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A characteristic property of circular cylinders

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Abstract. A translation surface in 3-dimensional Euclidean space is a surface that can be constructed as the sum of two regular curves α and β. Recently, the minimal translation surfaces were characterized in terms of the curvature and the torsion of the generating curves. In this paper, we characterize all translation surfaces with constant and non-zero mean curvature by proving that: The only translation surface in 3-dimensional Euclidean space $\mathbb{R}^3$ with constant and non-zero mean curvature H is the circular cylinder of radius $\frac{1}{2|H|}$.

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1. Introduction and statement of main result

The surfaces of our study have their origin in G. Darboux' classical text [1], where the so-called "*surfaces définies pour des propertiés cinématiques*" are considered; they are referred to as *Darboux surfaces* in the literature. A Darboux surface is defined kinematically as the movement of a curve by a uniparametric family of rigid motions of $\mathbb{R}^3$. Hence, a parameterization of such a surface is $\Psi(s,t) = A(t)\alpha(s) + \beta(t)$, where α and β are two space curves and $A(t)$ is an orthogonal matrix. In this paper we consider the case with $A(t)$ the identity. More precisely, we give the following definition.

Definition 1.1. A surface $S \subset \mathbb{R}^3$ is called *translation surface*, if it can, locally, be constructed as the sum $\Psi(s,t) = \alpha(s) + \beta(t)$ of two space curves $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ and $\beta : J \subset \mathbb{R} \to \mathbb{R}^3$. If α and β are plane curves lying on orthogonal planes, the surface is called translation surface of plane type.

The curves α and β are called the generating curves of S. The terminology "translation surface" is due to the fact that the surface S is obtained by the translation of α along β (or vice versa because the roles of α and β are interchangeable). In particular, all parametric curves $s =$ const. are congruent by translations (similarly for parametric curves $t =$ const.).

It is natural to ask for a classification of translation surfaces in $\mathbb{R}^3$, beyond translation surfaces of plane type, under some conditions on their curvatures. Recently, in collaboration with R. López [4] we completely classified all translation surfaces with constant Gaussian curvature K, proving that the only ones are cylindrical surfaces, and thus K must be zero. In [11], R. López and O. Perdomo characterized all minimal translation surfaces in $\mathbb{R}^3$ in terms of the curvature and torsion of the generating curves. Finally, in [5] jointly with R. López we offer an alternative approach to the study and construction of minimal translation surfaces. Consequently, it is natural to ask the following question.

Problem. *Which surfaces in the Euclidean space $\mathbb{R}^3$ with constant and non-zero mean curvature are sums of two space curves?*

The only known translation surface in $\mathbb{R}^3$ with constant mean curvature $H \neq 0$, is the circular cylinder of radius $\frac{1}{2|H|}$. A surface $z = f(x) + g(y)$ can be viewed as the sum of the plane curves $x \mapsto (x, 0, f(x))$ and $y \mapsto (0, y, g(y))$. Moreover, every translation surface S of plane type (see Definition 1.1) can be viewed as a surface of the above form. The following result was proved by Liu [7]: *Let S be a translation surface $z = f(x) + g(y)$* with constant mean curvature $H \neq 0$. Then, S is *congruent to the following surface or part of it*

$$z = -\frac{\sqrt{1+\alpha^2}}{2H}\sqrt{1-4H^2x^2} - \alpha y, \quad \alpha \in \mathbb{R}.$$

It is obvious that this surface is a circular cylinder of radius $\frac{1}{2|H|}$. In [3] we proved: Circular cylinders are the only translation surfaces of constant and non-zero mean curvature in $\mathbb{R}^3$, with a planar generating curve.

In the present paper we completely classify all translation surfaces with non-zero and constant mean curvature in $\mathbb{R}^3$ by proving the following result.

Theorem 1.1. *Circular cylinders are the only translation surfaces of constant and non-zero mean curvature in $\mathbb{R}^3$.*

Translation surfaces in ambient spaces besides $\mathbb{R}^3$ have been studied in [6–10] to mention just a few articles.

The paper is organized as follows: In Sect. 2, we recall some known formulas on the local theory of curves and surfaces of $\mathbb{R}^3$. Then, at any point $\Psi(s, t)$ of S we represent the rotation of Frenet trihedrons by Euler's angles and express the mean curvature of S by these angles. The Proof of Theorem 1.1 is given in Sect. 3 after some lengthy calculations.

2. Preliminaries

A general reference on curves and surfaces is [2]. All curves and surfaces considered will be assumed to be regular and of class C^∞. Let $x : I = (a, b) \to \mathbb{R}^3$ be a curve parameterized by arc length s, with curvature $k(s) > 0$, torsion $\tau(s)$, and oriented Frenet trihedron $\big(t(s), n(s), b(s)\big)$. We shall use the equations, referred to as Frenet formulas,

$$\begin{cases} t' = kn \\ n' = -kt + \tau b \\ b' = -\tau n. \end{cases} \tag{2.1}$$

We have omitted the argument s for convenience; by prime ($'$) we denote differentiation with respect to s.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3, s \mapsto \alpha(s)$, and $\beta : J \subset \mathbb{R} \to \mathbb{R}^3, t \mapsto \beta(t)$, be two curves in $\mathbb{R}^3$ parameterized by arc length with curvatures $k_\alpha(s), k_\beta(t)$, torsions $\tau_\alpha(s), \tau_\beta(t)$, and oriented Frenet trihedrons $(t_\alpha(s), n_\alpha(s), b_\alpha(s)), (t_\beta(t), n_\beta(t), b_\beta(t))$ for every $s \in I, t \in J$. In order that α and β be the generating curves of a regular translation surface $S \subset \mathbb{R}^3$, we suppose that $\alpha'(s) \times \dot\beta(t) \neq 0$ for all $(s, t) \in I \times J \subset \mathbb{R}^2$, where $\times$ represents the vector product of $\mathbb{R}^3$. By prime ($'$) we denote differentiation with respect to s, and by dot ($\dot{}$) differentiation with respect to t.

