J. Geom. (2020) 111:41

0047-2468/20/030001-16
published online September 17, 2020
https://doi.org/10.1007/s00022-020-00553-4

Journal of Geometry

A para-Kähler structure in the space of oriented geodesics in a real space form

Nikos Georgiou

Abstract. In this article, we construct a new para-Kähler structure $(\mathcal{G}, \mathcal{J}, \Omega)$ on the space of oriented geodesics of a n-dimensional (non-flat) real space form. We first show that the para-Kähler metric $\mathcal{G}$ is scalar flat and when $n = 3$, it is locally conformally flat. Furthermore, we prove that the space of oriented geodesics of hyperbolic n-space equipped with the constructed metric $\mathcal{G}$ is minimally isometrically embedded in the tangent bundle of hyperbolic n-space. We then study submanifold theory, and show that $\mathcal{G}$-geodesics correspond to minimal ruled surfaces in the real space form. Lagrangian submanifolds (with respect to the symplectic structure Ω) play an important role in the geometry of the space of oriented geodesics as they come from the Gauss map of hypersurfaces in the corresponding space form. We demonstrate that the Gauss map of a non-flat hypersurface of constant Gauss curvature is a minimal Lagrangian submanifold. Finally, we show that a Hamiltonian minimal submanifold is locally the Gauss map of a hypersurface Σ, which is a critical point of the functional $\mathcal{F}(\Sigma) = \int_\Sigma \sqrt{|K|}\, dV$, K denoting the Gaussian curvature of Σ.

Mathematics Subject Classification. Primary 53A35; Secondary 53C42.

Keywords. Para-Kähler structure, Real space form, Lagrangian submanifold.

1. Introduction

The geometry of the space $\mathbb{L}(\mathbb{S}^{n+1}(c))$ of oriented geodesics in a real space form $(\mathbb{S}^{n+1}(c), g)$, of constant sectional curvature c, has been the focus of a great deal of interest for the past two decades. In the celebrated article [7], Guilfoyle and Klingenberg constructed a Kähler structure in the space $\mathbb{L}(\mathbb{S}^3(0))$ of oriented lines in the Euclidean 3-space $\mathbb{S}^3(0) = \mathbb{R}^3$ and showed that the Kähler metric is of neutral signature. Additionally, it is invariant under the action of the Euclidean isometry group. A similar construction for hyperbolic 3-space $\mathbb{S}^3(-1)$ was established by Georgiou and Guilfoyle in [4]. Then, in [1], Anciaux extended this geometric construction to all non-flat real space forms by showing that $\mathbb{L}(\mathbb{S}^{n+1}(c))$ always admits a Kähler or a para-Kähler structure, where the metric is Einstein and invariant under the isometry group of $\mathbb{S}^{n+1}(c)$. In the same work, for $n = 2$, Anciaux proved that $\mathbb{L}(\mathbb{S}^{n+1}(c))$ admits another Kähler or para-Kähler structure, such that its metric $\overline{G}$, is of neutral signature, locally conformally flat and is invariant under the isometry group of $\mathbb{S}^{n+1}(c)$. This invariance, allows one to study problems in the base manifold $\mathbb{S}^{n+1}(c)$ by studying its space of oriented geodesics.

For example, the set of all oriented geodesics orthogonal to a hypersurface in $\mathbb{S}^{n+1}(c)$ corresponds to a Lagrangian submanifold in $\mathbb{L}(\mathbb{S}^{n+1}(c))$, with respect to the canonical symplectic structure Ω (see [1]). In particular, $\overline{G}$-flat Lagrangian surfaces in $\mathbb{L}(\mathbb{S}^3(c))$ are the oriented geodesics normal to a Weingarten surfaces in $\mathbb{S}^3(c)$, i.e. its principal curvatures are functionally related.

Consider a (para-) Kähler structure(M, J, ω), where J, ω are respectively the (para-) complex structure and the symplectic structure. If $f : \Sigma \to M$ is a Lagrangian immersion and H is the mean curvature, the one-form $f^*(JH \rfloor \omega)$ is called the *Maslov 1-form*. In this article, the Maslov 1-form will be denoted by a_H.

If $\mathbb{L}(\mathbb{S}^{n+1}(c))$ is equipped with the (para-) Kähler–Einstein structure, the Maslov 1-form is closed for any Lagrangian submanifold. This is generally true for any (para-) Kähler–Einstein manifold. In fact, if the Kähler metric is Einstein, then the Ricci 2-form ρ is proportional to the symplectic structure ω, that is,

$$\rho = \lambda\omega.$$

For a Lagrangian submanifold Σ the following identity holds [3]:

$$da_H = \rho|_\Sigma,$$

which yields,

$$da_H = \rho|_\Sigma = \lambda\omega|_\Sigma = 0.$$

It is natural then to ask whether the converse is true, i.e. considering a (para-) Kähler manifold such that every Lagrangian submanifold has closed Maslov 1-form, can we conclude that the Kähler metric is Einstein?

