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Absolute isotropic geometry

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Abstract. The term 'absolute geometry' was coined by János Bolyai to characterize the part of Euclidean geometry that does not depend on the parallel postulate. In the framework of Cayley–Klein geometries the parallel axiom characterizes three classical geometries, namely Euclidean, Galilean and Minkowskian planes. We study the part of *Galilean* geometry that does not depend on the parallel postulate (briefly called *absolute isotropic geometry*) and their models (*isotropic planes*). After an axiomatic foundation of absolute isotropic geometry, we develop the basic theory of isotropic planes, prove the theorem of Saccheri for the angle sum of a triangle, and construct models of the different types of planes (over fields and skew fields of characteristic $\neq 2$). Surprisingly, these models have counterparts in the theory of Hilbert planes. The article closes with a comparison between Hilbert planes and isotropic planes.

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1. Introduction

The term 'absolute geometry' was coined by János Bolyai to characterize the part of Euclidean geometry that does not depend on the parallel postulate. *Elementary* absolute geometry,[1] where in addition no continuity assumptions are made, can be characterized by the plane axioms of group I, II and III of Hilbert's *Grundlagen der Geometrie*. The models are commonly referred to as *Hilbert planes* (see Hessenberg and Diller [12], Bachmann [3], Hartshorne [11] and Pejas [20]).

In the framework of Cayley–Klein geometries the parallel axiom characterizes three classical geometries, namely Euclidean, Galilean and Minkowskian planes (see R. Struve [29]). In the present article we study the part of (elementary)

[1] We consider in this article only plane geometries.

Galilean geometry that does not depend on the parallel postulate. We call this theory *absolute isotropic geometry* and the models, briefly, *isotropic planes*.

The aim of the article is to propose an axiom system for real Galilean geometry, to introduce the notion of absolute isotropic geometry axiomatically, to develop the basic theory of isotropic planes and to construct 'classical' models of isotropic planes (over fields and skew fields of characteristic $\neq 2$). Throughout the paper we will highlight the parallels with the theory of Hilbert planes.

In Sect. 2 we provide an axiom system for real Euclidean planes from which in Sect. 3 an axiom system for Hilbert planes is derived.[2] We then recall some results about Hilbert planes and their models. Subsequently we will refer to these sections for a comparison between absolute isotropic geometry and Hilbert's absolute geometry.

In Sect. 4 we present an axiom system for real Galilean planes and introduce elementary absolute isotropic geometry as the theory which is defined by all of the axioms with the exception of the parallel axiom and the axioms of continuity.

In Sect. 5 we develop the basic theory of isotropic planes. As is well-known from Galilean geometry (see Yaglom [35, p. 43]), in an isotropic plane $\mathcal{E}$ there are no orthogonal lines, but the plane can be extended by 'trajectories' such that the extended plane $\mathcal{E}^*$ does have orthogonal lines. We show that $\mathcal{E}^*$ satisfies all axioms of Bachmann's plane absolute geometry [3] with the exception of the uniqueness of joining lines (see Theorem 5.5) and analyze the group of motions (see Theorem 5.7). Finally we prove that in every isotropic plane one of Saccheri's hypotheses is true: *If one triangle of an isotropic plane has an angle sum which is equal, less or greater than a straight angle, then so do all.*

Isotropic planes can thus be divided into three classes which correspond to the hypotheses of Saccheri. In Sect. 6 models of all three types of planes are constructed (over fields of characteristic $\neq 2$). Surprisingly, these models have counterparts among the Hilbert planes of Sect. 3. However, a remarkable difference exists: Every Hilbert plane has a *commutative* coordinatizing structure (a commutative field of characteristic $\neq 2$) whereas isotropic planes can also be constructed over (non-commutative) skew fields.

In Sect. 7 the analogies and differences between isotropic planes and Hilbert planes are summarized. The theory of Cayley–Klein geometries and Cayley–Klein groups provides an appropriate framework for this comparison.

[2] We follow the reflection-geometric approach of Hjelmslev [16] and Bachmann [3] and formulate axioms in group-theoretical terms (see Sect. 2).

2. An axiom system for real Euclidean geometry

In this section we introduce an axiom system for real Euclidean planes which is expressed in purely group-theoretical terms. We start with the general idea of a *geometry of involutory group elements* (see Bachmann [3, § 20,2]).

Basic Assumption. Let G be a group and S and P invariant subsets of involutions of G such that $S \cup P$ generates G.

Elements $a, b, c, \ldots$ of S are called *lines* and elements $A, B, C, \ldots$ of P *points*. If $A = b$ then A, b are *polar* to each other and the point A is called the *pole* of b and the line b the *polar* of A.

The 'stroke relation' $\alpha \mid \beta$ is an abbreviation of the statement that α, β and $\alpha\beta$ are involutory elements (i.e., group elements of order 2). The statement $\alpha, \beta \mid \delta$ is an abbreviation of $\alpha \mid \delta$ and $\beta \mid \delta$.

A point A and a line b are *incident* if $A \mid b$. The set of points on a line a is called a *row of points*. The set of lines through a point A is called a *pencil of lines*. Lines $a, b \in S$ are *orthogonal* if $a \mid b$. Points $A, B \in P$ are *polar* if $A \mid B$. A pair (A, b) is a *flag* if A, b are incident. Flags (A, a) and (B, b) are called *parallel* if $Aa = Bb$.

The mapping $a \to a^{\alpha}, A \to A^{\alpha}$ of S onto S and P onto P is called the *motion* induced by $\alpha \in G$ (we write β^{α} instead of $\alpha^{-1}\beta\alpha$). A point M is a *midpoint* of A and B (respectively of a and b) if $A^M = B$ (respectively if $a^M = b$). Dually, a line m is a *midline* of a and b (respectively of A and B) if $a^m = b$ (respectively if $A^m = B$). If a and b have a common point then a midline is also called a *bisector* of a and b. If A and B have a joining line, then a midline is also called a *right bisector* of A and B.

Double incidences are quadruples (A, B, c, d) with $A, B \mid c, d$ and $A \neq B$ and $c \neq d$. A *triangle* is a set of three (distinct) points A, B, C and three (distinct) lines a, b, c with $A \mid b, c$ and $B \mid a, c$ and $C \mid a, b$. A *quadrangle* is a set of four points A, B, C, D and four lines a, b, c, d with $a \mid A, B$ and $b \mid B, C$ and $c \mid C, D$ and $d \mid D, A$.

Remark 2.1. The 'geometry of involutory group elements' associates to any group-theoretical structure (G, S, P) a geometric structure (the *group plane*) of points, lines and geometric relations (such as incidence and orthogonality), and provides a translation from the associated group-theoretical language L to a geometric language L'.

As Pambuccian [19] points out, if one is interested in producing an axiomatic system, one should only use first-order logic.[3] As shown in H. Struve and R. Struve [31] and [32], there exist first-order versions L_P of L and L'_P of L', such that a theory $\mathcal{T}$, which is formulated in L_P, is *mutually faithfully interpretable*[4]

[3]This consensus has been reached by the end of the first half of the 20th century, based on the work of Skolem, Hilbert and Ackermann, Gödel, and Tarski.

[4]That is, a sentence φ is a theorem of $\mathcal{T}$ if and only if the translation of φ is a theorem of $\mathcal{T}'$, and vice versa.

with the theory $\mathcal{T}'$, which is obtained from $\mathcal{T}$ by a translation from L_P to L'_P. In other words, $\mathcal{T}$ and $\mathcal{T}'$ represent *the same theory*.[5]

This correspondence justifies to use the geometric interpretation of the group-theoretical structure (G, S, P) without any further ado (see the 'formulation of geometry in the group of motions' in [32, Sect. 3.4]). This should not cause any confusion since the context will always be clear.

Based on this group-theoretical approach, Bachmann [3] defines a common substratum of Euclidean, hyperbolic, and elliptic geometry by an axiom system whose models are called *Bachmann groups* or—in the geometric interpretation—*Bachmann planes*. No assumptions are made about order or continuity.

Bachmann groups are the models of an axiom system (denoted by $\mathcal{B}$) which consists of the Basic Assumption and the following axioms:

B0. If $a \,|\, b$ then $ab \in P$ and for every A there exist a, b with $A = ab$.
B1. For A, B there exists c with $A, B \,|\, c$.
B2. If $A, B \,|\, c, d$ then $A = B$ or $c = d$.
B3. If $a, b, c \,|\, D$ then $abc \in S$.
B4. If $a, b, c \,|\, d$ then $abc \in S$.
B5. There exists g, h, j with $g \,|\, h$ and $j \nmid g$ and $j \nmid h$ and $j \nmid gh$.

The axioms make the following statements: According to **B0** orthogonal lines a, b intersect in the point ab and through every point there exist two orthogonal lines. According to **B1** and **B2** any two points have a unique joining line and according to **B3** and **B4** the *theorem of three reflections* holds: If three lines have a common point or a common perpendicular, then the product of the reflections in these lines is a line reflection. According to **B5** there exist two orthogonal lines g and h and a line j which is neither orthogonal to g or h nor incident with gh.

A Bachmann plane is a *Euclidean plane* if the parallel postulate **Pax** holds:

Axiom Pax. For A, b with $A \nmid b$ there is one and only one line through A which has no common point with b.

According to [3] a Euclidean plane is a Pappian affine plane (with a commutative coordinate field K of characteristic $\neq 2$) which can be extended to a Pappian projective plane (the *projective ideal plane*). The orthogonality of lines induces in the projective ideal plane a polarity, which can be described in the projective coordinate plane by a symmetric bilinear form f of rank 2 and index[6] 0. The group G of Euclidean motions can be represented as the special orthogonal group $O_3^+(K, f)$ of degree 3 of orthogonal transformations with determinant 1.

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[5] This holds under mild hypotheses with respect to the axioms of $\mathcal{T}$, which all axiom systems of this article satisfy (see [32]).

[6] The *(Witt) index* of a quadratic space (V, f) is defined to be the maximal dimension of a totally isotropic subspace of $V/RadV$ (see [3, p. 311]).

A Euclidean plane (and more generally a Bachmann plane) has *free mobility* if any two points have a midpoint and if any two lines, which are incident with a common point, have a bisector, i.e., if the following axioms hold

B6. For A, B there exists M with $A^M = B$.
B7. If $a, b\,|\,C$ then there exists m with $a^m = b$.

A Euclidean plane with free mobility has a Pythagorean coordinate field.