Let $S = \{\alpha(s) + \beta(t) : s \in I, t \in J\} \subset \mathbb{R}^3$ be the set obtained by the sum of the curves α and β. Then, S is a regular translation surface, and $\Psi(s, t) = \alpha(s) + \beta(t)$ is a parameterization of S.

We will now proceed to calculate the mean curvature of S. For notational convenience, we omit the dependence of functions on s and t; it is implicitly understood. The derivatives of order 1 of Ψ are $\Psi_s = \alpha' = t_\alpha, \Psi_t = \dot\beta = t_\beta$ with $\Psi_s \times \Psi_t \neq 0$. Let $\varphi(s, t), 0 < \varphi(s, t) < \pi$, be the angle between $t_\alpha(s)$ and $t_\beta(t)$ at the point $\Psi(s, t)$, that is

$$\langle t_\alpha(s), t_\beta(t)\rangle = \cos\varphi(s, t), \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ stands for the usual scalar product on $\mathbb{R}^3$. The coefficients of the first fundamental form of S in the basis Ψ_s, Ψ_t are

$$E = 1, \quad F = \cos\varphi, \quad G = 1,$$

and the unit normal vector $N(s, t)$ at $\Psi(s, t)$ is

$$N(s, t) = \frac{t_\alpha(s) \times t_\beta(t)}{\sin\varphi(s, t)}.$$

The derivatives of order 2 of Ψ are

$$\Psi_{ss} = t'_\alpha = k_\alpha n_\alpha, \quad \Psi_{st} = 0, \quad \Psi_{tt} = \dot t_\beta = k_\beta n_\beta.$$

Hence, the coefficients of the second fundamental form of S are

$$\ell = \langle\Psi_{ss}, N\rangle = -\frac{k_\alpha\langle b_\alpha, t_\beta\rangle}{\sin\varphi}, \quad m = \langle\Psi_{st}, N\rangle = 0, \quad n = \langle\Psi_{tt}, N\rangle = \frac{k_\beta\langle t_\alpha, b_\beta\rangle}{\sin\varphi}.$$

Thus, from the well-known formula $2H = \frac{\ell G - 2Fm + nE}{EG - F^2}$ for the mean curvature H we have

$$-k_\alpha \langle b_\alpha, t_\beta \rangle + k_\beta \langle t_\alpha, b_\beta \rangle = 2H \sin^3 \varphi. \tag{2.3}$$

The orthogonal matrix

$$\mathcal{O} = \begin{pmatrix} \langle t_\alpha, t_\beta \rangle & \langle n_\alpha, t_\beta \rangle & \langle b_\alpha, t_\beta \rangle \\ \langle t_\alpha, n_\beta \rangle & \langle n_\alpha, n_\beta \rangle & \langle b_\alpha, n_\beta \rangle \\ \langle t_\alpha, b_\beta \rangle & \langle n_\alpha, b_\beta \rangle & \langle b_\alpha, b_\beta \rangle \end{pmatrix}$$

represents a rotation of the Frenet frame $(t_\alpha, n_\alpha, b_\alpha)$ to the frame $(t_\beta, n_\beta, b_\beta)$ at the point $\Psi(s,t)$ of S. As it is well known, any rotation can be described by three angles, the Euler angles. There are many ways to do this. Here, we proceed as follows. At the point $\Psi(s,t)$: (i) we rotate the frame $(t_\alpha, n_\alpha, b_\alpha)$ about t_α by an angle $\vartheta(s,t)$, (ii) we rotate about the new position of b_α by an angle $\varphi(s,t)$, and thus the new position of t_α coincides with t_β, (iii) finally, we rotate about the new position of t_α (that is, about t_β) by an angle $\omega(s,t)$. The final position of $(t_\alpha, n_\alpha, b_\alpha)$ is the frame $(t_\beta, n_\beta, b_\beta)$. Therefore, we have

$$\mathcal{O} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

or

$$\begin{aligned} &\langle t_\alpha, t_\beta \rangle = \cos\varphi, \quad \langle n_\alpha, t_\beta \rangle = \sin\varphi \cos\vartheta, \quad \langle b_\alpha, t_\beta \rangle = \sin\varphi \sin\vartheta, \\ &\langle t_\alpha, n_\beta \rangle = -\sin\varphi \cos\omega, \quad \langle n_\alpha, n_\beta \rangle = \cos\varphi \cos\vartheta \cos\omega - \sin\vartheta \sin\omega, \\ &\langle b_\alpha, n_\beta \rangle = \cos\varphi \sin\vartheta \cos\omega + \cos\vartheta \sin\omega, \quad \langle t_\alpha, b_\beta \rangle = \sin\varphi \sin\omega, \\ &\langle n_\alpha, b_\beta \rangle = -\cos\varphi \cos\vartheta \sin\omega - \sin\vartheta \cos\omega, \quad \langle b_\alpha, b_\beta \rangle = -\cos\varphi \sin\vartheta \sin\omega \\ &+ \cos\vartheta \cos\omega. \end{aligned} \tag{2.4}$$

