty), (\infty, c)\},$$
$$F := \{\overline{f_{a,b,c}} \mid a > 0, b, c \in \mathbb{R}\},$$
$$\overline{l_{s,t}} := \{(x, sx + t) \mid x \in \mathbb{R}\} \cup \{(\infty, \infty)\},$$
$$L := \{\overline{l_{s,t}} \mid s, t \in \mathbb{R}, s < 0\}.$$

For a set $\overline{f_{a,b,c}}$, *the convex branch of* $\overline{f_{a,b,c}}$ is the subset $\{(x, f_{a,b,c}(x)) \mid x > -b\}$, and *the concave branch of* $\overline{f_{a,b,c}}$ is the subset $\{(x, f_{a,b,c}(x)) \mid x < -b\}$.

We define $\mathcal{C}^-(f_1, f_2) := F \cup L$ and $\mathcal{C}^+(f_1, f_2) := \varphi(\mathcal{C}^-(f_1, f_2))$, where φ is the homeomorphism of the torus defined by $\varphi : (x, y) \mapsto (-x, y)$. In this section, we prove the following theorem.

Theorem 4.1. *For $i = 1..4$, let f_i be a strongly hyperbolic function. Let $\mathcal{C} := \mathcal{C}^-(f_1, f_2) \cup \mathcal{C}^+(f_3, f_4)$. Then $\mathcal{M}_f = \mathcal{M}(f_1, f_2; f_3, f_4) := (\mathcal{P}, \mathcal{C})$ is a flat Minkowski plane.*

In view of Theorem 2.1, to prove Theorem 4.1 it is sufficient to prove that $\mathcal{C}^-(f_1, f_2)$ is the negative half of a flat Minkowski plane. We verify that this is the case by showing that $\mathcal{C}^-(f_1, f_2)$ satisfies Axiom of Joining and Axiom of Touching in the next four subsections.

4.1. Axiom of Joining, existence

Three points $p_1, p_2, p_3 \in \mathcal{P}$ are in *admissible position* if they can be joined by an element of the negative half of the classical flat Minkowski plane $\mathcal{M}_C$. Furthermore, we say they are of type

1 if $p_1 = (\infty, \infty), p_2, p_3 \in \mathbb{R}^2$,
2 if $p_1 = (x_1, y_1), p_2 = (\infty, y_2), p_3 = (x_3, \infty), x_i, y_i \in \mathbb{R}$,
3 if $p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (x_3, \infty), x_i, y_i \in \mathbb{R}$,
4 if $p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (\infty, y_3), x_i, y_i \in \mathbb{R}$,
5 if $p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (x_3, y_3), x_i, y_i \in \mathbb{R}$.

We note that up to permutations, if three points are in admissible position, then they are in exactly one of the five admissible position types. The main theorem of this subsection is the following.

Theorem 4.2 (Axiom of Joining, existence). *Let $p_1, p_2, p_3 \in \mathcal{P}$ be three points in admissible position. Then there is at least one element in $\mathcal{C}^-(f_1, f_2)$ that contains p_1, p_2, p_3.*

We prove Theorem 4.2 in the following five lemmas.

Lemma 4.3. *Let p_1, p_2, p_3 be three points in admissible position type 1. Then there exist $s_0 < 0, t_0 \in \mathbb{R}$ such that $\overline{l_{s_0,t_0}}$ contains p_1, p_2, p_3.*

Proof. Since $\mathcal{C}^-(f_1, f_2)$ contains lines with negative slope extended by the point (∞, ∞), the choice of $\overline{l_{s_0, t_0}}$ is the line containing p_2, p_3. □

Lemma 4.4. *Let p_1, p_2, p_3 be three points in admissible position type 2. Then there exist $a_0 > 0, b_0, c_0 \in \mathbb{R}$ such that $\overline{f_{a_0, b_0, c_0}}$ contains p_1, p_2, p_3.*

Proof. Without loss of generality, we can assume $y_2 = x_3 = 0$ so that $p_2 = (\infty, 0), p_3 = (0, \infty)$. If $\overline{f_{a_0, b_0, c_0}}$ contains $(\infty, 0), (0, \infty)$, then $b_0 = c_0 = 0$. Since the points p_1, p_2, p_3 are in admissible position, there are two cases depending on x_1, y_1.

Case 1: $x_1, y_1 > 0$. The equation $y_1 = af_1(x_1)$ has a solution $a_0 = \dfrac{y_1}{f_1(x_1)}$, and we have $\overline{f_{a_0, b_0, c_0}}$ whose convex branch contains p_1.

Case 2: $x_1, y_1 < 0$. The equation $y_1 = af_1(x_1)$ has a solution $a_0 = -\dfrac{y_1}{f_2(-x_1)}$, and we have $\overline{f_{a_0, b_0, c_0}}$ whose concave branch contains p_1. □

Lemma 4.5. *Let p_1, p_2, p_3 be three points in admissible position type 3. Then there exist $a_0 > 0, b_0, c_0 \in \mathbb{R}$ such that $\overline{f_{a_0, b_0, c_0}}$ contains p_1, p_2, p_3.*

Proof. We can assume $x_3 = 0, y_2 = 0, x_1 > x_2$. Let $b_0 = 0$. There are three cases depending on the positions of p_1 and p_2. We show that there is a choice of a_0, c_0 in each case.

Case 1: $x_1 > x_2 > 0$, and $y_1 < 0$. We consider the system

$$\begin{cases} y_1 = af_1(x_1) + c \\ 0 = af_1(x_2) + c \end{cases}$$

in variables a, c. The solution for this system is $a_0 = \dfrac{y_1}{f_1(x_1) - f_1(x_2)}$ and $c_0 = -a_0 f_1(x_2)$. Then $\overline{f_{a_0, b_0, c_0}}$ contains p_1 and p_2 on its convex branch.