A Euclidean plane is *orderable* if every line is linearly orderable and if every line admits a partition into sides, which is compatible with the linear order of lines (see [28, Definition 3.9]). We note that this definition can be formulated in purely group-theoretical terms: A line g is orderable and admits a partition into sides, if the group of translations of the Euclidean plane is linearly orderable[7] such that the translations along g form a convex subgroup (see [27, Theorem 3.24]). In an ordered Euclidean plane the pencil of lines are cyclically ordered and the associated field K of coordinates is an ordered field (see [33] and [27, Sect. 4]).

If, in addition, the *continuity axiom* holds, which states, that the rows of points (respectively the group T_g of translations along a line g) are *Dedekind complete* (that is, *every non-empty subset which is bounded above has a supremum*), then the Euclidean plane is isomorphic to the Euclidean coordinate plane over the real numbers (cp. Behnke et al. [5, Chapt. 7, Sect. 2] and Prieß-Crampe [21]).[8]

3. Hilbert planes

The term 'absolute geometry' was coined by Bolyai to characterize the part of Euclidean geometry that does not depend on the parallel postulate. If, in addition, no continuity assumptions are assumed, this part of Euclidean geometry is the theory of the plane axioms of group I, II and III of Hilbert's *Grundlagen der Geometrie* or, equivalently, the theory of *ordered Bachmann planes with free mobility* (the so-called *Hilbert planes*). For the theory of Hilbert planes we refer to Pejas [20], Bachmann [3, § 20,13], Hessenberg and Diller [12] and Hartshorne [11].

Hilbert planes can be divided into three classes which correspond to the cases which Saccheri called the hypothesis of the right angle, the hypothesis of the acute angle, and the hypothesis of the obtuse angle.

Theorem 3.1. *If one triangle of a Hilbert plane has an angle sum which is equal, less or greater than a straight angle, then so do all.*

Proof. This holds according to Hartshorne [11, Theorem 34.7]. □

[7] A group is linearly orderable if and only if there exists a partition into cones; see [28, Theorem 3.3].

[8] The continuity axiom implies the Archimedean axiom which Hilbert used in [13]. Neither of these axioms can be stated in first-order logic.

Definition 3.2. A Hilbert plane is of **Type I**, **Type II** or **Type III** according as in every triangle the angle sum is equal, less or greater than a straight angle.

3.1. Models of Hilbert planes

Basic examples of the different types of Hilbert planes are the following models[9] (we recall that, if K denotes an ordered field, then $R = \{x \in K\colon |x| < n \text{ for a natural number } n\}$ is the ring of *finitely bounded* elements and $I = \{0\} \cup \{x \in K : x^{-1} \notin R\}$ is the ideal of *infinitely small* or *infinitesimal* elements of K).

3.1.1. Euclidean Hilbert planes. Ordered Euclidean planes with free mobility are Hilbert planes of **Type I**, which satisfy the Euclidean parallel axiom. They can be represented as *affine coordinate planes* $\mathcal{A}(K, k)$ over an ordered Pythagorean field K and a metric constant $k \in K$ with $-k \notin K^2$. Points are pairs (x, y) of elements of K and lines are triples $[u, v, w]$ with $u \neq 0$ or $v \neq 0$ (proportional triples represent the same line). A point (x, y) and a line $[u, v, w]$ are *incident* if $ux + vy + w = 0$. Lines $[u, v, w]$ and $[u', v', w']$ are *orthogonal* if $vv' + kuu' = 0$. The group G of Euclidean motions can be represented as the special orthogonal group $O_3^+(K, f)$ of degree 3 over K and a bilinear form f, which corresponds to the orthogonality of lines (see Bachmann [3, § 13,1]).

3.1.2. Semi-Euclidean Hilbert planes. These planes were discovered by Dehn [7]. They are subplanes of Euclidean planes and show that the Euclidean parallel axiom is not equivalent with the postulate, that in every triangle the angle sum is a straight angle. Semi-Euclidean Hilbert planes can be constructed over any non-Archimedean ordered Pythagorean field K. The points of the plane are the points with infinitesimal coordinates of the associated Euclidean plane (see Hessenberg and Diller [12, p. 218]). The lines of the Hilbert plane are all lines of the Euclidean plane, which are incident with at least one point of the Hilbert plane. They are called 'semi-Euclidean' since they are subplanes of a Euclidean plane, but do not satisfy the Euclidean parallel axiom. Semi-Euclidean Hilbert planes are of **Type I**.

3.1.3. Hyperbolic Hilbert planes. These planes can be represented as substructures (Klein models) of the projective coordinate plane over an ordered Euclidean field where a projective metric is given by a hyperbolic polarity π. Points of the hyperbolic plane are the points which are interior of the 'fundamental conic' κ of self-conjugate points of π. Lines of the hyperbolic plane are the secants of κ. The group G of hyperbolic motions can be represented as the special orthogonal group $O_3^+(K, f)$ of degree 3 over K and the bilinear form f, which is associated to the polarity π (see Bachmann [3]).

Hyperbolic Hilbert planes are of **Type II** and satisfy the *hyperbolic parallel axiom* of Hilbert's *Neue Begründung der Bolyai-Lobatschefskyschen Geometrie*

[9] There is no commonly agreed naming of Hilbert planes. We follow Bachmann [3] and Hessenberg and Diller [12].

[14] which states, that if a line a and a point B are not incident, then there exist two 'limiting lines' through B, which separate the lines through B, which have a common point with a, from those lines through B, which have no common point with a.

3.1.4. Semi-hyperbolic Hilbert planes. These planes were discovered by Schur. They are subplanes of hyperbolic planes and can be constructed over any non-Archimedean ordered Euclidean field K. The points of the plane are the points with infinitesimal coordinates of the associated hyperbolic plane. The Lotschnittaxiom ('A quadrangle with three right angles closes'; see [1]) is satisfied, but Hilbert's hyperbolic parallel axiom does not hold. These Hilbert planes of **Type II** are called 'semi-hyperbolic' since they are subplanes of a hyperbolic plane, but do not satisfy the hyperbolic parallel axiom.

3.1.5. Non-Legendrean Hilbert planes. These Hilbert planes of **Type III** were discovered by Dehn, who called them *non-Legendrean* (see Bachmann [2, § 9] and Hessenberg and Diller [12, § 65]). The coordinate field K is a non-Archimedean ordered Pythagorean field. The points of the plane are the points with infinitesimal coordinates of the associated elliptic plane over K.

We note that elliptic planes with order and free mobility are not Hilbert planes, since their rows of points are cyclically ordered.

4. An axiom system for real Galilean geometry

Galilean geometry is not a part of classical absolute geometry. For an axiomatic characterization of Galilean planes we generalize axiom system $\mathcal{B}$ of Sect. 2 and define a common substratum of the so-called Cayley–Klein geometries (including Euclidean, hyperbolic, elliptic, Galilean and Minkowskian geometry and their dual counterparts; see R. Struve [29]) by an axiom system whose models are called *Cayley–Klein groups* or—in the geometric interpretation—*Cayley–Klein planes*. No assumptions are made about order or continuity.

Cayley–Klein groups are the models of an axiom system (denoted by $\mathcal{A}$), which consists of the Basic Assumption and the following axioms:

- **A0.** If $a\,|\,b$ then $ab \in P$ and if $A\,|\,B$ then $AB \in S$.
- **A1.** For every pair (A, b) there exists (a, B) with $a\,|\,A$ and $B\,|\,b$ and $Aa = bB$ and if $A \neq b$ then (a, B) is unique.
- **A2.** If $A, B\,|\,c, d$ then $A = B$ or $c = d$.
- **A3.** If $A, B, C\,|\,d$ then $ABC \in P$ and if $a, b, c\,|\,D$ then $abc \in S$.
- **A4.** If $A\,|\,a$ and $B\,|\,b$ and $C\,|\,c$ and $Aa = Bb = Cc$ then $ABC \in P$ and $abc \in S$.
- **A5.** There exists a quadrangle.

The axioms make the following statements: According to **A0** orthogonal lines a, b are incident with the point ab and polar points A, B are incident with the line AB. The axiom **A1** states that if A is a point and b a line, then there exists a line a through A and a point B on b with $Aa = bB$ (a 'perpendicular' Aa from A to b with foot B), and that (a, B) is unique if A is not the pole of b. According to **A2** two distinct points have at most one joining line and two distinct lines have at most one common point. **A3** states that if A, B, C are collinear points then ABC is a point (the *fourth reflection point*) and that if a, b, c are copunctual lines then abc is a line (the *fourth reflection line*). **A4** states that parallel flags (A, a), (B, b) and (C, c) have a fourth reflection point ABC and a fourth reflection line abc. According to **A5** there exists a quadrangle.

The Basic Assumption and all axioms are self-dual. Hence in the theory of Cayley–Klein groups the principle of duality holds. If (G, S, P) is a Cayley–Klein group, then we get by interchanging points and lines the *dual Cayley–Klein group* (G, S', P') with $S' = P$ and $P' = S$. We note, that a Bachmann group is a Cayley–Klein group, which satisfies **B1**, and vice versa (see [29, Theorem 4.1]).

A Cayley–Klein plane is a *Galilean plane* if the parallel postulate **Pax** and the dual statement **Pax*** hold:

Axiom Pax*. For A, b with $A \nmid b$ there is one and only one point on b which has no joining line with A.

According to H. Struve [25] and R. Struve [29] the incidence structure of a Galilean plane can be obtained from an affine plane $\mathcal{A}$ by the removal of a pencil Π of parallel lines. The elements of Π are called *isotropic lines* and determine an isotropic direction. In every line b and in every point B of a Galilean plane there exists a reflection, i.e., an involutory collineation of $\mathcal{A}$ which leaves Π invariant and which fixes the points on b respectively the lines through B. The set S of line-reflections and the set P of point reflections generate the group G of motions. A Galilean plane can be coordinatized over a division ring R.[10]

A Galilean plane has *free mobility* if **B6** and **B7** hold. Both axioms are independent from $\mathcal{A} \cup \{\textbf{Pax}^*\}$.

An *order structure* for Galilean planes is introduced in [27], and will explicitly be introduced in Definition 5.8 (for the more general case of isotropic planes). Analogously to the Euclidean case, every line of an ordered Galilean plane is linearly orderable and every line admits a partition into sides, which is compatible with the linear order of lines.

The coordinatizing algebraic structure of an ordered Galilean plane is an ordered division ring R (cp. [27, Sect. 4]). If, in addition, the Archimedean axiom holds (which states in our group-theoretical approach, that the group T_g of translations along a line g is an Archimedean ordered group), then the division ring R is isomorphic to a subfield of the real numbers (see Prieß-Crampe [21, Chap. 2, § 3, Satz 3]). If the continuity axiom of Sect. 2 is satisfied (which

[10] In the literature division rings are also called 'distributive Veblen–Wedderburn systems' (see Hughes and Piper [17]).

implies the Archimedean axiom) then R is isomorphic to the field of real numbers.