Hence, relation (2.3) becomes

$$k_\beta \sin\omega = k_\alpha \sin\vartheta + 2H \sin^2\varphi. \tag{2.5}$$

For later use, we do some calculations. Differentiating $\langle t_\alpha, t_\beta \rangle = \cos\varphi$ with respect to s and taking into account the Frenet equations for α and (2.4), we have

$$k_\alpha \langle n_\alpha, t_\beta \rangle = -\sin\varphi \cdot \varphi_s$$

or

$$k_\alpha \sin\varphi \cos\vartheta = -\sin\varphi \cdot \varphi_s,$$

where φ_s stands for the partial derivative $\frac{\partial\varphi}{\partial s}$. So, we obtain

$$\varphi_s = -k_\alpha \cos\vartheta. \tag{2.6}$$

Moreover, from $\langle n_\alpha, t_\beta \rangle = \sin\varphi \cos\vartheta$ we have

$$-k_\alpha \langle t_\alpha, t_\beta \rangle + \tau_\alpha \langle b_\alpha, t_\beta \rangle = \cos\varphi \cos\vartheta \cdot \varphi_s - \sin\varphi \sin\vartheta \cdot \vartheta_s$$

or, in view of (2.4) and (2.6),

$$-k_\alpha \cos\varphi + \tau_\alpha \sin\varphi \sin\vartheta = -k_\alpha \cos\varphi \cos^2\vartheta - \sin\varphi \sin\vartheta \cdot \vartheta_s.$$

In the case where $\sin\vartheta \neq 0$, we have

$$\sin\varphi \cdot \vartheta_s = k_\alpha \cos\varphi \sin\vartheta - \tau_\alpha \sin\varphi. \tag{2.7}$$

Differentiating $\langle t_\alpha, n_\beta \rangle = -\sin\varphi \cos\omega$ with respect to s and using (2.6), we obtain

$$k_\alpha(\cos\varphi \cos\vartheta \cos\omega - \sin\vartheta \sin\omega) = -\cos\varphi \cos\omega(-k_\alpha \cos\vartheta) + \sin\varphi \sin\omega \cdot \omega_s$$

or, in the case where $\sin\omega \neq 0$,

$$\sin\varphi \cdot \omega_s = -k_\alpha \sin\vartheta. \tag{2.8}$$

In a similar way, differentiating with respect to t, we see that

$$\varphi_t = k_\beta \cos\omega, \quad \sin\varphi \cdot \vartheta_t = k_\beta \sin\omega, \quad \sin\varphi \cdot \omega_t = -k_\beta \cos\varphi \sin\omega + \tau_\beta \sin\varphi. \tag{2.9}$$

3. Proof of Theorem 1.1

We, firstly, recall that the parametric curves $t =$ const. are parallel translations and congruent to α (similarly, the parametric curves $s =$ const. are congruent to β). In what follows, we suppose that the translation surface S has constant and non-zero mean curvature H.

If the curvature of a generating curve vanishes everywhere, that is, if a generating curve is a line ε, then S is cylindrical. Thus, if its mean curvature is constant $H \neq 0$, the section of S by a plane normal to ε is a circle of radius $\frac{1}{2|H|}$, and S is a circular cylinder.

Moreover, if a generating curve is a plane curve, that is, if its torsion vanishes everywhere, then, as we have proved in [3], S is also a circular cylinder.

Therefore, and since the problem is of local nature, in what follows we assume that

$$k_\alpha(s) > 0 \quad \text{and} \quad \tau_\alpha(s) \neq 0 \tag{3.1}$$

everywhere.

We need the following auxiliary result.

Lemma 3.1. *Under the above assumptions, we have*

$$(i) \quad \sin\vartheta \neq 0$$
$$(ii) \quad \sin\omega \neq 0$$

almost everywhere in $I \times J$.

Proof. (i) Indeed, if this were not the case, then (2.4) would yield $\langle b_\alpha, t_\beta \rangle = 0$ in an open subset of $I \times J$. Differentiating with respect to s and using the fact that $\tau_\alpha(s) \neq 0$, we obtain $\langle n_\alpha, t_\beta \rangle = 0$. So t_β is parallel to $n_\alpha \times b_\alpha = t_\alpha$, which contradicts $0 < \varphi(s,t) < \pi$.

(ii) In fact, if this were not the case, then from (2.4) we would have $\langle t_\alpha, b_\beta\rangle = 0$ in an open subset of $I \times J$. Differentiating with respect to s and using (3.1), we have $\langle n_\alpha, b_\beta\rangle = 0$. So $b_\alpha = t_\alpha \times n_\alpha = \pm b_\beta$, from which by differentiating with respect to s, we have $\tau_\alpha = 0$. This contradicts our assumption (3.1). □