Case 2: $x_1 > 0 > x_2$, and $y_1 > 0$. We claim that there exists $\overline{f_{a_0, b_0, c_0}}$ whose convex branch contains p_1 and whose concave branch contains p_2, that is, the system

$$\begin{cases} y_1 = af_1(x_1) + c \\ 0 = -af_2(-x_2) + c \end{cases}$$

has a solution a_0, c_0. It is sufficient to show that the function $g : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$g(a) = (f_1(x_1) + f_2(-x_2))a - y_1,$$

has a root a_0, which is immediate from the IVT.

Case 3: $0 > x_1 > x_2$, and $y_1 < 0$. One can show that there exists $\overline{f_{a_0, b_0, c_0}}$ whose concave branch contains p_1 and p_2. This is similar to Case 1. □

Lemma 4.6. *Let p_1, p_2, p_3 be three points in admissible position type 4. Then there exist $a_0 > 0, b_0, c_0 \in \mathbb{R}$ such that $\overline{f_{a_0,b_0,c_0}}$ contains p_1, p_2, p_3.*

Proof. We can assume $y_3 = 0, x_2 = 0, y_1 > y_2$. Let $c_0 = 0$. There are three cases.

Case 1: $y_1 > y_2 > 0$, and $x_1 < 0$. We show that there exists $\overline{f_{a_0,b_0,c_0}}$ whose convex branch contains p_1 and p_2, by showing that the system

$$\begin{cases} y_1 = af_1(x_1 + b) \\ y_2 = af_1(b) \end{cases} \tag{1}$$

has a solution a_0, b_0. Eliminating the variable a, we get

$$\frac{y_1}{y_2} = \frac{f_1(x_1 + b)}{f_1(b)}. \tag{2}$$

We consider the function $g : (-x_1, +\infty) \to \mathbb{R}$ defined by $g(b) = \dfrac{f_1(x_1 + b)}{f_1(b)}$. We have that g is continuous, $\lim_{b \to +\infty} g(b) = 1$ by Definition 3.1, and $\lim_{b \to -x_1+} g(b) = +\infty$. By the IVT, there exists b_0 such that $g(b_0) = \dfrac{y_1}{y_2} > 1$, that is, b_0 is a root of (2). This shows that (1) also has a solution.

Case 2: $y_1 > 0 > y_2$, and $x_1 > 0$. It can be shown that there exists $\overline{f_{a_0,b_0,c_0}}$ whose convex branch contains p_1 and whose concave branch contains p_2. This is similar to Case 1.

Case 3: $0 > y_1 > y_2$, and $x_1 < 0$. It can be shown that there exists $\overline{f_{a_0,b_0,c_0}}$ whose concave branch contains p_1 and p_2. This is similar to Case 1. $\qquad\square$

Lemma 4.7. *Let p_1, p_2, p_3 be three points in admissible position type 5. Then either there exist $a_0 > 0, b_0, c_0 \in \mathbb{R}$ such that $\overline{f_{a_0,b_0,c_0}}$ contains p_1, p_2, p_3; or, there exist $s_0 < 0, t_0 \in \mathbb{R}$ such that $\overline{l_{s_0,t_0}}$ contains p_1, p_2, p_3.*

Proof. We can assume $x_3 = y_3 = 0$ and $0 < x_1 < x_2$. For nontriviality, we further assume that the three points are not collinear. There are four cases depending on the values of y_1 and y_2.

Case 1: $y_2 < y_1 < 0$ and $y_1 < \dfrac{x_1}{x_2} y_2 < 0$. We claim that there exists $\overline{f_{a_0,b_0,c_0}}$ whose convex branch contains p_1, p_2, p_3, by showing that the system

$$\begin{cases} y_1 = af_1(x_1 + b) + c \\ y_2 = af_1(x_2 + b) + c \\ 0 = af_1(b) + c \end{cases}$$

has a solution a_0, b_0, c_0. The third equation gives $c = -af_1(b)$. Substituting this into the first two equations and rearranging, we get

$$a = \frac{y_1}{f_1(x_1 + b) - f_1(b)} = \frac{y_2}{f_1(x_2 + b) - f_1(b)}. \tag{3}$$

We consider the function $g : (0, +\infty) \to \mathbb{R}$ defined by

$$g(b) = \frac{f_1(x_2 + b) - f_1(b)}{f_1(x_1 + b) - f_1(b)} = 1 + \frac{f_1(x_2 + b) - f_1(x_1 + b)}{f_1(x_1 + b) - f_1(b)}.$$

We note that g is continuous and $\lim_{b \to 0+} g(b) = 1$. Also, $\lim_{b \to +\infty} g(b) = \dfrac{x_2}{x_1}$ by Lemma 3.2. The conditions of x_i, y_i imply that $1 < \dfrac{y_2}{y_1} < \dfrac{x_2}{x_1}$. By the IVT, there exists b_0 such that $g(b_0) = \dfrac{y_2}{y_1}$. This shows that the second equality in (3) has a solution and the claim follows.

Case 2: $y_1 > y_2 > 0$. It can be shown that exists $\overline{f_{a_0, b_0, c_0}}$ whose convex branch contains p_1, p_2 and whose concave branch contains p_3. This is similar to Case 1.

Case 3: $y_1 < 0$, and $y_2 > 0$. It can be shown that there exists $\overline{f_{a_0, b_0, c_0}}$ whose concave branch contains p_1, p_3 and whose convex branch contains p_2. This is similar to Case 1.

Case 4: $y_2 < y_1 < 0$ and $\dfrac{x_1}{x_2} y_2 < y_1 < 0$. It can be shown that there exists $\overline{f_{a_0, b_0, c_0}}$ whose concave branch contains p_1, p_2, p_3. This is similar to Case 1.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. Axiom of Joining, uniqueness

In this subsection we prove the following.