Galilean geometry is studied in detail in the literature. We refer to Yaglom [35], Giering [9], Strubecker [23] and Sachs [22] and the references given there.

5. Absolute isotropic geometry

We now study the part of Galilean geometry which does not depend on the parallel postulate and any continuity assumption. This is the theory of Cayley–Klein groups which satisfy the dual parallel postulate and the axioms of order and free mobility. We call this theory *absolute isotropic geometry*. The models are called *groups of absolute isotropic geometry* and the associated group planes *planes of absolute isotropic geometry* (or, for brevity, *isotropic planes*).

More explicitly, a group of absolute isotropic geometry (G, S, P) is a model of the **Basic Assumption** and the axioms **A0**–**A5**, **Pax***, **B6**, **B7** and **axioms of order**, for which we refer to Definition 5.8.

In an isotropic plane there exist points which have no joining line. We define:

Definition 5.1. Points A and B are called *parallel* if A and B have no joining line or if $A = B$.

A characterization of parallel points by a positive first-order sentence is given in the next theorem.

Theorem 5.2. *Distinct points A and B of an isotropic plane are parallel if and only if $a\,|\,A$ implies $Aa\,|\,B$ for all $a \in S$.*

Proof. This holds according to [29, Theorem 7.2]. □

Immediate consequences are, that parallelism is an equivalence relation on the set of points (cp. [29, Theorem 7.4]) and that the equivalence class of a point A is the set $\{B \in P : \text{There exists a line } a \text{ with } a\,|\,A \text{ and } B\,|\,Aa\}$.

According to the dual parallel postulate, through any point E of an isotropic plane there is a distinguished 'isotropic direction', given by the set of points which are parallel to E.

If H is a subset of G then we denote the set of involutions of H by $I(H)$. The involutions of PS are called *trajectories*. Let $J = I(PS)$. We say that a trajectory $\tau \in J$ and a point A (respectively a line b) are *incident* (respectively *orthogonal*) if $A\,|\,\tau$ (respectively $b\,|\,\tau$).

Theorem 5.3. *In an isotropic plane the following holds:*

(a) *The set of points of a trajectory is an equivalence class of parallel points.*
(b) *Trajectories, which have a common point, carry the same points.*

Proof. For a proof see Theorems 3.15 and 7.4 in [29]. □

Theorem 5.4. *In an isotropic plane the following statements hold:*

(a) *There are no orthogonal lines.*
(b) *There are no polar points.*
(c) *The sets S, P and J are disjoint.*

Proof. Statement (a) holds by [29, Theorem 7.5]. For a proof of (b), suppose that there exist points A, B with $A \mid B$. Then $AB \in S$. If $g := AB$ then AB is an involution in the group $T(g)$ of translations along g. Hence $T(g)$ is not linearly orderable, which is a contradiction to our axioms of order, which imply that every row of points is linearly orderable. Statement (c) holds by [29, Theorem 3.8]. □

5.1. The extended isotropic plane

Trajectories can be considered as 'ideal lines' precisely, to every group (G, S, P) of absolute isotropic geometry we associate the structure (G, S^*, P) with $S^* = S \cup J$. In this way we obtain from an isotropic plane *the extended isotropic plane*. For the extension of a Galilean plane by trajectories we refer to Bachmann [4, § 7,5].

Basic properties of an extended isotropic plane are summarized in the next theorem. Please observe that, following the idea of the geometry of involutory group elements, in this section the elements of S^* are denoted by $a, b, c, \ldots$.

Theorem 5.5. *In (G, S^*, P) with $S^* = S \cup J$ the Basic Assumption and the following axioms hold:*

(a) $P = I(S^* \cdot S^*)$: **B0**
(b) Existence of joining lines: **B1**
(c) Theorem of three reflections: **B3**, **B4**
(d) Existence of three lines in general position: **B5**
(e) Weak uniqueness of joining lines: If $A \neq B$ and $A, B, C \mid c$ and $A, B \mid d$ then $C \mid d$.
(f) Existence of perpendiculars: For A, b there exists c with $A, b \mid c$.
(g) Uniqueness of perpendiculars: If $A, b \mid c, d$ then $c = d$.
(h) Existence of double incidences: There exist A, B, c, d with $A \neq B$ and $c \neq d$ and $A, B \mid c, d$.
(i) Existence of points with at most one joining line: There exist A, B such that $A, B \mid c, d$ implies $c = d$.

Proof. For a proof of the Basic Assumption and of the statements (a), (b), (c), (f), (g), (h), (i) we can refer to the proof of Theorem 7.7 in [29], where these statements are proved for Galilean planes, but without using the parallel postulate **Pax**. Statement (d) is an immediate consequence of (f), (h) and (i). Statement (e) holds by Theorem 5.3 and axiom **A2**. □

(G, S^*, P) satisfies all axioms of a Bachmann group with the exception of the uniqueness of joining lines, which is replaced by the weaker version (e). According to (a), (c), (d), (f) and (g), (G, S^*, P) satisfies the defining axioms of a *Hjelmslev group*, which is a generalization of the notion of a Bachmann group (the existence and uniqueness of joining lines is substituted by the existence and uniqueness of perpendiculars; see Bachmann [4]).

Moreover, the following theorem holds (we recall that a Hjelmslev group is called *singular* if *every quadrilateral with three right angles is a rectangle*; see Bachmann [4, § 4.3]).

Theorem 5.6. *Let (G, S, P) be a group of absolute isotropic geometry and J the set of trajectories.*

(a) *$(G, S \cup J, P)$ and $(G, S \cup P, J)$ and $(G, J \cup P, S)$ are Hjelmslev groups.*
(b) *$(G, J \cup P, S)$ is a singular Hjelmslev group.*
(c) *$(G, S \cup P, J)$ is a non-singular Hjelmslev group.*

Proof. For a proof of (a) let (G, S, P) be a group of absolute isotropic geometry and J the set of trajectories. Then $(G, S \cup J, P)$ is a Hjelmslev group which satisfies, in addition, the properties (e), (h) and (i) of Theorem 5.5. Lines of J with a common point carry the same points whereas lines of S are not elements of a double incidence. Hence $(G, S \cup P, J)$ and $(G, J \cup P, S)$ are also Hjelmslev groups (according to H. Struve and R. Struve [24, Theorem 12]) and (a) holds.

All three Hjelmslev groups are non-elliptic since S, P and J are disjoint sets (by Theorem 5.4), whereas in the elliptic case every point-reflection is a line-reflection and vice versa. Hence none of the sets S, P and J contains elements α, β with $\alpha \,|\, \beta$.

In $(G, S \cup J, P)$ elements of J with a common point (an element of P) carry the same points. This implies for $(G, P \cup J, S)$ that lines of J, which have a common perpendicular (which is necessarily an element of P), have the same set of perpendiculars. Hence every quadrilateral with three right angles is a rectangle, i.e., the Hjelmslev group $(G, J \cup P, S)$ is singular and (b) holds.

For a proof of (c) suppose that both $(G, J \cup P, S)$ and $(G, S \cup P, J)$ are singular Hjelmslev groups, i.e., Hjelmslev groups where lines with a common perpendicular have the same set of perpendiculars (see Bachmann [4, § 4.3]). This implies that in $(G, S \cup J, P)$ lines of S (respectively lines of J), which have a common point of P, carry the same points. This is a contradiction to Theorem 5.5, (i) that there exist points with at most one joining line. □

The next theorem states basic group-theoretical properties of $(G, S \cup J, P)$.

Theorem 5.7. *In $(G, S \cup J, P)$ the following group-theoretical properties hold:*

(a) $I(G) = S \cup P \cup J$
(b) $S^3 \subseteq S$ *and* $J^3 \nsubseteq J$.
(c) *Every element of G is a product of two or three elements of $S \cup J$.*

(d) $Z(G) = \{1\}$
(e) *The centralizer of a flag* (A, b) *is* $\{1, A, b, Ab\}$ *(Rigidity Theorem).*

Proof. Since $(G, S \cup J, P)$ is a Hjelmslev group, (a), (c), (d) and (e) hold by Theorems 3.3, 3.7, 3.29 and 3.12 of [4]. Statement (b) holds since $(G, J \cup P, S)$ is a singular Hjelmslev group and $(G, S \cup P, J)$ is a non-singular Hjelmslev group. □

Since (G, S^*, P) is a Hjelmslev group, we can define that (G, S^*, P) is orderable, if the Hjelmslev group is orderable (according to R. Struve [28]).

Definition 5.8. An extended isotropic plane is *orderable*, if (a) every line is linearly orderable and (b) every line admits a partition into sides, which is compatible with the linear order of lines (see [28]). An isotropic plane is orderable, if the associated extended plane is orderable.

5.2. The theorem of Saccheri in the extended isotropic plane

In Hilbert planes the Theorem of Saccheri holds: If one Saccheri quadrilateral has acute (respectively obtuse) angles, then so do all. If one is a rectangle, then all Saccheri quadrilaterals are rectangles (see Hartshorne [11, Theorem 34.5]). We will show that a corresponding theorem holds in extended isotropic planes and start this section with the introduction of the necessary order structure.

Let (G, S^*, P) be an (extended) isotropic plane with $S^* = S \cup J$. According to our axiomatic assumptions, (G, S^*, P) is an ordered Hjelmslev group. For the order structure of Hjelmslev groups we refer to H. Struve and R. Struve [27] and [28].

Let A, B be a pair of points which have a joining line c. We define the *line segment* $\overline{AB}$ to be the set of points consisting of A, B and all points lying between A and B. We call $\overline{AB}$ a *segment of the first kind*, if $c \in S$ and of the *second kind* if $c \in J$.

Segments $\overline{AB}$ and $\overline{CD}$ are called *congruent* if there exists a motion $\alpha \in G$ which maps $\overline{AB}$ onto $\overline{CD}$. Congruence is an equivalence relation on the set of line segments. Congruent segments are of the same kind (since S and J are invariant subsets of G). We define the *length of a line segment* to be the associated congruence class.

Free mobility implies that the classes of congruent segments of the first kind can be represented by the segments $\overline{AB}$ of a fixed line $g \in S$ (that is, with $A, B \mid g$), which, in turn, can be represented by the elements of the positive cone T_g^+ of the abelian group T_g of translations along g. Since T_g is an ordered abelian group (according to our axiomatic assumptions), the classes of congruent segments can be endowed with an addition and an order relation '$<$' such that they generate an ordered abelian group (isomorphic to T_g). These definitions are equivalent with the definitions of a sum of segments and of an order relation in Hilbert planes (see Hartshorne [11, Sect. 8]).