Obviously, given a translation surface $\Psi(s,t) = \alpha(s) + \beta(t)$ with constant non-zero mean curvature H and a non-zero constant $\lambda \in \mathbb{R}$, the surface $Z(s,t) = \lambda\Psi(s,t)$ is also a translation surface. The mean curvature $H_Z(s,t)$ of $Z(s,t)$ is the non-zero constant $H_Z(s,t) = \frac{H}{\lambda}$; for $\lambda = 2H$, we have $H_Z(s,t) = \frac{1}{2}$. So, without loss of generality, we may suppose that $H = \frac{1}{2}$, and this hypothesis will be assumed without further comment. Therefore, from (2.5) we obtain

$$k_\beta \sin\omega = k_\alpha \sin\vartheta + \sin^2\varphi. \tag{3.2}$$

Differentiating (3.2) with respect to s and taking (2.6), (2.7) and (2.8) into account, we get

$$\begin{aligned} &k_\beta \frac{\cos\omega}{\sin\varphi}(-k_\alpha \sin\vartheta) = k'_\alpha \sin\vartheta + k_\alpha \frac{\cos\vartheta}{\sin\varphi}(k_\alpha \cos\varphi \sin\vartheta - \tau_\alpha \sin\varphi) \\ &+2\sin\varphi\cos\varphi(-k_\alpha\cos\vartheta) \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} &-k_\alpha k_\beta \cos\omega \sin\vartheta = k'_\alpha \sin\vartheta \sin\varphi + k_\alpha^2 \cos\varphi\cos\vartheta\sin\vartheta - k_\alpha\tau_\alpha\cos\vartheta\sin\varphi \\ &-2k_\alpha\cos\varphi\sin^2\varphi\cos\vartheta. \end{aligned} \tag{3.4}$$

Hence, in view of (3.1) and Lemma 3.1, we have

$$k_\beta\cos\omega = -\frac{k'_\alpha}{k_\alpha}\sin\varphi - k_\alpha\cos\varphi\cos\vartheta + \tau_\alpha\frac{\cos\vartheta}{\sin\vartheta}\sin\varphi + 2\cos\varphi\sin^2\varphi\frac{\cos\vartheta}{\sin\vartheta}. \tag{3.5}$$

Differentiating (3.5) with respect to s again and taking (3.2) into account, we have

$$\begin{aligned} &k_\alpha^2\frac{\sin^2\vartheta}{\sin\varphi} + k_\alpha\sin\vartheta\sin\varphi = -\left(\frac{k'_\alpha}{k_\alpha}\right)'\sin\varphi + k_\alpha^2\frac{\cos^2\varphi\sin^2\vartheta}{\sin\varphi} - k_\alpha\tau_\alpha\cos\varphi\sin\vartheta \\ &- k_\alpha^2\sin\varphi\cos^2\vartheta + \tau'_\alpha\frac{\cos\vartheta\sin\varphi}{\sin\vartheta} - k_\alpha\tau_\alpha\frac{\cos\varphi}{\sin\vartheta} + \tau_\alpha^2\frac{\sin\varphi}{\sin^2\vartheta} - k_\alpha\tau_\alpha\frac{\cos^2\vartheta\cos\varphi}{\sin\vartheta} \\ &+ 2k_\alpha\frac{\cos^2\vartheta\sin^3\varphi}{\sin\vartheta} - 4k_\alpha\frac{\cos^2\varphi\cos^2\vartheta\sin\varphi}{\sin\vartheta} - 2k_\alpha\frac{\cos^2\varphi\sin\varphi}{\sin\vartheta} + 2\tau_\alpha\frac{\cos\varphi\sin^2\varphi}{\sin^2\vartheta} \end{aligned}$$

or, multiplying by $\frac{\sin^2\vartheta}{\tau_\alpha}$ and collecting the terms,

$$\begin{aligned} &\frac{\left(\frac{k'_\alpha}{k_\alpha}\right)' + k_\alpha^2}{\tau_\alpha}\sin\varphi\sin^2\vartheta - \frac{\tau'_\alpha}{\tau_\alpha}\cos\vartheta\sin\vartheta\sin\varphi - \tau_\alpha\sin\varphi - 2\cos\varphi\sin^2\varphi \\ &+ \frac{2k_\alpha}{\tau_\alpha}(\tau_\alpha\cos\varphi - \sin^3\varphi + 3\cos^2\varphi\sin\varphi)\sin\vartheta \\ &+ \frac{k_\alpha}{\tau_\alpha}(2\sin^3\varphi + \sin\varphi - 4\cos^2\varphi\sin\varphi)\sin^3\vartheta = 0. \end{aligned}$$

Dividing by $\sin^3\varphi$, setting $Z := \frac{\cos\varphi}{\sin\varphi}$, and using the relation $\frac{1}{\sin^2\varphi} = 1 + Z^2$, we see that

$$\begin{aligned}&\frac{\left(\frac{k'_\alpha}{k_\alpha}\right)' + k_\alpha^2}{\tau_\alpha}\sin^2\vartheta(1+Z^2) - \frac{\tau'_\alpha}{\tau_\alpha}\cos\vartheta\sin\vartheta(1+Z^2) - \tau_\alpha(1+Z^2) - 2Z\\&+\frac{2k_\alpha}{\tau_\alpha}\left(\tau_\alpha Z(1+Z^2) - 1 + 3Z^2\right)\sin\vartheta + \frac{3k_\alpha}{\tau_\alpha}(1-Z^2)\sin^3\vartheta = 0\end{aligned}$$

or

$$\begin{aligned}&2k_\alpha\sin\vartheta Z^3 + \left(\frac{\left(\frac{k'_\alpha}{k_\alpha}\right)' + k_\alpha^2}{\tau_\alpha}\sin^2\vartheta - \frac{\tau'_\alpha}{\tau_\alpha}\cos\vartheta\sin\vartheta - \tau_\alpha + \frac{6k_\alpha}{\tau_\alpha}\sin\vartheta - \frac{3k_\alpha}{\tau_\alpha}\sin^3\vartheta\right)Z^2\\&+(2k_\alpha\sin\vartheta - 2)Z\\&+\left(\frac{\left(\frac{k'_\alpha}{k_\alpha}\right)' + k_\alpha^2}{\tau_\alpha}\sin^2\vartheta - \frac{\tau'_\alpha}{\tau_\alpha}\cos\vartheta\sin\vartheta - \tau_\alpha - \frac{2k_\alpha}{\tau_\alpha}\sin\vartheta + \frac{3k_\alpha}{\tau_\alpha}\sin^3\vartheta\right) = 0.\end{aligned} \tag{3.6}$$