Theorem 4.8 (Axiom of Joining, uniqueness). *Two distinct elements $C, D \in \mathcal{C}^-(f_1, f_2)$ have at most two intersections.*

Proof. We note that the theorem holds if at least one of C or D has the form $\overline{l_{s,t}}$. In the remainder of the proof, let $C = \overline{f_{a_1, b_1, c_1}}$ and $D = \overline{f_{a_2, b_2, c_2}}$, where $a_1, a_2 > 0, b_1, b_2, c_1, c_2 \in \mathbb{R}, (a_1, b_1, c_1) \neq (a_2, b_2, c_2)$. We assume that C and D have two intersections p, q and show that they can have no other intersection. There are three cases depending on the coordinates of p.

Case 1: $p = (b, \infty), b \in \mathbb{R}$. Then $b_1 = b_2 = -b$. If $q = (x_q, y_q) \in \mathbb{R}^2$, then the claim follows from Lemma 3.6. Otherwise, $q = (\infty, c)$ so that $c_1 = c_2 = c$. In this case, $a_1 \neq a_2$ and the claim follows also from Lemma 3.6.

Case 2: $p = (\infty, c), c \in \mathbb{R}$. This case can be treated similarly to the previous case.

Case 3: $p = (x_p, y_p) \in \mathbb{R}^2$. By symmetry, we may also assume that $q = (x_q, y_q) \in \mathbb{R}^2$. It is sufficient to show that if p, q are on convex branches of C and D, then C and D have no other intersections. Comparing the cases in Lemmas 3.6 and 3.7, we have $b_1 \neq b_2$ and $c_1 \neq c_2$, which shows that C and D have no intersections at infinity. Also, the equation

$$a_1 f_1(x + b_1) + c_1 = a_2 f_1(x + b_2) + c_2$$

has two solutions x_p, x_q. By Lemma 3.7, the equation

$$-a_1 f_2(-x - b_1) + c_1 = -a_2 f_2(-x - b_2) + c_2$$

has no solution. This implies that the concave branch of C does not intersect the concave branch of D. We note that the convex branch of C cannot intersect the concave branch of D and vice versa. This completes the proof. □

4.3. Axiom of touching, existence

We say that two distinct elements C, D of $\mathcal{C}^-(f_1, f_2)$ *touch* at p if $C \cap D = \{p\}$. As a preparation for the proof of the main theorem of this subsection, we have the following.

Lemma 4.9. *Let* $a_1, a_2 > 0, b_1, b_2, c_1, c_2 \in \mathbb{R}, (a_1, b_1, c_1) \neq (a_2, b_2, c_2)$. *If one of the following conditions holds, then* $D_1 = \overline{f_{a_1, b_1, c_1}}$ *and* $D_2 = \overline{f_{a_2, b_2, c_2}}$ *touch at a point* p.

1. $p = (b, \infty)$, $a_1 = a_2$, $b_1 = b_2 = -b$ *and* $c_1 \neq c_2$.
2. $p = (\infty, c)$, $a_1 = a_2$, $b_1 \neq b_2$, *and* $c_1 = c_2 = c$.
3. $p = (x_p, y_p) \in \mathbb{R}^2$, $f_{a_1, b_1, c_1}(x_p) = f_{a_2, b_2, c_2}(x_p) = y_p$ *and* $f'_{a_1, b_1, c_1}(x_p) = f'_{a_2, b_2, c_2}(x_p)$.

Proof.   1. Assume that the first condition holds. Note that $p \in D_1 \cap D_2$. By Lemma 3.6, D_1 and D_2 have no intersection on their convex and concave branches. They cannot have an intersection of the form (∞, c) either, since $c_1 \neq c_2$. Therefore, p is the only common point of D_1 and D_2. A similar conclusion can be made for the second condition.

2. Assume that the third condition holds. Since $f_{a_1, b_1, c_1}(x_p) = f_{a_2, b_2, c_2}(x_p) = y_p$, the point p is an intersection of D_1 and D_2. If p is on the convex branch of D_1 and the concave branch of D_2, then it is easy to check that D_1 and D_2 have no other intersection. Consider the case p is on the convex branches of D_1 and D_2. Then the function $\check{f} : (\max\{-b_1, -b_2\}, +\infty) \to \mathbb{R}$ defined by

$$\check{f}(x) = a_1 f_1(x + b_1) + c_1 - a_2 f_1(x + b_2) - c_2$$

has at least one root x_p such that $\check{f}'(x_p) = 0$.
If $b_1 = b_2, c_1 \neq c_2$ or $b_1 \neq b_2, c_1 = c_2$, then from Lemma 3.6 we obtain a contradiction. Hence $b_1 \neq b_2, c_1 \neq c_2$. By Lemma 3.7, p is the only finite intersection of D_1 and D_2. Also, D_1 and D_2 have no intersection at infinity, and so p is the only common point of D_1 and D_2. This completes the proof. □

Lemma 4.10. *Let* $C = \overline{f_{a_0, b_0, c_0}}$ *and* $p = (-b_0, \infty)$. *Let* q *be a point such that* $q \notin C, q \not\Vert p$. *Then there exist* $a_1 > 0, b_1, c_1 \in \mathbb{R}$ *such that* $D = \overline{f_{a_1, b_1, c_1}}$ *contains* p, q *and touches* C *at* p.

Proof. Let $a_1 = a_0, b_1 = b_0$. Depending on the position of q, we let c_1 be described as follows.

Case 1: $q = (\infty, y_q)$. Let $c_1 = y_q$.

Case 2: $q = (x_q, y_q), x_q > -b_1$. Let $c_1 = y_q - a_1 f_1(x_q + b_1)$.

Case 3: $q = (x_q, y_q), x_q < -b_1$. Let $c_1 = y_q + a_1 f_2(-x_q - b_1)$.