For segments of the second kind corresponding results hold: The associated congruence classes can be endowed with an addition and an order relation such that they generate an ordered abelian group which is isomorphic to the group of translations along a line $h \in J$. The length of line segments of different kinds are not comparable (with respect to $<$).

Every line $g \in S^*$ admits a partition into sides which divides the set P of points into three classes, namely, the points of g and the two 'sides' of g (also called *halfplanes*; see [28]). We denote the sides of g by g^+ respectively g^-. An *angle* is an unordered quadruple, which is given by two lines a, b with a unique point of intersection (called the *vertex* of the angle), and by a halfplane a^+ of a and a halfplane b^+ of b (in symbols $\angle(a, b, a^+, b^+)$ or, for brevity, $\angle(a^+, b^+)$). The *interior* of an angle $\angle(a^+, b^+)$ is the set of points $a^+ \cap b^+$.

There are two kinds of angles, namely $\angle(a^+, b^+)$ with $a, b \in S$ (first kind) and $\angle(a^+, b^+)$ with $a \in S$ and $b \in J$ (second kind). There are no angles $\angle(a^+, b^+)$ with $a, b \in J$ since there are no lines $a, b \in J$ with a unique point of intersection (see Theorem 5.3). In (G, S, P) every line is an element of S and every angle is of the first kind. For this reason we are mainly interested in geometric properties of angles of the first kind. Angles of the second type are introduced for technical reasons.

The introduction of angles can, equivalently, be based on the notion of rays. A ray $\vec{u}$ of a line u is the set of points of u consisting of a point O on u (the origin of the ray) plus all points on u that are on the same side of O (with respect to the linear order on u; see [11]). For any two rays $\vec{u}$ and $\vec{v}$ with $u, v \in S$ (or $u, v \in J$) there exists a motion $\alpha \in G$, which maps $\vec{u}$ onto $\vec{v}$ (since the origins of $\vec{u}$ and $\vec{v}$ have a midpoint and since any two lines of S respectively J, which have a point of intersection, have a midline, and since the reflection in the vertex of $\vec{v}$ maps $\vec{v}$ onto the associated opposite ray). According to the Rigidity Theorem 5.7, (e) the centralizer of a flag (O, u) is $\{1, O, u, Ou\}$. Since two of these motions reverse the orientation of u the following holds:

(‡) *There are exactly two motions which leave a line u and a ray $\vec{u}$fixed, namely the identity and the reflection in u.*

With these definitions in mind, an angle can equivalently be defined as an unordered quadruple $(u, v, \vec{u}, \vec{v})$ of two distinct rays $\vec{u}$ and $\vec{v}$ and their associated lines u and v, originating at the same point (in symbols $\angle(u, v, \vec{u}, \vec{v})$ or, for brevity, $\angle\vec{u}, \vec{v})$). This definition corresponds to the definition of angles in Hilbert planes (see [11]). Since the lines u and v are distinct, there are no 'straight angles' and since the rays $\vec{u}$ and $\vec{v}$ are distinct, the lines u and v are not incident with the same points (that is, not both lines u and v are elements of J).

An angle with vertex B is also given by three points A, B, C and joining lines g of A, B and h of B, C, in symbols $\angle(A, B, C, g, h)$. We use the various representations of angles freely without further ado.

Angles $\angle(\vec{u},\vec{v})$ and $\angle(\vec{x},\vec{y})$ are called *congruent* if there exists a motion $\alpha \in G$ which maps $\{u,v\}$ onto $\{x,y\}$ and $\{\vec{u},\vec{v}\}$ onto $\{\vec{x},\vec{y}\}$. Congruence is an equivalence relation on the set of angles. The angles of an equivalence class are either all of the first kind or all of the second kind (since S and J are invariant subsets of G). We define the *magnitude of an angle* $\angle(\vec{u},\vec{v})$ to be the associated congruence class (in symbols $\measuredangle(\vec{u},\vec{v})$).

We now consider angles of the first kind and define the following notions analogously to the theory of Hilbert planes (see Hartshorne [11, Chapter 9]). If $\angle(\vec{u},\vec{v})$ and $\angle(\vec{x},\vec{y})$ are angles of the first kind (that is, $u,v,x,y \in S$) then $\varphi = \measuredangle(\vec{u},\vec{v})$ is *less than* $\vartheta = \measuredangle(\vec{x},\vec{y})$ (in symbols $\varphi < \vartheta$) if the following condition holds:

(†) *There exists a ray $\vec{z}$ in the interior of $\angle(\vec{x},\vec{y})$ such that $\angle(\vec{u},\vec{v})$ and $\angle(\vec{x},\vec{z})$ or $\angle(\vec{u},\vec{v})$ and $\angle(\vec{z},\vec{y})$ are congruent.*

Since angles of the first kind have an angular bisector, the two conditions '*there exists a ray $\vec{z}$ such that $\angle(\vec{u},\vec{v})$ and $\angle(\vec{x},\vec{z})$ are congruent*' respectively '*there exists a ray $\vec{z}$ such that $\angle(\vec{u},\vec{v})$ and $\angle(\vec{z},\vec{y})$ are congruent*' of statement (†) are equivalent. If $\varphi < \vartheta$ then ϑ is *greater than* φ (in symbols $\vartheta > \vartheta$). The $<$-relation induces a $<$-relation on the congruence classes of angles of the first kind, which is a total linear order (the proof is identical with the proof in Hilbert planes; see [11, Proposition 9.5]).

We now consider angles $\angle(\vec{u},\vec{v})$ and $\angle(\vec{x},\vec{y})$ of the second kind (with $u,x \in S$ and $v,y \in J$) and define the relation '$<$ ' in the same way as for angles of the first kind, i.e., by condition (†). There exists no ray $\vec{z}$ in the interior of $\angle(\vec{x},\vec{y})$ such that $\angle(\vec{u},\vec{v})$ and $\angle(\vec{x},\vec{z})$ are congruent (since this would imply $z \in J$, but then z and y are incident with the same points, which is a contradiction to the assumption that $\vec{z}$ is in the interior of $\angle(\vec{x},\vec{y})$). Hence the statement $\angle(\vec{u},\vec{v}) < \angle(\vec{x},\vec{y})$ is equivalent with the existence of a ray $\vec{z}$ (with $z \in S$) in the interior of $\angle(\vec{x},\vec{y})$ such that $\angle(\vec{u},\vec{v})$ and $\angle(\vec{z},\vec{y})$ are congruent.

We show that this relation is irreflexive: Let $\alpha = \angle(\vec{u},\vec{v})$ be an angle of the second kind with vertex O and let us assume $\alpha < \alpha$. Then there exists a ray $\vec{z}$ in the interior of $\angle(\vec{u},\vec{v})$ such that $\angle(\vec{u},\vec{v})$ and $\angle(\vec{z},\vec{v})$ are congruent. Thus there exists a motion $\beta \in G$ with $\vec{u}^{\beta} = \vec{z}$ and $\vec{v}^{\beta} = \vec{v}$. According to (‡) it is $\beta = v$. This is a contradiction to our assumption that $\vec{z}$ is a ray in the interior of $\angle(\vec{u},\vec{v})$, since the reflection in v interchanges the sides of v (see R. Struve [28, Theorem 3.7]). Hence there are no elements $\alpha \in G$ with $\alpha < \alpha$ and '$<$' is an irreflexive relation. Since '$<$' is transitive, it is a partial linear order, which induces a partial linear order on the set of congruence classes of angles of the second kind. Free mobility implies, that any two of these congruence classes are comparable, i.e., the partial order is a total linear order.

Finally we define the relation '$<$' for arbitrary angles of the first or second kind by condition (†). As is easily verified, this is a total linear order on the set of congruence classes of angles of the first or second kind.

Following the terminology of Hilbert planes, we say that an angle $\angle(\vec{u},\vec{v})$ is the *sum* of angles $\angle(\vec{u},\vec{w})$ and $\angle(\vec{w},\vec{v})$ if the ray $\vec{w}$ lies in the interior of $\angle(\vec{u},\vec{v})$ (see [11, Sect. 9]). The sum of angles induces a corresponding definition on the classes of congruent angles.

Now let $\angle(B,A,C,g,h)$ be an angle with vertex A and joining lines g of A,B and h of A,C. If D is a point on h on the other side of A from C, then the angles $\angle(B,A,C,g,h)$ and $\angle(B,A,D,g,h)$ are *supplementary* (cp. [11, p. 92]). An angle, which is congruent to one of its supplementary angles, is called a *right angle* (cp. [11, p. 94]). Free mobility ensures that any two right angles are congruent.

Lemma 5.9. *An angle $\angle(\vec{u},\vec{v})$ is a right angle if and only if u and v are orthogonal lines.*

Proof. Let $\varphi = \angle(B,A,C,g,h)$ and $\psi = \angle(B,A,D,g,h)$ be supplementary angles with $g\,|\,A,B$ and $h\,|\,A,C$. Then D is a point on h on the other side of A from C, in other words, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are 'opposite' rays of h.

If $g\,|\,h$ then $g^g = g$ and $(\overrightarrow{AB})^g = \overrightarrow{AB}$ and $h^g = h$ and $(\overrightarrow{AC})^g = \overrightarrow{AD}$, i.e., φ and ψ are congruent and hence right angles.

Now let $g \nmid h$ and let us assume that there exists $\alpha \in G$ with $g^\alpha = g$ and $h^\alpha = h$ and $A^\alpha = A$ or $\beta \in G$ with $g^\beta = h$ and $h^\beta = g$ and $A^\beta = A$. In the first case the flags (A,g) and (A,h) are fixed and $\alpha = 1$ or $\alpha = A$, according to the Rigidity Theorem 5.7, (e). In the second case the flags (A,g) and (A,h) are fixed by βw, if $w \in S$ denotes the angular bisector of g and h, and this implies $\beta w = 1$ or $\beta w = A$, that is $\beta = w$ or $\beta = Aw$. None of the motions of $\{1,A,w,Aw\}$ maps $\overrightarrow{AC}$ onto $\overrightarrow{AD}$ and leaves $\overrightarrow{AB}$ fixed, and hence the angles φ and ψ are not congruent and not right angles. This completes the proof. □

An angle is called *acute* if it is less than a right angle, and *obtuse* if it is greater than a right angle. We are now in a position to formulate the Theorem of Saccheri in an extended isotropic plane.

Definition 5.10. A quadrilateral with vertices A,B,C,D and sides $a,b,c,d \in S^*$ and $a \mid A,B$ and $b \mid B,D$ and $c \mid C,A$ and $d \mid D,C$ and $a\,|\,b$ and $a\,|\,c$ is called a *Saccheri quadrilateral* if there exists a line m of symmetry with $A^m = B$ and $C^m = D$.