Henceforth, for notational convenience, we set

$$\begin{aligned}&k := k_\alpha, \quad \tau := \tau_\alpha, \quad \varSigma := \left(\frac{k'}{k}\right)' + k^2 - \tau^2, \quad B := \frac{2k'}{k} + \frac{\tau'}{\tau}\\&C := \frac{\varSigma}{\tau} - \tau, \quad D := \frac{\varSigma}{\tau} + \tau, \quad X := \cos\vartheta, \quad Y := \sin\vartheta.\end{aligned} \tag{3.7}$$

Inserting these notations in (3.6) and using the identity $\sin^3\vartheta = \sin\vartheta(1 - \cos^2\vartheta)$, we obtain

$$\begin{aligned}&2kYZ^3 + \left(\frac{3k}{\tau}X^2Y + DY^2 - \frac{\tau'}{\tau}XY + \frac{3k}{\tau}Y - \tau\right)Z^2 + (2kY-2)Z\\&+(-\frac{3k}{\tau}X^2Y + DY^2 - \frac{\tau'}{\tau}XY + \frac{k}{\tau}Y - \tau) = 0.\end{aligned} \tag{3.8}$$

Remark 3.1. We note that $Z \neq 0$ almost everywhere in $I \times J$. Indeed, if this were not the case, then (2.1) would yield $\langle t_\alpha, t_\beta\rangle = 0$. Differentiating this twice with respect to s, and using (3.1) and the Frenet formulas (2.1), we are lead to a contradiction $\langle n_\alpha, t_\beta\rangle = \langle b_\alpha, t_\beta\rangle = 0$.

Before we proceed to some more calculations, we need the following obvious consequence of (2.6) and (2.7).

Lemma 3.2. *For the derivatives of X, Y, Z with respect to s we have*

$$\begin{aligned}X_s &= -kY^2Z + \tau Y,\\Y_s &= kXYZ - \tau X,\\Z_s &= kX(1+Z^2).\end{aligned}$$

Differentiating (3.8) with respect to s and taking Lemma 3.2 into account, we get

$$\begin{aligned}
&8k^2XYZ^4+\Big(\frac{15k^2}{\tau}X^3Y-\frac{4k\tau'}{\tau}X^2Y+4kDXY^2+\frac{3k^2}{\tau}XY-4k\tau X+kBY\Big)Z^3\\
&+\Big(-9kX^3+3\Big(\frac{k}{\tau}\Big)'X^2Y+C'Y^2+(8k^2-B')XY+kX+3\Big(\frac{k}{\tau}\Big)'Y\Big)Z^2\\
&+\Big(-\frac{3k^2}{\tau}X^3Y+4kDXY^2-\frac{4k\tau'}{\tau}X^2Y+\frac{13k^2}{\tau}XY-4k\tau X+kBY\Big)Z\\
&+\Big(-3\Big(\frac{k}{\tau}\Big)'X^2Y-9kXY^2+C'Y^2-B'XY+\Big(\frac{k}{\tau}\Big)'Y\Big)=0,
\end{aligned}\tag{3.9}$$

where in some steps we have used the relation $X^2+Y^2=1$, the notation (3.7) and the relation $2k^2-\big(\frac{\tau'}{\tau}\big)'-2\tau D=-B'$. Now, multiplying (3.8) by $4kXZ$ and subtracting from (3.9), and dividing the result by Y, which is non-zero by Lemma 3.1, we obtain

$$\begin{aligned}
&k\Big(\frac{3k}{\tau}X^3-\frac{9k}{\tau}X+B\Big)Z^3+\Big(3\Big(\frac{k}{\tau}\Big)'X^2+9kXY+C'Y-B'X+3\Big(\frac{k}{\tau}\Big)'\Big)Z^2\\
&+k\Big(\frac{9k}{\tau}X^3+\frac{9k}{\tau}X+B\Big)Z+\Big(-3\Big(\frac{k}{\tau}\Big)'X^2-9kXY+C'Y-B'X+\Big(\frac{k}{\tau}\Big)'\Big)=0.
\end{aligned}\tag{3.10}$$

Elimination of Z^3 between equations (3.8) and (3.10) yields an equation of the form

$$b_2Z^2+b_1Z+b_0=0.\tag{3.11}$$

This is a polynomial equation of second degree with respect to Z, with coefficients