We note that the condition $q \notin C$ implies that $c_1 \neq c_0$ in each of the cases above. Let $D = \overline{f_{a_1, b_1, c_1}}$. Then D contains p, q, and by Lemma 4.9, touches C at p. $\qquad\square$

Lemma 4.11. *Let $C = \overline{f_{a_0, b_0, c_0}}$ and $p = (\infty, c_0)$. Let q be a point such that $q \notin C, q \nparallel p$. Then there exist $a_1 > 0, b_1, c_1 \in \mathbb{R}$ such that $D = \overline{f_{a_1, b_1, c_1}}$ contains p, q and touches C at p.*

Proof. Let $a_1 = a_0, c_1 = c_0$. There are three cases depending on the position of q.

Case 1: $q = (x_q, \infty)$. Let $b_1 = x_q$. We note that $b_1 = x_q \neq b_0$, since $q \notin C$.

Case 2: $q = (x_q, y_q) \in \mathbb{R}^2, y_q > c_1$. Since f_1 is surjective on $\mathbb{R}^+$, there exists $b_1 \in (-x_q, \infty)$ such that $a_1 f_1(x_q + b_1) + c_1 = y_q$. Since $q \notin C$, it follows that $y_q \neq a_0 f_1(x_q + b_0) + c_0$, which implies $f_1(x_q + b_1) \neq f_1(x_q + b_0)$. In particular, $b_1 \neq b_0$ since f_1 is injective on $\mathbb{R}^+$.

Case 3: $q = (x_q, y_q), y_q < c_1$. This is similar to Case 2.

Let $D = \overline{f_{a_1, b_1, c_1}}$. Then D contains p, q, and by Lemma 4.9, touches C at p.

$\qquad\square$

Lemma 4.12. *Let $p = (x_p, y_p) \in \mathbb{R}^2$ and $s < 0$. Let q be a point such that $q \nparallel p$. Then exactly one of the following is true.*

1. *There exists $t \in \mathbb{R}$ such that $D = \overline{l_{s,t}}$ contains p and q.*
2. *There exist $a_1 > 0, b_1, c_1 \in \mathbb{R}$ such that $D = \overline{f_{a_1, b_1, c_1}}$ contains p, q and $f'_{a_1, b_1, c_1}(x_p) = s$.*

Proof. We can assume $p = (0, 0)$. If q is on the line $y = sx$ or $q = (\infty, \infty)$, then $D = \overline{l_{s,0}}$ contains p and q. We now consider other cases.

Case 1: $q = (x_q, \infty), x_q \neq 0$. Let $b_1 = -x_q$. If $x_q < 0$, we consider the system

$$\begin{cases} 0 = a f_1(b_1) + c \\ s = a f_1'(b_1) \end{cases}$$

in variables a, c. The solution is $a_1 = \dfrac{s}{f_1'(b_1)}, c_1 = -a_1 f_1(b_1)$. Then $D = \overline{f_{a_1, b_1, c_1}}$ contains p, q and satisfies $f'_{a_1, b_1, c_1}(x_p) = s$. The case $x_q > 0$ is similar.

Case 2: $q = (\infty, y_q), y_q \neq 0$. Let $c_1 = y_q$. If $y_q < 0$, then we consider the system

$$\begin{cases} 0 = a f_1(b) + c_1 \\ s = a f_1'(b) \end{cases} \tag{4}$$

in variables a, b. Eliminating a, we obtain $-s/c_1 = f_1'(b)/f_1(b)$. Let $g : (0, +\infty) \to \mathbb{R}$ be defined by

$$g(b) = \frac{f_1'(b)}{f_1(b)}.$$

Since f_1 is continuously differentiable, g is continuous. By Lemma 3.2, $\lim_{b\to+\infty} g(b) = 0$ and g is unbounded below. As $-s/c_1 < 0$, by the IVT, (4) has a solution. Then $D = \overline{f_{a_1,b_1,c_1}}$ contains p, q with p on its convex branch, and $f_{a_1,b_1,c_1}'(x_p) = s$. Similarly, when $y_q > 0$, $D = \overline{f_{a_1,b_1,c_1}}$ contains p, q with p on its concave branch, and $f_{a_1,b_1,c_1}'(x_p) = s$.

Case 3: $q = (x_q, y_q) \in \mathbb{R}^2$, where $x_q > 0, sx_q < y_q < 0$, or $x_q < 0, 0 < sx_q < y_q$. We claim that there exists $\overline{f_{a_1,b_1,c_1}}$ containing p, q on its convex branch and $f_{a_1,b_1,c_1}'(x_p) = s$, that is, the system

$$\begin{cases} 0 = a f_1(b) + c \\ y_q = a f_1(x_q + b) + c \\ s = a f_1'(b) \end{cases}$$

has a solution a_1, b_1, c_1. From the first equation, we get $c = -a f_1(b)$. Substituting this into the remaining equations and rearranging, we have

$$a = \frac{y_q}{f_1(x_q + b) - f_1(b)} = \frac{s}{f_1'(b)}. \tag{5}$$

Let $g : (\max\{-x_q, 0\}, +\infty) \to \mathbb{R}$ be defined by

$$g(b) = \frac{f_1(x_q + b) - f_1(b)}{f_1'(b)}.$$

We consider the case $x_q > 0, sx_q < y_q < 0$. Note that $g(b)$ is continuous and $\lim_{b\to 0+} g(b) = 0$. By Lemma 3.2, $\lim_{b\to+\infty} g(b) = x_q$. By the IVT, there exists b_1 such that $g(b_1) = \frac{y_q}{s} \in (0, x_q)$. This shows that (5) has a solution. The case $x_q < 0, 0 < sx_q < y_q$ is similar. The claim follows.

Case 4: $q = (x_q, y_q) \in \mathbb{R}^2$, where $x_q < 0, y_q < 0$. It can be shown that there exists $\overline{f_{a_1,b_1,c_1}}$ containing p on its convex branch, q on its concave branch, and $f_{a_1,b_1,c_1}'(x_p) = s$.