In a Saccheri quadrilateral the angles at A and B are right angles and the angles at C and D are equal. Hence, if a Saccheri quadrilateral has an acute angle then two of the angles are acute and none of the angles is obtuse.

Theorem 5.11 (Saccheri's Theorem). *If one Saccheri quadrilateral of an extended isotropic plane has acute (respectively obtuse) angles, so do they all. If one is a rectangle, then all Saccheri quadrilaterals are rectangles.*

We follow the proof of Saccheri's Theorem in Hilbert planes (see [11, Sect. 34]), which proceeds in several steps. In a first step we prove the following lemma (see [11, Proposition 10.3]).

Lemma 5.12 (Exterior Angle Theorem). *In any triangle an exterior angle is greater than either of the opposite interior angles.*

Proof. Let A, B, C be the vertices of the given triangle and $a, b, c \in S^*$ the sides with $A \mid b, c$ and $B \mid a, c$ and $C \mid a, b$. Let D be a point on a, such that C is a point between B and D. We show that the exterior angle $\angle(A, C, D, b, a)$ (with vertex C) is greater than the interior angle $\angle(C, A, B, b, c)$ with vertex A.

Let E be the midpoint of A and C and $F = B^E$. The reflection in E maps the angle $\angle(E, A, B, b, c)$ (with vertex A) onto the angle $\angle(E, C, F, b, c^E)$ with vertex C. Since $\angle(C, A, B, b, c) = \angle(E, A, B, b, c) = \angle(E, C, F, b, c^E) = \angle(A, C, F, b, c^E)$, it is sufficient for a proof of the theorem, to show that $\angle(A, C, F, b, c^E)$ is less than $\angle(A, C, D, b, a)$.

We show that F is an interior point of $\angle(A, C, D, b, a)$. The points B and D (and also B and F) are on opposite sides of b. Hence D and F are on the same side of b. We now consider the sides of a. The point E is a point of the segment $\overline{AC}$. Hence A, E and also A, F are on the same side of a. This shows, that F is an interior point of $\angle(A, C, D, b, a)$ and this implies that $\angle(A, C, F, b, c^E)$ is less than $\angle(A, C, D, b, a)$, as required. □

Proof of Theorem 5.11. Hartshorne shows in [11] that the Exterior Angle Theorem implies in Hilbert planes successively Propositions 34.2 and 34.3 and 34.4 of [11] and finally the Theorem of Saccheri (Proposition 34.5). The proofs are identical for (extended) isotropic planes. □

We note as an immediate consequence of Saccheri's Theorem, that in an extended isotropic plane the existence of a rectangle implies that any quadrilateral with three right angles is a rectangle. This corollary holds not only in isotropic planes but also in Hilbert planes and in Bachmann's plane absolute geometry (see [3, § 6,8]).

There is a close relationship between triangles and Saccheri quadrilaterals, as the next theorem shows (cp. Hartshorne [11, Proposition 34.6]).

Theorem 5.13. *To any triangle there exists a Saccheri quadrilateral for which the sum of its two top angles is equal to the sum of the three angles of the triangle.*

Proof. We follow in outline the proof [11, p. 310] for Hilbert planes and denote points as in the figures, which are provided there. However, the proofs are not identical, since the axiomatic bases are distinct.

Let A, B, C, a, b, c be a triangle with $A \,|\, b, c$ and $B \,|\, a, c$ and $C \,|\, a, b$. Let $A^D = B$ and $A^E = C$ and n the perpendicular bisector with $n \,|\, a$ and $B^n = C$ (it is $n = Ma$, if M denotes the midpoint of B and C).

It is $A^{EnD} = A$. Hence EnD a glide reflection (by [4, Satz 2.20]) with fixed point A, and $EnD \in S$ (according to [4, Satz 2.30]). Hence there exists a line m with $m \,|\, E, n, D$ (by [4, Satz 2.23]). If $g := EnD$ then $g \,|\, A, m$, that is, g is the uniquely determined perpendicular from A to m.

Analogously, it is $B^{DEn} = B$ and this implies that $f := DEn$ is a line with $f \,|\, B, m'$ for a line m' with $m' \,|\, D, E, n$. Since there is a unique perpendicular from D to n, it is $m' = m$ and f the perpendicular from B to m. Finally, it is $C^{nDE} = C$ and $h := nDE$ the perpendicular from C to m. It is $f^n = (DEn)^n = n(DEn)n = nDE = h$. If F, G and H denote the feet of the perpendiculars f, g, h from A, B, C to m, respectively, then $F^n = H$.

It is $D^D = D$ and $c^D = c$ and $A^D = B$ and $m^D = m$, which implies $f^D = g$ and $F^D = G$. Thus $\overline{AG}$ and $\overline{CH}$ are congruent segments and $\angle(D, A, G, c, g)$ and $\angle(D, B, F, c, f)$ are congruent angles. Analogously, $\overline{BF}$ and $\overline{AG}$ are congruent segments and $\angle(E, A, G, b, g)$ and $\angle(E, C, H, b, h)$ are congruent angles.

The quadrilateral $FHBC$ with sides f, h, a, m has right angles at F and H, and n is a line of symmetry (it is $B^n = C$ and $F^n = H$, as just shown), so it is a Saccheri quadrilateral.

The angles of the quadrilateral at B and C are the sum of the angles of the triangle at B and C, plus angles that are congruent to the two parts of the angle at A, divided by the line g. Hence the angles at B and C of the quadrilateral equal the angle sum of the triangle. □

5.3. The geometry of isotropic planes

A characteristic property of isotropic geometry is that the group of rotations around a point O and the group of translations in the isotropic direction (i.e., along a trajectory of parallel points) are isomorphic.

For a proof let (G, S, P) be a group of absolute isotropic geometry. If $S(O) := \{a : a \,|\, O\}$ denotes the set of lines, which are incident with a point O, then $S(O)^3 \subseteq S(O)$. The set $D(O) := \{ab : a, b \,|\, O\}$ of rotations around O forms an abelian group. Free mobility implies that all groups $D(O)$ with $O \in P$ are conjugated.

If $\tau \in J$ then $P(\tau) := \{A : A \mid \tau\}$ is, with respect to the Hjelmslev group $(G, S \cup J, P)$, the set of points which are incident with the line τ, and thus $P(\tau)^3 \subseteq P(\tau)$. Hence the set $T(\tau) := \{AB : A, B \mid \tau\}$ of translations along a trajectory τ forms an abelian group. Free mobility implies that all groups $T(\tau)$ with $\tau \in J$ are conjugated.

Theorem 5.14. *In an isotropic plane the group $D(O)$ of rotations around a point O and the group $T(\tau)$ of translations along a trajectory τ are isomorphic to each other, for any $O \in P$ and $\tau \in J$.*

Proof. Let (G, S, P) be a group of absolute isotropic geometry. Then there are no orthogonal lines (by Theorem 5.4) and the existence and uniqueness of perpendiculars in the Hjelmslev group $(G, S \cup P, J)$ imply that an element of S and an element $\tau \in J$ have a unique common element of P.

Now let O be a point of (G, S, P) and $\tau \in J$ a trajectory with $O \nmid \tau$. Let φ be the mapping from the pencil of lines through O onto the set of points of τ which associates to a line a with $a \,|\, O$ the point A with $A \,|\, a, \tau$. Then φ is bijective (since the points of τ have no joining line and **Pax*** holds).

φ induces a mapping χ from $D(O)$ onto $T(\tau)$ with $\chi(ab) = \varphi(a)\varphi(b)$. We show that χ is well-defined. Let $a, b, c, d \mid O$ and $ab = cd$ and $A = \varphi(a)$ and $B = \varphi(b)$ and $C = \varphi(c)$ and $D = \varphi(d)$. We denote the trajectories $Aa, Bb, Cc, Dd \in J$ by a', b', c', d', respectively. By Theorem 5.3, (b) these trajectories carry the same points (namely the points which are incident with τ). Hence for all points E with $E \mid \tau$ it is $E^{AB} = E^{a'a \cdot b'b} = E^{ab'b} = E^{ab}$ and $E^{CD} = E^{c'c \cdot d'd} = E^{cd'd} = E^{cd}$. Since $ab = cd$ it is $E^{AB} = E^{CD}$ and $E^{EAB} = E^{ECD}$.

Since $(G, S \cup J, P)$ is a Hjelmslev group, $E, A, B, C, D \,|\, \tau$ implies $EAB, ECD \in P$. Since the Hjelmslev group is non-elliptic (since there are no polar points according to Theorem 5.4) any two points have at most one midpoint (see [4, Satz 2.35]) and $E^{EAB} = E^{ECD}$ implies $EAB = ECD$ and $AB = CD$. Hence $ab = cd$ implies $\varphi(a)\varphi(b) = \varphi(c)\varphi(d)$ and χ is well-defined. As an immediate consequence we note, that $abd = c$ implies $\varphi(a)\varphi(b)\varphi(d) = \varphi(c)$.

χ is bijective since φ is bijective. It remains to show that χ is a group homomorphism. If $a, b, c, d \mid O$ then $\chi(ab \cdot cd) = \chi(abc \cdot d) = \varphi(abc)\varphi(d) = \varphi(a)\varphi(b)\varphi(c)\varphi(d) = \chi(ab) \cdot \chi(cd)$. This proves the theorem. □

In Hilbert planes the length a, b, c of the sides of a triangle satisfy the triangle inequality $a + b \geq c$. In isotropic planes a 'triangle equality' holds.

Theorem 5.15. *If a, b, c are the length of the sides of a triangle of an isotropic plane and $a \leq b \leq c$ then $a + b = c$.*

Proof. Let (G, S, P) be a group of absolute isotropic geometry and A, B, C, a, b, c a triangle with $A \,|\, b, c$ and $B \,|\, a, c$ and $C \,|\, a, b$ and $a \leq b \leq c$.

Let D be the point on the line c which is parallel to C. If w denotes the line through B with $a^w = c$ then $C^w = D$ and (since the reflection in w preserves lengths) $\overline{BC} = \overline{BD}$. Analogously it is $\overline{AC} = \overline{AD}$. Hence $\overline{BC} + \overline{AC} = \overline{BD} + \overline{DA} = \overline{AB}$ (since D is a point between A and B) and $a + b = c$. □

Thus in isotropic planes there are no equilateral triangles: If two sides of a triangle are congruent, then the third side is twice as long. We now study the angle sum of a triangle in isotropic planes.