$$\begin{aligned}
b_2&=\frac{9k^2}{\tau^2}X^5Y+\frac{3kD}{\tau}X^3Y^2-\frac{3k\tau'}{\tau^2}X^4Y-\frac{18k^2}{\tau^2}X^3Y-3kX^3\\
&\quad+\frac{18k\tau'}{\tau^2}X^2Y-\Big(\frac{9kD}{\tau}+18k\Big)XY^2+(BD-2C')Y^2\\
&\quad+\Big(2B'-\frac{B\tau'}{\tau}-\frac{27k^2}{\tau^2}\Big)XY+9kX+\frac{9k\tau'}{\tau^2}Y-\tau B,\\
b_1&=-\frac{12k^2}{\tau}X^3Y-\frac{6k}{\tau}X^3-\frac{36k^2}{\tau}XY+\frac{18k}{\tau}X-2B,\\
b_0&=-\frac{9k^2}{\tau^2}X^5Y+\frac{3kD}{\tau}X^3Y^2-\frac{3k\tau'}{\tau^2}X^4Y+\frac{30k^2}{\tau^2}X^3Y-3kX^3\\
&\quad-\Big(\frac{9kD}{\tau}-18k\Big)XY^2+(BD-2C')Y^2+\Big(2B'-\frac{B\tau'}{\tau}-\frac{9k^2}{\tau^2}\Big)XY\\
&\quad+9kX+\frac{3k\tau'}{\tau^2}Y-\tau B.
\end{aligned}$$

Next, we shall eliminate Z^3 between equation (3.8) and $b_2Z^3+b_1Z^2+b_0Z=0$. The resulting polynomial equation is of second degree with respect to Z and has the form

$$c_2Z^2+c_1Z+c_0=0,\tag{3.12}$$

with coefficients

$$
\begin{aligned}
c_2 = {} & \frac{27k^3}{\tau^3}X^7Y^2 + \frac{18k^2D}{\tau^2}X^5Y^3 - \frac{18k^2\tau'}{\tau^3}X^6Y^2 + \Big(-\frac{27k^3}{\tau^3} + \frac{3k(\tau')^2}{\tau^3}\Big)X^5Y^2 \\
& + \frac{3kD^2}{\tau}X^3Y^4 - \frac{6k\tau'D}{\tau^2}X^4Y^3 - \frac{18k^2}{\tau}X^5Y + \frac{63k^2\tau'}{\tau^3}X^4Y^2 - \Big(\frac{36k^2D}{\tau^2} + \frac{54k^2}{\tau}\Big)X^3Y^3 \\
& + \Big(\frac{27k\tau'D}{\tau^2} + \frac{3k}{\tau}(BD - 2C') + \frac{18k\tau'}{\tau}\Big)X^2Y^3 + \frac{6k\tau'}{\tau}X^4Y - D\Big(\frac{9kD}{\tau} + 18k\Big)XY^4 \\
& + \Big(\frac{6kB'}{\tau} - \frac{3k\tau'B}{\tau^2} - 6kD - \frac{135k^3}{\tau^3} + \frac{24k^3}{\tau} - \frac{18k(\tau')^2}{\tau^3}\Big)X^3Y^2 + \frac{48k^2}{\tau}X^3Y \\
& + D(BD - 2C')Y^4 + \Big(\frac{108k^2\tau'}{\tau^3} - \frac{\tau'}{\tau}(2B' - \frac{B\tau'}{\tau})\Big)X^2Y^2 \\
& + \Big(2DB' - \frac{2\tau'}{\tau}(BD - C') - \frac{54k^2D}{\tau^2} - \frac{54k^2}{\tau}\Big)XY^3 + 3k\tau X^3 - \Big(6k' + \frac{30k\tau'}{\tau}\Big)X^2Y \\
& + \Big(\frac{12k\tau'D}{\tau^2} + \frac{6k'D}{\tau} - \frac{6kC'}{\tau}\Big)Y^3 \\
& + \Big(18kD + 18k\tau + \frac{72k^3}{\tau} - \frac{81k^3}{\tau^3} - \frac{9k(\tau')^2}{\tau^3} + \frac{6kB'}{\tau} - \frac{3k\tau'B}{\tau^2}\Big)XY^2 \\
& + \Big(\frac{27k^2\tau'}{\tau^3} - 2\tau BD + 2\tau C'\Big)Y^2 + \Big(\frac{18k^2}{\tau} + 2\tau'B - 2\tau B'\Big)XY - 9k\tau X \\
& + \Big(2k' - \frac{8k\tau'}{\tau}\Big)Y + \tau^2 B, \\
c_1 = {} & \frac{36k^3}{\tau^2}X^5Y^2 - \frac{18k^2}{\tau^2}X^5Y - \Big(\frac{96k^3}{\tau^2} + \frac{6kD}{\tau}\Big)X^3Y^2 + \frac{6k\tau'}{\tau^2}X^4Y + \frac{36k^2\tau'}{\tau^2}X^2Y^2 \\
& - 72k^2XY^3 + \frac{36k^2}{\tau^2}X^3Y + 6kX^3 - \frac{36k\tau'}{\tau^2}X^2Y + \Big(\frac{18kD}{\tau} + 36k - \frac{36k^3}{\tau^2}\Big)XY^2 \\
& + \Big(\frac{12k^2\tau'}{\tau^2} - 2BD + 4C'\Big)Y^2 - 2\Big(2B' - \frac{B\tau'}{\tau} - \frac{27k^2}{\tau^2}\Big)XY - 18kX \\
& - \frac{18k\tau'}{\tau^2}Y + 2\tau B, \\
c_0 = {} & -\frac{27k^3}{\tau^3}X^7Y^2 + \Big(\frac{63k^3}{\tau^3} + \frac{3k(\tau')^2}{\tau^3}\Big)X^5Y^2 + \frac{3kD^2}{\tau}X^3Y^4 - \frac{6k\tau'D}{\tau^2}X^4Y^3 \\
& - \frac{39k^2\tau'}{\tau^3}X^4Y^2 + \Big(\frac{54k^2}{\tau} + \frac{12k^2D}{\tau^2}\Big)X^3Y^3 + \frac{6k\tau'}{\tau}X^4Y - D\Big(\frac{9kD}{\tau} + 18k\Big)XY^4 \\
& + \Big(-\frac{3k}{\tau}\Big(2B' - \frac{B\tau'}{\tau} - \frac{27k^2}{\tau^2}\Big) - 6kD - \frac{18k(\tau')^2}{\tau^3} - \frac{18k^3}{\tau^3}\Big)X^3Y^2 \\
& + \Big(-\frac{3k}{\tau}(BD - 2C') + \frac{27k\tau'D}{\tau^2} + \frac{18k\tau'}{\tau}\Big)X^2Y^3 + D(BD - 2C')Y^4 - \frac{12k^2}{\tau}X^3Y \\
& + \Big(D\Big(2B' - \frac{B\tau'}{\tau} - \frac{27k^2}{\tau^2}\Big) - \frac{\tau'}{\tau}(BD - 2C') - \frac{k}{\tau}\Big(\frac{9kD}{\tau} + 18k\Big)\Big)XY^3 \\
& - \Big(\frac{9k^2\tau'}{\tau^3} + \frac{\tau'}{\tau}\Big(2B' - \frac{B\tau'}{\tau} - \frac{27k^2}{\tau^2}\Big)\Big)X^2Y^2 + \Big(3kB - \frac{27k\tau'}{\tau}\Big)X^2Y + 3k\tau X^3 \\
& + \Big(\frac{9k\tau'D}{\tau^2} + \frac{k}{\tau}(BD - 2C')\Big)Y^3 + \Big(\frac{k}{\tau}\Big(2B' - \frac{B\tau'}{\tau} - \frac{27k^2}{\tau^2}\Big) + 18kD + 18k\tau \\
& - \frac{9k(\tau')^2}{\tau^3}\Big)XY^2 + \Big(\frac{9k^2\tau'}{\tau^3} - 2\tau(BD - C')\Big)Y^2 + \Big(\frac{36k^2}{\tau} + 2(\tau'B - \tau B')\Big) \\
& \times XY - 9k\tau X - \Big(2k' + \frac{10k\tau'}{\tau}\Big)Y + \tau^2 B.
\end{aligned}
$$