Case 5: $q = (x_q, y_q) \in \mathbb{R}^2$, where $0 < x_q, y_q < sx_q < 0$, or $x_q < 0, 0 < y_q < sx_q$. It can be shown that there exists $\overline{f_{a_1,b_1,c_1}}$ containing p, q on its on concave branch and $f_{a_1,b_1,c_1}'(x_p) = s$.

Case 6: $q = (x_q, y_q) \in \mathbb{R}^2$, where $x_q > 0, y_q > 0$. It can be shown that there exists $\overline{f_{a_1,b_1,c_1}}$ containing p on its concave branch and q on its convex branch, and $f_{a_1,b_1,c_1}'(x_p) = s$. $\qquad\square$

Theorem 4.13 (Axiom of Touching, existence). *Given $C \in \mathcal{C}^-(f_1, f_2)$ and two points $p \in C, q \notin C$ such that $q \nparallel p$, there exists $D \in \mathcal{C}^-(f_1, f_2)$ that contains p, q and touches C at p.*

Proof. There are four cases depending on p and C.

Case 1: $p = (\infty, \infty)$, $C = \overline{l_{s_0, t_0}}$. Then $q \in \mathbb{R}^2$. Let D be the Euclidean line goes through q and parallel to C. Then D satisfies the requirements.

Case 2: $p = (-b_0, \infty)$, $C = \overline{f_{a_0, b_0, c_0}}$. The existence of D follows from Lemma 4.10.

Case 3: $p = (\infty, c_0)$, $C = \overline{f_{a_0, b_0, c_0}}$. The existence of D follows from Lemma 4.11.

Case 4: $p = (x_p, y_p) \in \mathbb{R}^2$. Let s be the slope of the tangent of C at p. By Lemma 4.12, there exist $a_1 > 0, b_1, c_1 \in \mathbb{R}$ such that $D = \overline{f_{a_1, b_1, c_1}}$ contains p, q and satisfies $f'_{a_1, b_1, c_1}(x_p) = s$. Then by Lemma 4.9, D touches C at p. $\qquad\square$

4.4. Axiom of touching, uniqueness

In this subsection, we prove the following theorem.

Theorem 4.14 (Axiom of Touching, uniqueness). *Let $C \in \mathcal{C}^-(f_1, f_2)$, $p \in C$, $q \notin C$, $q \nparallel p$. Then there is at most one element $D \in \mathcal{C}^-(f_1, f_2)$ that contains p, q and touches C at p.*

Lemma 4.15. *Let $D_1 = \overline{f_{a_1, b_1, c_1}}$ and $D_2 = \overline{f_{a_2, b_2, c_2}}$ touch at a point p.*

1. *If $p = (b, \infty)$, $b \in \mathbb{R}$, then $a_1 = a_2$, $b_1 = b_2 = -b$, and $c_1 \neq c_2$.*
2. *If $p = (\infty, c)$, $c \in \mathbb{R}$, then $a_1 = a_2$, $b_1 \neq b_2$, and $c_1 = c_2 = c$.*
3. *If $p = (x_p, y_p) \in \mathbb{R}^2$, then $f'_{a_1, b_1, c_1}(x_p) = f'_{a_2, b_2, c_2}(x_p)$.*

Proof. 1. If p is a point at infinity, then $b_1 = b_2, c_1 \neq c_2$ or $b_1 \neq b_2, c_1 = c_2$. Suppose for a contradiction that $a_1 \neq a_2$. Since D_1 and D_2 touch at p, their convex branches have no intersections. This implies that the equation

$$a_1 f_1(x + b_1) + c_1 = a_2 f_1(x + b_2) + c_2$$

has no roots. By Lemma 3.6, the equation

$$-a_1 f_2(-x - b_1) + c_1 = -a_2 f_2(-x - b_2) + c_2$$

has exactly one root. This implies that D_1 and D_2 have one intersection on their concave branches, which is a contradiction. Thus, $a_1 = a_2$.

2. Assume that $p = (x_p, y_p) \in \mathbb{R}^2$. Since p is the only intersection of D_1 and D_2, we have $b_1 \neq b_2, c_1 \neq c_2$. We note that if p is on the convex branch of D_1 and the concave branch of D_2, then $f'_{a_1, b_1, c_1}(x_p) = f'_{a_2, b_2, c_2}(x_p)$. Consider the case p is on the convex branches of both D_1 and D_2. Let $\check{f} : (\max\{-b_1, -b_2\}, +\infty) \to \mathbb{R}$ defined by

$$\check{f}(x) = a_1 f_1(x + b_1) + c_1 - a_2 f_1(x + b_2) - c_2.$$

By checking the cases in Lemma 3.7, we see that the derivative $\check{f}'$ also has a root at x_0. This implies that $f'_{a_1, b_1, c_1}(x_p) = f'_{a_2, b_2, c_2}(x_p)$. The case p on the concave branches of D_1 and D_2 is similar. $\qquad\square$

Proof of Theorem 4.14. Assume that there are $D_1, D_2 \in \mathcal{C}^-(f_1, f_2)$ that contain p, q and touch C at p. We will show that $D_1 = D_2$. There are four cases depending on the coordinates of p.

Case 1: $p = (\infty, \infty)$. Then C, D_1, D_2 are Euclidean lines extended by p. The lines D_1 and D_2 are parallel to C and go through q so they must be the same line.

Case 2: $p = (b, \infty), b \in \mathbb{R}$. Since C, D_1, D_2 contain p, they cannot be Euclidean lines extended by (∞, ∞). Let $C = \overline{f_{a_0,b_0,c_0}}$, $D_1 = \overline{f_{a_1,b_1,c_1}}$ and $D_2 = \overline{f_{a_2,b_2,c_2}}$. It follows that $b_1 = b_2 = b_0$. Since the two pairs of circles C, D_1 and C, D_2 touch at p, by Lemma 4.15, $a_1 = a_2 = a_0$.