Remark 5.16. Hilbert and Hartshorne introduce the notion of an angle as the union of two rays not lying on the same line. Then the interior of an angle

$\angle BAC$ consists of all points D such that D and C are on the same side of the line through A and B, and such that D and B are on the same side of the line through A and C. Hartshorne [11, p. 77] notes, that in this approach there is no 'straight angle' (particularly, the interior of a straight angle, consisting of two opposite rays of a line, cannot be defined).

An alternative approach is, to use halfplanes for a definition of angles (see Section 5.2) and to call $\angle(a, a, a^+, a^+)$, where a^+ denotes one of the halfplanes of a line a, a *straight angle*. The interior $a^+ \cap a^+$ of this angle is a halfplane, as one would expect, and any two straight angles are congruent (since free mobility holds).[11] We propose this approach and extend the definition of an angle, given at the beginning of Section 5.2, by the notion of a straight angle.

Based on this terminology, the following theorem holds.

Theorem 5.17. *If in an isotropic plane there exists a triangle with an angle sum, which is less (respectively equal or greater) than a straight angle, then every triangle has an angle sum, which is less (respectively equal or greater) than a straight angle.*

Proof. If in an isotropic plane there exists a triangle with angle sum less than a straight angle, then the associated Saccheri quadrilateral in the extended isotropic plane must have acute angles (see Theorem 5.13). According to Theorem 5.11 then every Saccheri quadrilateral has acute angles and by Theorem 5.13 every triangle has an angle sum less than a straight angle.

The proofs of the two other cases (there exists a triangle with an angle sum which is equal respectively greater than a straight angle) are analogous. □

Hence, just as in the theory of Hilbert planes, isotropic planes can be divided into three classes which correspond to the cases which Saccheri called the hypothesis of the right angle, the hypothesis of the acute angle, and the hypothesis of the obtuse angle.

Definition 5.18. An isotropic plane is of **Type I** or **Type II** or **Type III**, depending on whether in every triangle the angle sum is equal to a straight angle or less or greater than a straight angle.

6. Models of isotropic planes

We use the framework of quadratic spaces and orthogonal groups for the construction of isotropic planes. Let V be a 3-dimensional vector space over a field K of characteristic $\neq 2$, f a symmetric bilinear form, (V, f) the associated quadratic space and $O_3^+(K, f)$ the *special orthogonal group* (of degree 3)

[11] $\angle(a, a, a^+, a^-)$ is a *zero angle* (whose interior $a^+ \cap a^-$ is void). Any two zero angles are congruent.

of orthogonal transformations with determinant 1, which we will also denote by $O_3^+(V, f)$.

If (V, f) is a regular quadratic space (with radical $radV = \{0\}$), then every element of $O_3^+(K, f)$ can be represented as a product of symmetries $\sigma_1, \ldots, \sigma_r$, and the *spinor norm* on $O_3^+(K, f)$ can be defined as the homomorphism η from $O_3^+(K, f)$ into the factor group $\dot{K}/\dot{K}^2$ (of the multiplicative group $\dot{K}$ of K over the group $\dot{K}^2$ of non-zero squares) with $\eta(\sigma_1 \cdots \sigma_r) = \eta(\sigma_1) \cdots \eta(\sigma_r)$ and $\eta(\sigma_i) = f(u_i, u_i)\dot{K}^2$, where u_i denotes a non-zero element of the 1-dimensional fixed space of the symmetry σ_i (for $i = 1, \ldots, r$). As is well-known two symmetries have the same spinor norm if and only if they are conjugated. For the theory of quadratic spaces we refer to O'Meara [18].

We now generalize the notion of a spinor norm to the non-regular case. Let (V, f) be a quadratic space with $radV \neq \{0\}$. Then the quotient space $V' = V/radV$ inherits a regular metric from V by $f'(u + radV, v + radV) := f(u, v)$ for $u, v \in V$. In other words, (V', f') is a regular quadratic space.

An orthogonal transformation $\alpha \in O_3^+(V, f)$ induces an orthogonal transformation $\alpha' \in O_3^+(V', f')$ by $\alpha'(v + radV) := \alpha(v) + radV$. The mapping $\varphi : \alpha \rightarrowtail \alpha'$ is a group homomorphism from $O_3^+(V, f)$ into $O_3^+(V', f')$. If η denotes the spinor norm on the regular quadratic space $O_3^+(V', f')$ then the product $\eta^* := \eta \circ \varphi$ is a homomorphism from $O_3^+(V, f)$ into $\dot{K}/\dot{K}^2$, which we call the *spinor norm* on $O_3^+(V, f)$.

From a geometric point of view, quadratic spaces and the associated orthogonal groups $O_3^+(V, f)$ correspond to projective spaces with a Cayley–Klein metric (see H. Struve and R. Struve [26]). The isotropic planes, which we will construct, correspond to subplanes of projective planes with a Cayley–Klein metric.

We call a metric plane *complete* if the group G of motions can be represented as a full group $O_3^+(K, f)$ (see Bachmann [3, § 18,3]). The complete Hilbert planes are the Euclidean, hyperbolic and non-Legendrean Hilbert planes. A subplane of a metric plane $\mathcal{E}$ is called *locally complete* (*lokalvollständig*, see [3, p. 346]), if the subplane contains with a point of $\mathcal{E}$ all lines of $\mathcal{E}$, which are incident with that point.

6.1. Galilean planes

Ordered Galilean planes with free mobility are introduced in Sect. 4. They are isotropic planes of **Type I** since in the extended Galilean plane every quadrilateral with three right angles is a rectangle. They satisfy the Euclidean parallel axiom (see Bachmann [4, § 7]).

6.2. Locally Galilean planes

These planes can be constructed over any non-Archimedean ordered field K. Let I be the (maximal) ideal of infinitesimal elements of K. The points of the plane are those points of the associated Galilean coordinate plane $\mathcal{E}$ over K, whose first coordinate is infinitesimal and whose second coordinate is an arbitrary element of K. The subplane of $\mathcal{E}$ with this set of points and with the set of lines of $\mathcal{E}$ which are incident with at least one of these points, is an isotropic plane of **Type I**, which does *not* satisfy the Euclidean parallel postulate.

We note, that the associated group (G, S, P) of absolute isotropic geometry can be represented as a group of certain (3×3)-matrices, whose entries are elements of subgroups A, B and C of the additive group of K (see [24, Sect. 5]), namely $B = (I, +)$ and $A = C = (K, +)$.

We call these planes *locally Galilean*, since they are locally complete subplanes of a Galilean plane and in any neighborhood of a point[12] the Galilean geometry holds. Every Galilean plane is locally Galilean.

6.3. Locally co-Minkowskian planes

Let $G = O_3^+(K, f)$ be the special orthogonal group over a field K (of characteristic $\neq 2$) and a symmetric bilinear form f of rank 2 and index 1. Then there exist isotropic vectors, which are not elements of the 1-dimensional radical, and f can be represented in the *normal form* $f((u, v, w), (u', v', w')) = vv' + k \cdot uu'$ for all $(u, v, w), (u', v', w') \in K^3$ and an *orthogonality constant* k with $-k \in \dot{K}^2$ (see Wolff [34] and Bachmann [3]).

Theorem 6.1. *Let V be a 3-dimensional vector space over an ordered Euclidean field K, which is endowed with a symmetric bilinear form f of rank 2 and index 1. Let k be the associated orthogonality constant and $-k \in \dot{K}^2$. Then the involutions of $G = O_3^+(V, f)$ can be divided into three classes with the following properties:*

1. *The involutions of two classes I_1^- and I_2^- have a negative spinor norm and the involutions of one class I_3^+ have a positive norm.*
2. *(G, S', P') with $S' = I_1^-$ and $P' = I_2^- \cup I_3^+$ is a co-Minkowkian Cayley–Klein group and the dual Cayley–Klein group is a Minkowskian group.*
3. *(G, S^*, P^*) and (G, S^*, P^{**}) with $S^* = I_1^-$ and $P^* = I_2^-$ and $P^{**} = I_3^+$ are isotropic planes of **Type II**, which are isomorphic.*

Proof. Let K, k, V, f satisfy the assumptions of the theorem. If S denotes the set of reflections in 1-dimensional subspaces of (V, f), and if $G = O_3^+(V, f)$ and $P = I(S^2)$, then (G, S, P) is a Minkowskian group (see Wolff [34]). The associated Minkowskian plane is orderable (since K is orderable; see H. Struve and R. Struve [27]), and since $k \notin \dot{K}^2$ no right angle has a bisector (cp. [24, Sect. 2]) and the Exchange Theorem can be applied (see [4, § 14] and [24, Satz 2]).

[12]The neighborhood is defined with respect to the order topology.

According to our assumptions $-k \in \dot{K}^2$. Hence $-k > 0$ and $k < 0$ and $k \notin \dot{K}^2$. The spinor norm of an element of P is the element $k \cdot \dot{K}^2$ of $\dot{K}/\dot{K}^2$ and hence negative. If $I_1^- := P$ and if I_2^- and I_3^+ are the sets of elements of S with a negative (respectively positive) spinor norm, then I_1^-, I_2^-, I_3^+ satisfiy statement (1).

Statement (2) holds since $(G, I_2^- \cup I_3^+, I_1^-)$ is a Minkowskian group, which implies that the dual structure $(G, I_1^-, I_2^- \cup I_3^+)$ is a co-Minkowskian Cayley–Klein group (see [29]).

For a proof of (3) we note that, according to [24, Satz 3], (G, S', P') with $S' = I_1^- \cup I_2^-$ and $P' = I_3^+$ is a Hjelmslev group, which satisfies the conditions of the Exchange Theorem [4, § 14], that is, I_1^- and I_2^- are invariant subsets of G which contain with three copunctual lines the fourth reflection line, but no pair of orthogonal lines, and for every pair (α, β) with $\alpha \in I_1^-$ and $\beta \in I_2^-$ there exists a unique element $\gamma \in I_3^-$ with $\alpha, \beta | \gamma$.

Hence $(G, I_1^- \cup I_3^+, I_2^-)$ is also a Hjelmslev group and $(G, S^*, P^*)$and(G, S^*, P^{**}) with $S^* = I_1^-$ and $P^* = I_2^-$ and $P^{**} = I_3^+$ are Cayley–Klein groups. They are isotropic planes (of **Type II**) since $(G, I_2^- \cup I_3^+, I_1^-)$ is an orderable Minkowskian group with free mobility, which satisfies the Euclidean parallel postulate. They are isomorphic since the extended isotropic planes are isomorphic (by [24, p. 405]). □

The isotropic planes of the theorem are locally complete subplanes of a co-Minkowkian plane. We call isotropic planes with this property *locally co-Minkowkian*. Their group G of motions is the full orthogonal group $O_3^+(V, f)$, that is, the isotropic planes are complete.