Before proving Theorem 1.1, for completeness, let us comment on the case that at least one of the coefficients b_i, c_i $(i = 0, 1, 2)$ is zero.

Proposition 3.1. *If at least one of the coefficients* b_i, c_i $(i = 0, 1, 2)$ *is zero in an open subset of* $I \times J$*, then the curvature of the generating curve* β *is zero, that is* $k_\beta = 0$.

Proof. Since the proofs are similar in all cases, we shall only consider the case $b_2 = 0$ in an open subset of $I \times J$. Inserting $Y^2 = 1 - X^2$ in the expression of b_2 we have

$$\Big(\frac{9k^2}{\tau^2}X^5 - \frac{3k\tau'}{\tau^2}X^4 - \frac{18k^2}{\tau^2}X^3 + \frac{18k\tau'}{\tau^2}X^2 + \Big(2B' - \frac{B\tau'}{\tau} - \frac{27k^2}{\tau^2}\Big)X + \frac{9k\tau'}{\tau^2}\Big)Y + \Big(-\frac{3kD}{\tau}X^5 + \Big(\frac{12kD}{\tau} + 15k\Big)X^3 - (BD - 2C')X^2 - \Big(\frac{9kD}{\tau} + 9k\Big)X + BD - 2C' - \tau B\Big) = 0.$$

The last equation is of the form

$$P(X)Y + Q(X) = 0,$$

where P and Q are polynomials of one variable X and their coefficients are functions of s. The leading terms are $\frac{9k^2}{\tau^2}X^5$ and $-\frac{3kD}{\tau}X^5$, respectively. Squaring $P(X)Y = -Q(X)$ and inserting $Y^2 = 1 - X^2$, we obtain

$$P^2(X)(X^2 - 1) + Q^2(X) = 0, \tag{3.13}$$

a polynomial equation in X of degree 12 with leading coefficient $\frac{81k^4}{\tau^4} > 0$, since $k > 0$ and $\tau \neq 0$. Thus, the root $X = \cos\vartheta$ is a function $f(s)$ of s, and hence we have

$$\cos\vartheta = f(s). \tag{3.14}$$

Differentiating (3.14) with respect to t, we see that $\sin\vartheta \cdot \vartheta_t = 0$; consequently, in view of Lemma 3.1, we have $\vartheta_t = 0$. From the second relation of (2.9) we obtain $k_\beta = 0$. □

Now, we are ready to prove Theorem 1.1

Proof of Theorem 1.1. Using the expressions for b_i, c_i $(i = 0, 1, 2)$ we calculate