Suppose for a contradiction that $c_1 \neq c_2$. Then q must be either on the convex branches or the concave branches of both D_1, D_2. But this is impossible by the condition $a_1 = a_2$ and Lemma 3.6. Hence $c_1 = c_2$ and $(a_1, b_1, c_1) = (a_2, b_2, c_2)$, so that $D_1 = D_2$.

Case 3: $p = (\infty, c)$. This can be treated similarly to Case 2.

Case 4: $p = (x_p, y_p) \in \mathbb{R}^2$. We only consider the nontrivial case when $C = \overline{f_{a_0,b_0,c_0}}$, $D_1 = \overline{f_{a_1,b_1,c_1}}$ and $D_2 = \overline{f_{a_2,b_2,c_2}}$. Since C and D_1 touch at a finite point $p \in \mathbb{R}^2$, by Lemma 4.15, $f'_{a_0,b_0,c_0}(x_p) = f'_{a_1,b_1,c_1}(x_p)$. Similarly, we have $f'_{a_0,b_0,c_0}(x_p) = f'_{a_2,b_2,c_2}(x_p)$. It follows that $f'_{a_1,b_1,c_1}(x_p) = f'_{a_2,b_2,c_2}(x_p)$.

Suppose for a contradiction that $(a_1, b_1, c_1) \neq (a_2, b_2, c_2)$. By Lemma 3.6, it cannot be the case that $b_1 \neq b_2, c_1 = c_2$ or $b_1 = b_2, c_1 \neq c_2$. If $b_1 \neq b_2, c_1 \neq c_2$, then we obtain a contradiction from Lemma 3.7. Hence $b_1 = b_2, c_1 = c_2$. Then D_1 and D_2 have at least three points in common (which are p and two points at infinity), and so $D_1 = D_2$. □

5. Isomorphism classes, automorphisms and Klein–Kroll types

We say a flat Minkowski plane $\mathcal{M}_f = \mathcal{M}(f_1, f_2; f_3, f_4)$ is *normalised* if $f_1(1) = f_3(1) = 1$. Every flat Minkowski plane $\mathcal{M}_f$ can be described in a normalised form and we will assume that all planes $\mathcal{M}_f$ under consideration are normalised.

Let $\mathrm{Aut}(\mathcal{M}_f)$ be the full automorphism group of $\mathcal{M}_f$. Let Σ_f be the connected component of $\mathrm{Aut}(\mathcal{M}_f)$. From the definition of $\mathcal{M}_f$, the group Φ_∞ is contained in Σ_f.

Lemma 5.1. *Let ϕ be an isomorphism between two flat Minkowski planes $\mathcal{M}_f = \mathcal{M}(f_1, f_2; f_3, f_4)$ and $\mathcal{M}_g = \mathcal{M}(g_1, g_2; g_3, g_4)$. If $\phi(\infty, \infty) \neq (\infty, \infty)$, then both $\mathcal{M}_f$ and $\mathcal{M}_g$ are isomorphic to the classical Minkowski plane.*

Proof. Assume that $\phi(\infty, \infty) = (x_0, y_0)$, $x_0, y_0 \in \mathbb{R} \cup \{\infty\}, (x_0, y_0) \neq (\infty, \infty)$. We first note that $\phi \Sigma_f \phi^{-1} = \Sigma_g$. If $\dim \Sigma_f = \dim \Sigma_g = 3$, then since both groups Σ_f and Σ_g are connected and contain Φ_∞, it follows that $\Sigma_f = \Sigma_g =$

Φ_∞, and so $\phi\Phi_\infty\phi^{-1} = \Phi_\infty$. This is a contradiction, since $\phi\Phi_\infty\phi^{-1}$ fixes (x_0, y_0), but Φ_∞ can only fix (∞, ∞).

If $\dim\Sigma_f = \dim\Sigma_g = 4$, then by Theorem 2.5, $\mathcal{M}_f$ is isomorphic to either a proper half-classical plane $\mathcal{M}_{HC}(f, id)$ or a proper generalized Hartmann plane $\mathcal{M}_{GH}(r_1, s_1; r_2, s_2)$. The former case cannot occur because the plane $\mathcal{M}_{HC}(f, id)$ does not admit Φ_∞ as a group of automorphisms. In the latter case, we have

$$\Sigma_f = \{(x, y) \mapsto (rx + a, sy + b) \mid r, s > 0, a, b \in \mathbb{R}\},$$

which fixes only the point (∞, ∞). This is also a contradiction similar to the previous case. Therefore $\dim\Sigma_f = \dim\Sigma_g = 6$, and both $\mathcal{M}_f$ and $\mathcal{M}_g$ are isomorphic to the classical Minkowski plane $\mathcal{M}_C$. $\square$

We now consider the case that ϕ maps (∞, ∞) to (∞, ∞). Since translations in $\mathbb{R}^2$ are automorphisms of both planes $\mathcal{M}_f$ and $\mathcal{M}_g$, we can also assume that ϕ maps $(0, 0)$ to $(0, 0)$. Since ϕ induces an isomorphism between the two Desarguesian derived planes, we can represent ϕ by matrices in $\mathrm{GL}(2, \mathbb{R})$. Since ϕ maps parallel classes to parallel classes, ϕ is of the form $\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$ or $\begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix}$, for $u, v \in \mathbb{R}\backslash\{0\}$. We rewrite

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} \cdot A,$$

and

$$\begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot A,$$

where $r = |u|, s = |v|$, and

$$A \in \mathfrak{A} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Let the matrices in $\mathfrak{A}$ be A_1, A_2, A_3, A_4, respectively. Then ϕ is a composition of maps of the form $\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or A_i, where $i = 1..4$. To describe ϕ, it is sufficient to describe each of these maps. This is done in Lemmas 5.2, 5.3, 5.4.