The planes satisfy the *hyperbolic parallel axiom* of Hilbert's *Neue Begründung der Bolyai-Lobatschefskyschen Geometrie* [14] which we stated explicitly in Section 3.1.3.

Models of isotropic planes, which are locally co-Minkowskian but which are not complete and do not satisfy the hyperbolic parallel axiom, can be constructed over any non-Archimedean ordered Euclidean field K. Let R be the ring of finitely bounded elements of K and let I be the maximal ideal of infinitesimal elements of R. The subplane of the co-Minkowskian plane over K, whose points have infinitesimal coordinates, is an isotropic plane (of **Type II**) with these properties.

6.4. Locally co-Euclidean planes

Let $G = O_3^+(K, f)$ be the special orthogonal group over a field K (of characteristic $\neq 2$) and a symmetric bilinear form f of rank 2 and index 0. Then every isotropic vector is an element of the 1-dimensional radical, and f can be represented in the *normal form* $f((u, v, w), (u', v', w')) = vv' + k \cdot uu'$ for all $(u, v, w), (u', v', w') \in K^3$ and an *orthogonality constant* k with $-k \notin \dot{K}^2$ (see Bachmann [4, § 7.4] and [3, § 10,1]). The field K is called *'k-Pythagorean'* if

$1 + kx^2 \in \dot{K}^2$ for all $x \in K$. For the construction of fields of this type we refer to Gröger and Sörensen [10, Sect. 3].[13]

Theorem 6.2. *Let V be a 3-dimensional vector space over an ordered field K, which is endowed with a symmetric bilinear form f of rank 2 and index 0. Let k be the associated orthogonality constant and $k < 0$ and $-k \notin \dot{K}^2$. If K is k-Pythagorean then the involutions of $G = O_3^+(V, f)$ can be divided into three classes with the following properties:*

1. *The involutions of two classes I_1^- and I_2^- have a negative spinor norm and the involutions of one class I_3^+ have a positive norm.*
2. *(G, S', P') with $S' = I_1^-$ and $P' = I_2^- \cup I_3^+$ is a co-Euclidean Cayley–Klein group and the dual Cayley–Klein group is a Euclidean group.*
3. *(G, S^*, P^*) and (G, S^*, P^{**}) with $S^* = I_1^-$ and $P^* = I_2^-$ and $P^{**} = I_3^+$ are isotropic planes of **Type III**, which are isomorphic.*

Proof. Let K, k, V, f satisfy the assumptions of the theorem. If S denotes the set of reflections in 1-dimensional subspaces of (V, f), and if $G = O_3^+(V, f)$ and $P = I(S^2)$, then (G, S, P) is a Euclidean Bachmann group (see [3, § 9,1; Theorem 4]). The associated Euclidean plane is orderable, since K is orderable. Since $k \notin \dot{K}^2$ no right angle has a bisector (cp [3, p. 216]) and the Exchange Theorem can be applied (see [4, § 14] and [24, Satz 2]).

The spinor norm of an element of P is the element $k \cdot \dot{K}^2$ of $\dot{K}/\dot{K}^2$ and hence negative. If $I_1^- := P$ and if I_2^- and I_3^+ are the sets of elements of S with a negative (respectively positive) spinor norm, then I_1^-, I_2^-, I_3^+ satisfiy statement (1).

Statement (2) holds since $(G, I_2^- \cup I_3^+, I_1^-)$ is a Euclidean Bachmann group, which implies that the dual structure $(G, I_1^-, I_2^- \cup I_3^+)$ is a co-Euclidean Cayley–Klein group (see [29]).

For a proof of (3) we note that, according to [24, Satz 3], (G, S', P') with $S' = I_1^- \cup I_2^-$ and $P' = I_3^+$ is a Hjelmslev group, which satisfies the conditions of the Exchange Theorem [4, § 14], that is, I_1^- and I_2^- are invariant subsets of G which contain with three copunctual lines the fourth reflection line, but no pair of orthogonal lines, and for every pair (α, β) with $\alpha \in I_1^-$ and $\beta \in I_2^-$ there exists a unique element $\gamma \in I_3^-$ with $\alpha, \beta | \gamma$.

Hence $(G, I_1^- \cup I_3^+, I_2^-)$ is also a Hjelmslev group and $(G, S^*, P^*)$and(G, S^*, P^{**}) with $S^* = I_1^-$ and $P^* = I_2^-$ and $P^{**} = I_3^+$ are Cayley–Klein groups. They are isotropic planes (of **Type III**) since $(G, I_2^- \cup I_3^+, I_1^-)$ is a Euclidean group with free mobility, which is orderable and satisfies the Euclidean parallel postulate. They are isomorphic since the extended isotropic planes are isomorphic (by [24, p. 405]). □

An isotropic plane of Theorem 6.2, (3) is a subplane of a co-Euclidean plane and contains with a given point all lines of the co-Euclidean plane which are

[13] In [10] the field K and the metric constant k are denoted by M respectively κ.

incident with this point (the co-Euclidean pencil of lines). We call isotropic planes with this property *locally co-Euclidean.* Their group G of motions is the full orthogonal group $O_3^+(V, f)$, that is, the isotropic planes are complete.

Co-Euclidean planes are locally co-Euclidean. However, they are not isotropic planes since their rows of points are cyclically ordered.

6.5. Isotropic planes over skew fields

Let $\mathcal{P}$ be an arguesian projective plane, which satisfies the Fano axiom. Let g and h be two 'fundamental lines' with the common point F. The pair of lines is a degenerate conic $\kappa = \{g, h\}$. A harmonic (and hence involutory) homology, which leaves κ invariant, has either an axis, which is incident with F, or the center of the homology is F. The set of harmonic homologies, which leave κ invariant, can thus be divided into two classes $\mathcal{H}$ and $\mathcal{H}^*$, depending on whether the axis is incident with F or the center is the point F.

If $\mathcal{P}$ is an ordered arguesian Fanoian projective plane then a separation relation is defined on the pencil of lines. This relation induces an equivalence relation $\sim$ on the set of lines through F, which are distinct from g and h: Two lines are called equivalent if they are not separated by g, h. There are two equivalence classes. Similarly, the set of points of $\mathcal{P}$, which are not incident with g or h, can be divided into two disjoint classes $\mathcal{W}$ and $\mathcal{W}'$ with the property, that the joining lines of F with points of the same class are equivalent with respect to the relation $\sim$.

$\mathcal{P}$ can be represented as a projective coordinate plane over an ordered skew field (cp. Beutelspacher and Rosenbaum [6]). The points and lines are the 1-dimensional respectively 2-dimensional subspaces of a 3-dimensional left vector space $V = K^3$ over a skew field K of characteristic $\neq 2$. Using homogenous coordinates the points can be represented by triples (x, y, z) with $x, y, z \in K$ and $(x, y, z) \neq (0, 0, 0)$. The triples (x, y, z) and $(\lambda x, \lambda y, \lambda z)$ with $\lambda \neq 0$ represent the same point. The homogenous coordinates of those points, which are incident with a given line, are the solutions of a homogenous equation with coefficients in K. Thus we can represent the lines as triples $[u, v, w]$ with $u, v, w \in K$ and $[u, v, w] \neq [0, 0, 0]$. The triples $[u, v, w]$ and $[u\lambda, v\lambda, w\lambda]$ with $\lambda \neq 0$ represent the same line. A point (x, y, z) is incident with a line $[u, v, w]$ if and only if $xu + yv + zw = 0$.

Let $g = [1, 0, 0]$ and $h = [0, 1, 0]$ be the two 'fundamental' lines with the common point $F = (0, 0, 1)$. The 'isotropic' lines of $\mathcal{P}$ through F are $[1, c, 0]$ with $c \neq 0$. The points of $\mathcal{P}$ which are not incident with one of the fundamental lines g and h are $(c, -1, d)$ with $c, d \in K$ and $c \neq 0$. The set of points $\{(c, -1, d) : c, d \in K; c \neq 0\}$ can be divided into two classes $\mathcal{W}^<$ and $\mathcal{W}^>$ depending on whether $c < 0$ or $c > 0$. We call these classes *angular spaces.* In an affine interpretation the points of an angular space are the points of a 'strip', i.e., the points 'between' two parallel lines g and h, or the points of a halfplane.

Now let K be a Euclidean ordered skew field of characteristic $\neq 2$. Then $\dot{K}^2$ is the set of positive elements of K and $\dot{K} = \dot{K}^2 \cup -\dot{K}^2$. The center L of K is a field of characteristic 0. The restriction of the order relation of K to L is a Euclidean order on L.

Let $S = \mathcal{H}^*$ and let P be the set of homologies of $\mathcal{H}$ which satisfy two conditions: (a) the center is a point of the angular space $\mathcal{W}^<$ and hence of the form $(c, -1, d)$ with $c, d \in K$ and $c < 0$, and (b) the coordinate c is an element of L. Let G be the group of collineations of $\mathcal{P}$ which is generated by $S \cup P$. If we represent the elements of G as linear mappings of $V = K^3$ then G is the group of the following matrices:

$$M(c,a,b)=\begin{pmatrix} 0 & c^{-1} & a \\ c & 0 & b \\ 0 & 0 & 1 \end{pmatrix};\ N(c,a,b)=\begin{pmatrix} c & 0 & a \\ 0 & c^{-1} & b \\ 0 & 0 & 1 \end{pmatrix}\ (c \in \dot{L} \text{ and } a, b \in K)$$

and $S=\{N(-1,a,b) : a,b \in K\}$; $P=\{M(c,-c^{-1}d,d) : c \in L; c < 0; d \in K\}$.

The matrix representation allows to verify that (G, S, P) is an isotropic plane. If K is a field, then $K = L$ and (G, S', P') with $P' = S$ and $S'=\{M(c,-c^{-1}d,d)$: $c, d \in K; c \neq 0\}$ is a Minkowskian group (see [4, § 14.8]).

7. A comparison between Hilbert planes and isotropic planes

7.1. The axiomatic perspective

Plane Euclidean geometry and plane Galilean geometry have a common axiomatic basis. In the reflection-geometric approach they both can be characterized by Cayley–Klein groups, which satisfy the Euclidean parallel axiom **Pax** and one additional axiom, namely axiom **B1** (the existence of a joining line) respectively **Pax*** (the dual parallel axiom; see [29, Sects. 4.2, 7.2]).

Hilbert planes and isotropic planes inherit the common axiomatic basis of Euclidean and Galilean geometry. They are both Cayley–Klein planes which satisfy the following axiom:

- **E.** For a, B with $a \nmid B$ there is at most one point on a which has no joining line with B.[14]

The order structure of Hilbert planes and isotropic planes can both be introduced by Definition 5.8. In Hilbert planes the set of trajectories coincides with the set of lines and thus Definition 5.8 coincides with the definition of an order structure for Bachmann planes in [28, Definition 3.9].