$$
\begin{aligned}
&b_2c_0 - b_0c_2 = \\
&= \Big(\frac{9k^2}{\tau^2}X^5Y + \frac{3kD}{\tau}X^3Y^2 - \frac{3k\tau'}{\tau^2}X^4Y - \frac{18k^2}{\tau^2}X^3Y + \cdots\Big)\cdot \\
&\quad \cdot \Big(-\frac{27k^3}{\tau^3}X^7Y^2 + \Big(\frac{63k^3}{\tau^3} + \frac{3k(\tau')^2}{\tau^3}\Big)X^5Y^2 + \frac{3kD^2}{\tau}X^3Y^4 - \frac{6k\tau' D}{\tau^2}X^4Y^3 + \cdots\Big) \\
&\quad - \Big(-\frac{9k^2}{\tau^2}X^5Y + \frac{3kD}{\tau}X^3Y^2 - \frac{3k\tau'}{\tau^2}X^4Y + \frac{30k^2}{\tau^2}X^3Y + \cdots\Big)\cdot \\
&\quad \cdot \Big(\frac{27k^3}{\tau^3}X^7Y^2 + \frac{18k^2D}{\tau^2}X^5Y^3 - \frac{18k^2\tau'}{\tau^3}X^6Y^2 + \Big(-\frac{27k^3}{\tau^3} + \frac{3k(\tau')^2}{\tau^3}\Big)X^5Y^2 \\
&\quad + \frac{3kD^2}{\tau}X^3Y^4 - \frac{6k\tau' D}{\tau^2}X^4Y^3 + \cdots\Big) = P_1(X,Y), \\
&b_2c_1 - b_1c_2 = \\
&= \Big(\frac{9k^2}{\tau^2}X^5Y + \cdots\Big)\Big(\frac{36k^3}{\tau^2}X^5Y^2 + \cdots\Big) - \Big(-\frac{12k^2}{\tau}X^3Y + \cdots\Big)\Big(\frac{27k^3}{\tau^3}X^7Y^2 + \cdots\Big) \\
&= \frac{18\cdot 36k^5}{\tau^4}X^{10}Y^3 + P_2(X,Y), \\
&b_1c_0 - b_0c_1 = \\
&= \Big(-\frac{12k^2}{\tau}X^3Y + \cdots\Big)\Big(-\frac{27k^3}{\tau^3}X^7Y^2 + \cdots\Big) \\
&\quad - \Big(-\frac{9k^2}{\tau^2}X^5Y + \cdots\Big)\Big(\frac{36k^3}{\tau^2}X^5Y^2 + \cdots\Big) \\
&= \frac{18\cdot 36k^5}{\tau^4}X^{10}Y^3 + P_3(X,Y),
\end{aligned} \tag{3.15}
$$

where $P_i(X,Y)$ $(i = 1, 2, 3)$ are polynomials of two variables X, Y and of total degree at most 12. The coefficients of $P_i(X,Y)$ are functions of s.

Consider the system

$$
\begin{cases} b_2Z^2 + b_1Z + b_0 = 0 \\ c_2Z^2 + c_1Z + c_0 = 0 \end{cases} \tag{3.16}
$$

of two polynomial equations of second degree with respect to Z.

If at least one of the coefficients b_i, c_i $(i = 0, 1, 2)$ is zero in an open subset of $I \times J$, then, by Proposition 3.1, we have $k_\beta = 0$ and thus S is a circular cylinder.

In the sequel we assume that the coefficients b_i, c_i $(i = 0, 1, 2)$ are non-zero almost everywhere on $I \times J$. System (3.16) possesses at least one solution at any point $(s,t) \in I \times J$. We distinguish two cases.

Case I The system has two solutions and thus the two equations coincide up to a multiplicative factor. Then, we must have

$$b_2c_0 - b_0c_2 = b_2c_1 - b_1c_2 = b_1c_0 - b_0c_1 = 0.$$

In particular, we have

$$b_2c_1 - b_1c_2 = \frac{18\cdot 36k^5}{\tau^4}X^{10}Y^3 + P_2(X,Y) = 0.$$

Inserting $Y^2 = 1 - X^2$ and proceeding as in the proof of Proposition 3.1, we obtain an equation of the form

$$P(X)Y + Q(X) = 0,$$

where P and Q are polynomials of one variable X. The leading term of P is $-\frac{18 \cdot 36 k^5}{\tau^4} X^{12}$ and the degree of Q is at most 12. Squaring $P(X)Y = -Q(X)$ and proceeding as before we conclude that $k_\beta = 0$; hence, S is a circular cylinder.

Case II System (3.16) has exactly one solution. Then, the resultant of the two equations of (3.16) must vanish, that is, we have

$$(b_2c_0 - b_0c_2)^2 - (b_1c_0 - b_0c_1)(b_2c_1 - b_1c_2) = 0. \tag{3.17}$$

Inserting (3.15) in (3.17), we get

$$-18^2 \cdot 36^2 \frac{k^{10}}{\tau^8} X^{20} Y^6 + Q_1(X, Y) = 0, \tag{3.18}$$

where $Q_1(X, Y)$ is a polynomial of two variables X, Y and of total degree at most 25. Inserting $Y^2 = 1 - X^2$ in (3.18), we obtain an equation of the form

$$P(X) + Q(X)Y = 0, \tag{3.19}$$

where P and Q are polynomials of one variable X. The leading term of P is $18^2 \cdot 36^2 \, \frac{k^{10}}{\tau^8} X^{26}$ and the degree of Q is at most 24. Squaring $P(X) = -Q(X)Y$ and inserting $Y^2 = 1 - X^2$ we obtain

$$P^2(X) + (X^2 - 1)Q^2(X) = 0, \tag{3.20}$$

a polynomial equation in X of degree 52, with leading coefficient $18^4 \cdot 36^4 \, \frac{k^{20}}{\tau^{16}} > 0$, since $k > 0$ and $\tau \neq 0$. All coefficients of this polynomial equation are functions of s. Continuing as in the proof of Proposition 3.1, we have $k_\beta = 0$ and thus S is a circular cylinder.

This completes the Proof of Theorem 1.1.

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