Lemma 5.2. *For $r, s > 0$, let $\phi : \mathcal{P} \to \mathcal{P} : (x, y) \mapsto (rx, sy)$. If ϕ is an isomorphism between $\mathcal{M}_f$ and $\mathcal{M}_g$, then*

$$\begin{cases} g_1(r)f_1\left(\dfrac{1}{r}\right) = 1, g_2(r)f_2\left(\dfrac{1}{r}\right) = 1 \\[2mm] g_1(x) = \dfrac{f_1(x/r)}{f_1(1/r)} \\[2mm] g_2(x) = \dfrac{f_2(x/r)}{f_1(1/r)} \\[2mm] g_3(x) = \dfrac{f_3(x/r)}{f_2(1/r)} \\[2mm] g_4(x) = \dfrac{f_4(x/r)}{f_2(1/r)} \end{cases} \tag{6}$$

holds. Conversely, if there exists $r > 0$ such that (6) holds, then for every $s > 0$, the map $\phi : \mathcal{P} \to \mathcal{P} : (x, y) \mapsto (rx, sy)$ is an isomorphism between $\mathcal{M}_f$ and $\mathcal{M}_g$.

Proof. We only prove the equations for g_i, where $i = 1, 2$, as the cases $i = 3, 4$ are similar. Under ϕ, the convex branch $\{(x, f_1(x)) \mid x > 0\}$ of the circle $\{(x, f_1(x))\}$ is mapped onto $\{(x, sf_1(x/r)) \mid x > 0\}$. On the other hand, since $r, s > 0$ (so that points in the first quadrant are mapped onto the first quadrant) and ϕ maps circles to circles, the convex branch is mapped onto $\{(x, ag_1(x)) \mid x > 0\}$, for some $a > 0$. This implies

$$sf_1(x/r) = ag_1(x)$$

for all $x > 0$. For $x = 1$ and $x = r$, the normalisation condition gives

$$f_1\left(\frac{1}{r}\right) = \frac{1}{g_1(r)} = \frac{a}{s},$$

so that $g_1(r)f_1\left(\dfrac{1}{r}\right) = 1$ and $g_1(x) = \dfrac{f_1(x/r)}{f_1(1/r)}$. Similarly, for the concave branch, we obtain

$$sf_2\left(\frac{x}{r}\right) = ag_2(x),$$

for $x > 0$. This gives $g_2(r)f_2\left(\dfrac{1}{r}\right) = 1$ and $g_2(x) = \dfrac{f_2(x/r)}{f_1(1/r)}$.

The converse direction is easily verified. $\square$

Lemma 5.3. Let $\phi : \mathcal{P} \to \mathcal{P} : (x, y) \mapsto (y, x)$. If ϕ is an isomorphism between $\mathcal{M}_f$ and $\mathcal{M}_g$, then

$$\begin{cases} g_1 = f_1^{-1} \\[2mm] g_2 = f_2^{-1} \\[2mm] g_3 = \dfrac{1}{f_4^{-1}(1)}f_4^{-1} \\[2mm] g_4 = \dfrac{1}{f_4^{-1}(1)}f_3^{-1} \end{cases} \tag{7}$$

holds.

Lemma 5.4. *For $A \in \mathfrak{A}$, let $\phi : \mathcal{P} \to \mathcal{P} : (x, y) \mapsto (x, y) \cdot A$. If ϕ is an isomorphism between $\mathcal{M}_f$ and $\mathcal{M}_g$, then one of the following occurs.*

1. *If $A = A_1$, then*

$$
\begin{cases}
g_1 = f_1 \\
g_2 = f_2 \\
g_3 = f_3 \\
g_4 = f_4
\end{cases}
\tag{8}
$$

    holds.
2. *If $A = A_2$, then*

$$
\begin{cases}
g_1 = \dfrac{1}{f_4(1)} f_4 \\
g_2 = \dfrac{1}{f_4(1)} f_3 \\
g_3 = \dfrac{1}{f_2(1)} f_2 \\
g_4 = \dfrac{1}{f_2(1)} f_1
\end{cases}
\tag{9}
$$

    holds.
3. *If $A = A_3$, then*

$$
\begin{cases}
g_1 = f_3 \\
g_2 = f_4 \\
g_3 = f_1 \\
g_4 = f_2
\end{cases}
\tag{10}
$$

    holds.
4. *If $A = A_4$, then*

$$
\begin{cases}
g_1 = \dfrac{1}{f_2(1)} f_2 \\
g_2 = \dfrac{1}{f_2(1)} f_1 \\
g_3 = \dfrac{1}{f_4(1)} f_4 \\
g_4 = \dfrac{1}{f_4(1)} f_3
\end{cases}
\tag{11}
$$

    holds.

Lemmas 5.3 and 5.4 can be verified by direct calculations.

Theorem 5.5. *A flat Minkowski plane $\mathcal{M}_f$ is isomorphic to the classical Minkowski plane $\mathcal{M}_C$ if and only if $f_i(x) = 1/x$, for $i = 1..4$.*

Proof. The "if" direction is straightforward. For the converse direction, let $g_i(x) = 1/x$, where $i = 1..4$. Then $\mathcal{M}_g = \mathcal{M}(g_1, g_2; g_3, g_4)$ is isomorphic to the classical flat Minkowski plane and

$$\Sigma_g \cong \langle PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R}), \{(x,y) \mapsto (y,x)\}\rangle.$$