Theorem 7.1. *Hilbert planes and isotropic planes are exactly the Cayley–Klein planes which satisfy axiom* **E** and the axioms of free mobility **B6** and **B7**, and which can be endowed with an order structure according to Definition 5.8.

[14]The axiom was introduced in R. Struve [30] for a characterization of the part of Euclidean geometry which satisfies the principle of duality.

Proof. Obviously, Hilbert planes and isotropic planes satisfy the axioms of the theorem. Now let (G, S, P) be a Cayley–Klein group with **E**, **B6** and **B7**, which can be endowed with an order structure according to Definition 5.8.

In a Cayley–Klein group with axiom **E** either any two points have a joining line or for every pair (a, B) with $a \nmid B$ there is exactly one point on a which has no joining line with B, that is, **B1** or **Pax*** holds (see [30, Theorem 3.2]; the proof of Theorem 3.2 in [30] is based on an axiom system for Cayley–Klein groups which satisfy **E** and the dual axiom **E***, but **E*** is not used in the proof of this theorem). Hence (G, S, P) is an ordered Bachmann group with free mobility (see [29, Theorem 4.1]) or a group of absolute isotropic geometry. This proves the theorem. □

As a major difference between Hilbert planes and isotropic planes, we note that in both cases there are no polar points, but isotropic planes have also no orthogonal lines. However, extended isotropic planes (see Sect. 5.1) do have orthogonal lines and they satisfy all axioms of a Hjelmslev group. The theory of Hjelmslev groups (see [4]) can hence be applied both to Hilbert planes and to isotropic planes. As an example we note that in both cases one of the hypotheses of Saccheri holds, that is, the hypothesis of the acute, obtuse or right angle (see Theorems 3.1 and 5.17).

7.2. The model perspective

From a model perspective, Hilbert planes and isotropic planes fit into the framework of Cayley–Klein geometries.

The parameters m and n in Table 1 and their values 0 or 1 or 2 can be interpreted in the following way: If $n = 0$ then the 'elliptic parallel axiom' holds (*Any two lines have a common point*). If $n = 1$ then the Euclidean parallel axiom **Pax** holds and if $n = 2$ then Hilbert's hyperbolic parallel axiom is satisfied (see Sect. 6.3).[15] The parameter m has the value 0 or 1 or 2 according as the corresponding dual axiom holds.

Models of Hilbert planes can be constructed as subplanes of the geometries of the first row of Table 1 (see Sect. 3). Models of isotropic planes can be constructed as subplanes of the geometries of the second row of Table 1 (see Sect. 6).

For a classification of the models of Hilbert planes and isotropic planes we generalize the definition of Sect. 6: A subplane of an elliptic, Euclidean, hyperbolic or any other Cayley–Klein geometry $\mathcal{E}$ of Table 1 is called *locally elliptic*, *locally Euclidean*, *locally hyperbolic*, and so on, if it contains with a point A all lines of $\mathcal{E}$ which are incident with A. The intended geometric interpretation is, that in the neighborhood of any point of these subplanes the associated Cayley–Klein geometry holds.

[15] In other words, if A is a point, which is not incident with a line b then $n = 0$ or $n = 1$ or $n = 2$ according as there are 0, 1 or at least 2 lines through A which are 'parallel' to b (i.e., which have no common point with b).

TABLE 1 Plane ordered Cayley–Klein geometries (see [29])

	$n = 0$	$n = 1$	$n = 2$
$m = 0$	Elliptic planes	Euclidean planes	Hyperbolic planes
$m = 1$	Co-Euclidean planes	Galilean planes	Co-Minkowskian planes
$m = 2$	Cohyperbolic planes	Minkowskian planes	Doubly-hyperbolic planes

The Hilbert planes, which were constructed in Sect. 3.1, are locally Euclidean (the Euclidean and semi-Euclidean Hilbert planes) or locally hyperbolic (the hyperbolic and semi-hyperbolic Hilbert planes) or locally elliptic (the non-Legendrean Hilbert planes).[16] The Euclidean, hyperbolic and non-Legendrean Hilbert planes are complete. There are no elliptic Hilbert planes since their rows of points are cyclically ordered.

Correspondingly, the isotropic planes, which were constructed in Sect. 6, are locally Galilean or locally co-Minkowskian or locally co-Euclidean and each of these classes contain isotropic planes which are complete (as is shown there). There are no co-Euclidean isotropic planes since their rows of points are cyclically ordered.

7.3. The algebraic perspective

From an algebraic point of view the constructions of Hilbert planes and isotropic planes in Sects. 3.1 and 6 are analogical, namely based on an orthogonal group or on a coordinate plane over a non-Archimedean ordered field K, where the coordinates of points are restricted to the subset of infinitesimal or finitely bounded elements of K.

However, a remarkable difference exists: Every Hilbert plane has a *commutative* field of characteristic $\neq 2$ as coordinatizing structure,[17] whereas isotropic planes can be constructed, in addition, over skew fields (see Sect. 6.5).

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

[16] A non-Legendrean Hilbert plane is locally elliptic, since it is a locally complete subplane of the associated elliptic ideal plane (see Bachmann [3, § 18,3]).

[17] This holds also for plane absolute geometry [3] and for Hjelmslev *Allgemeine Kongruenzlehre* [16].

References

[1] Bachmann, F.: *Zur Parallelenfrage.* Abh. Math. Sem. Univ. Hamburg **27**, 173–192 (1964)

[2] Bachmann, F.: *Modelle der ebenen absoluten Geometrie.* J.-Ber. dtsch. Math.-Verein. **66**, 152–170 (1964)

[3] Bachmann, F.: Aufbau der Geometrie aus dem Spiegelungsbegriff, 2nd edn. Springer, Heidelberg (1973)

[4] Bachmann, F.: Ebene Spiegelungsgeometrie. Eine Vorlesung über Hjelmslev-Gruppen. BI-Wissenschaftsverlag, Mannheim (1989)

[5] Behnke, H., Bachmann, F., et al.: Fundamentals of Mathematics, vol. II. Geometry. MIT Press, London (1974)

[6] Beutelspacher, A., Rosenbaum, U.: Projektive Geometrie, 2nd edn. Springer, Wiesbaden (2004)

[7] Dehn, M.: *Die Legendreschen Sätze über die Winkelsumme im Dreieck.* Math. Ann. **53**, 404–439 (1900)

[8] Dress, A.: *Trägheitsstrukturen quadratischer Formen.* Math. Ann. **157**, 326–331 (1964)

[9] Giering, O.: Vorlesungen über höhere Geometrie. Vieweg, Braunschweig (1982)

[10] Gröger, D., Sörensen, K.: *Two remarks on ordering in absolute planes.* J. Geom. **111**, 6 (2020)

[11] Hartshorne, R.: Geometry: Euclid and Beyond. Springer, Heidelberg (2000)

[12] Hessenberg, G., Diller, J.: Grundlagen der Geometrie, 2nd edn. Walter de Gruyter, Berlin (1967)

[13] Hilbert, D.: *Grundlagen der Geometrie.* Leipzig, Teubner (1899)—translated by L. Unger, Open Court, La Salle, Ill., under the title: *Foundations of Geometry.* (1971)

[14] Hilbert, D.: *Neue Begründung der Bolyai-Lobatschefskyschen Geometrie.* Math. Ann. **57**, 137–150 (1903)

[15] Hjelmslev, J.: *Neue Begründung der ebenen Geometrie.* Math. Ann. **64**, 449–474 (1907)

[16] Hjelmslev, J.: *Einleitung in die allgemeine Kongruenzlehre.* Danske Vid. Selsk., mat-fys. Medd. **8**, Nr. 11 (1929); **10**, Nr. 1 (1929); **19**, Nr. 12 (1942); **22**, Nr. 6, Nr. 13 (1945); **25**, Nr.10 (1949)

[17] Hughes, D.R., Piper, F.C.: Projective Planes. Springer, Heidelberg (1973)

[18] O'Maera, O.T.: Introduction to Quadratic Forms. Springer, New York (1963)

[19] Pambuccian, V.: Axiomatizations of hyperbolic and absolute geometries. In: Prékopa, A., Molnár, E. (eds.) Non-Euclidean Geometries: Janos Bolyai Memorial Volume, pp. 119–153. Springer, New York (2006)

[20] Pejas, W.: *Die Modelle des Hilbertschen Axiomensystems der absoluten Geometrie.* Math. Ann. **14**, 212–235 (1961)

[21] Prieß-Crampe, S.: Angeordnete Strukturen: Gruppen, Körper, projektive Ebenen. Springer, Berlin (1983)

[22] Sachs, H.: Ebene isotrope Geometrie. Vieweg, Braunschweig (1987)

[23] Strubecker, K.: *Geometrie in einer isotropen Ebene I–III.* MNU **15**, 297–306, 343–351, 385–394 (1962).

[24] Struve, H., Struve, R.: *Hjelmslevgruppen, in denen sich die Punkte gegen Geraden austauschen lassen*. Geom. Dedicata **13**, 399–417 (1983)

[25] Struve, H.: *Ein spiegelungsgeometrischer Aufbau der Galileischen Geometrie*. Beiträge zur Algebra und Geometrie **17**, 197–211 (1984)

[26] Struve, H., Struve, R.: *Projective spaces with Cayley–Klein metrics*. J. Geom. **81**, 155–167 (2004)

[27] Struve, H., Struve, R.: *Ordered groups and ordered geometries*. J. Geom. **105**, 419–447 (2014)

[28] Struve, R.: *Ordered metric geometry*. J. Geom. **106**, 551–570 (2015)

[29] Struve, R.: *An axiomatic foundation of Cayley–Klein geometries*. J. Geom. **107**, 225–248 (2016)

[30] Struve, R.: *The principle of duality in Euclidean and in absolute geometry*. J. Geom. **107**, 707–717 (2016)

[31] Struve, R., Struve, H.: *The Thomsen–Bachmann correspondence in metric geometry I*. J. Geom. **110**, 9 (2019)

[32] Struve, R., Struve, H.: *The Thomsen–Bachmann correspondence in metric geometry II*. J. Geom. **110**, 14 (2019)

[33] Struve, R.: *Cyclic order: a geometric analysis*. Beitr. Algebra Geom. **61**, 647–669 (2020)

[34] Wolff, H.: *Minkowskische und absolute Geometrie I, II*. Math. Ann. **171**, 144–193 (1967)

[35] Yaglom, I.M.: A Simple Non-Euclidean Geometry and Its Physical Basis. Springer, Heidelberg (1979)

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