Assume $\mathcal{M}_f$ and $\mathcal{M}_g$ are isomorphic. Let ϕ be an isomorphism between $\mathcal{M}_f$ and $\mathcal{M}_g$. Since Σ_g is transitive on the torus, for $(x_0,y_0) \in \mathbb{S}_1 \times \mathbb{S}_1$, there exists an automorphism $\psi \in \Sigma_g$ such that $\psi((x_0,y_0)) = (\infty,\infty)$. If $\phi((\infty,\infty)) = (x_0,y_0) \neq (\infty,\infty)$, then $\psi\phi$ is an isomorphism between $\mathcal{M}_f$ and $\mathcal{M}_g$ that maps (∞,∞) to (∞,∞). From Lemmas 5.2, 5.3, and 5.4, $f_i(x) = 1/x$. □

Let $\mathfrak{A}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathfrak{A}$. We now determine isomorphism classes of the planes $\mathcal{M}_f$. We shall say that two flat Minkowski planes $\mathcal{M}_f$ and $\mathcal{M}_g$ are *simply isomorphic* if there is an isomorphism from $\mathcal{M}_f$ to $\mathcal{M}_g$ of the form $(x,y) \mapsto (rx, sy)$, for some $r, s > 0$. In general, an isomorphism is a composition of a simple automorphism and a map from $\mathfrak{A}$ or $\mathfrak{A}'$. The effect of the latter map is described by Lemmas 5.3 and 5.4. For the simple isomorphism class, we have the following description.

Theorem 5.6. *Two flat Minkowski planes $\mathcal{M}_f$ and $\mathcal{M}_g$ are simply isomorphic if and only if there exists $r > 0$ such that* (6) *holds. (cp. Lemma 5.2).*

Proof. Let ϕ be an isomorphism between $\mathcal{M}_f$ and $\mathcal{M}_g$. If both $\mathcal{M}_f$ and $\mathcal{M}_g$ are isomorphic to the classical flat Minkowski plane, then the proof follows from Theorem 5.5. Otherwise, from Lemma 5.1, ϕ maps (∞,∞) to (∞,∞). The claim now follows from Lemma 5.2. □

From Theorem 5.6, we can determine when $\mathcal{M}_f$ is simply isomorphic to a generalised Hartmann plane (cp. Example 2.3) as follows.

Theorem 5.7. *A proper flat Minkowski plane $\mathcal{M}_f$ is simply isomorphic to a generalised Hartmann plane $\mathcal{M}_{GH}(r_1, s_1; r_2, s_2)$ if and only if $f_1(x) = x^{-r_1}$, $f_2(x) = s_1^{-1} x^{-r_1}, f_3(x) = x^{-r_2}, f_4(x) = s_2^{-1} x^{-r_2}$, where $r_1, s_1, r_2, s_2 > 0$, $(r_1, s_1, r_2, s_2) \neq (1,1,1,1)$.*

From previous results, we obtain the following group dimension classification.

Theorem 5.8. *A flat Minkowski plane $\mathcal{M}_f = \mathcal{M}(f_1, f_2; f_3, f_4)$ has group dimension*

6 if and only if $f_i(x) = 1/x$, for $i = 1..4$;
4 if and only if there exist $r_1, s_1, r_2, s_2 > 0$, $(r_1, s_1, r_2, s_2) \neq (1,1,1,1)$, such that $f_1(x) = x^{-r_1}, f_2(x) = s_1 x^{-r_1}, f_3(x) = x^{-r_2}, f_4(x) = s_2 x^{-r_2}$;
3 in all other cases.

Proof. Let n be the group dimension of $\mathcal{M}_f$. Then $n \geq 3$, since it admits the group Φ_∞ as a group of automorphisms. By Theorem 2.5, $n \geq 5$ if and only if $n = 6$ if and only if $\mathcal{M}_f$ is isomorphic to the classical Minkowski plane. The form of f_i follows from Theorem 5.5. Furthermore, $n = 4$ if and only if $\mathcal{M}_f$ is isomorphic to either a proper half-classical plane $\mathcal{M}(f, id)$ or a proper

generalized Hartmann plane. The former case cannot occur, however, since the plane $\mathcal{M}(f, id)$ does not admit Φ_∞ as a group of automorphisms. In the latter case, the form of f_i follows from Theorem 5.7 together with Lemmas 5.3 and 5.4. Observe here that the isomorphisms from $\mathfrak{A}$ or $\mathfrak{A}'$, applied to $\mathcal{M}_f$, yield a plane $\mathcal{M}_g$ where g_i has the same form as f_i. □

Regarding the Klein–Kroll types, we obtain the following.

Theorem 5.9. *A flat Minkowski plane $\mathcal{M}_f$ has Klein–Kroll type*

VII.F.23 *if it is isomorphic to the classical flat Minkowski plane;*
III.C.19 *if it is isomorphic to a generalised Hartmann plane $\mathcal{M}_{GH}(r, 1; r, 1)$, $r \neq 1$;*
III.C.1 *in all other cases.*

Proof. Let $p = (\infty, \infty)$. The connected component Σ of the full automorphism group $\mathrm{Aut}(\mathcal{M}_f)$ has a subgroup $\{\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + a, y + b) \mid a, b \in \mathbb{R}\}$ of Euclidean translations, which is transitive on $\mathcal{P} \backslash \{[p]_+ \cup [p]_-\}$. Hence, a plane $\mathcal{M}_f$ has Klein–Kroll type at least III.C.

By Theorem 2.6, the only possible types are III.C.1, III.C.18, III.C.19, or VII.F.23. By Theorem 2.7, if $\mathcal{M}_f$ is of type III.C.18, then it is isomorphic to an Artzy–Groh plane with group dimension 3. In this case,

$$\Sigma = \{(x, y) \mapsto (ax + b, ay + c) \mid a, b, c \in \mathbb{R}, a > 0\}.$$

On the other hand, Σ contains the group Φ_∞ and since $\dim \Sigma = \dim \Phi_\infty$, it follows that $\Sigma = \Phi_0$, which is a contradiction. The claim now follows from Theorem 2.7. □

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