In this article, we show that the converse is not true. In particular, we construct a para-Kähler (non-Einstein) structure $(\mathcal{G}, \mathcal{J}, \Omega)$ in $\mathbb{L}(\mathbb{S}^{n+1}(c))$ (for $c \neq 0$), where $\mathcal{G}, \mathcal{J}$ and Ω are respectively the metric, the paracomplex structure and the canonical symplectic structure such that all Lagrangian submanifolds have closed Maslov 1-form. We use the fact that $\mathbb{L}(\mathbb{S}^{n+1}(c))$ is identified with the Grassmannian of oriented 2-planes of $\mathbb{R}^{n+2}$ and thus, they are submanifolds of the set of all bivectors $\Lambda^2(\mathbb{R}^{n+2}) = \{x \wedge y | x, y \in \mathbb{R}^{n+2}\}$ in $\mathbb{R}^{n+2}$. In particular, a tangent vector at $x \wedge y \in \mathbb{L}(\mathbb{S}^{n+1}(c)) \subset \Lambda^2(\mathbb{R}^{n+2})$ can be written as $x \wedge X + y \wedge Y$, where X, Y are vectors in $\mathbb{R}^{n+2}$ normal to the oriented plane $x \wedge y$ (see [1] for more details).

We then show the following:

Theorem 1. *The metric $\mathcal{G}$ on $\mathbb{L}(\mathbb{S}^{n+1}(c))$, defined by*

$$\mathcal{G}(x \wedge X_1 + y \wedge Y_1, x \wedge X_2 + y \wedge Y_2) = g(X_1, Y_2) + g(X_2, Y_1),$$

where $x \wedge X_1 + y \wedge Y_1,\ x \wedge X_2 + y \wedge Y_2 \in T_{x\wedge y}\mathbb{L}(\mathbb{S}^{n+1}(c))$, is scalar flat and non-Einstein. Furthermore, $\mathcal{G}$ is locally conformally flat if and only if $n = 2$.

It is proven in [5] that $(\mathbb{L}(\mathbb{S}^3(0)), \overline{G})$ is minimally isometrically embedded in the tangent bundle $(T\mathbb{S}^3(0), \mathcal{G}_0)$, where $\mathcal{G}_0$ is a neutral, scalar flat and locally conformally flat metric. For the hyperbolic case, $\mathbb{H}^n = \mathbb{S}^n(-1)$, we have the following remark:

Remark 1. The isometric embedding $f : (\mathbb{L}(\mathbb{H}^{n+1}), \mathcal{G}) \longrightarrow (T\mathbb{H}^{n+1}, \mathcal{G}_0) : x \wedge y \mapsto (x, -y)$ is minimal.

The reason Remark 1 holds in any dimension, while in the Euclidean case it only holds for $n = 2$, is because $\mathbb{L}(\mathbb{H}^{n+1})$ admits invariant (para-) Kähler structures for any n. The space $\mathbb{L}(\mathbb{R}^3)$ of oriented lines in $\mathbb{R}^{n+1}$ admits an invariant (para-) Kähler structure only when $n = 3$ and 7 (see [9]). On the other hand, there is no similar result for the spherical cases since the spheres are not Hadamard and therefore the embedding f is not well-defined.

A curve in $\mathbb{L}(\mathbb{S}^3(c))$ corresponds to a 1-parameter family of oriented geodesics, i.e. it corresponds to a ruled surface in the real space form $\mathbb{S}^3(c)$. We then have:

Remark 2. A curve γ in $(\mathbb{L}(\mathbb{S}^3(c)), \mathcal{G})$ is a geodesic if and only if the corresponding ruled surface in $\mathbb{S}^3(c)$ is minimal.

Let $\phi : \Sigma \to \mathbb{S}^{n+1}(c)$ be an immersion of a hypersurface in $\mathbb{S}^{n+1}(c)$. The set of oriented geodesics normal to $\phi(\Sigma)$ is immersed by the mapping

$$\Phi : \Sigma \to \mathbb{L}(\mathbb{S}^{n+1}(c)) : x \mapsto \phi(x) \wedge N(x), \tag{1}$$

where N is the unit normal vector field along $\phi(\Sigma)$. The map Φ is called the *Gauss map* of the immersion ϕ. It is already known that the image $\Phi(\Sigma)$ of the Gauss map of a hypersurface in $\mathbb{S}^3(c)$ form a Lagrangian submanifold in $(\mathbb{L}(\mathbb{S}^3(c)), \Omega)$, where Ω is the canonical symplectic structure. The following remark describes all minimal Lagrangian submanifolds:

Remark 3. Every Lagrangian submanifold in $(\mathbb{L}(\mathbb{S}^{n+1}(c)), \mathcal{G}, \Omega)$ has closed Maslov 1-form. If Σ is a non-flat hypersurface of $\mathbb{S}^{n+1}(c)$ then it is of constant Gauss curvature if and only if the oriented geodesics normal to Σ form a minimal Lagrangian submanifold of $(\mathbb{L}(\mathbb{S}^{n+1}(c)), \mathcal{G}, \Omega)$.

A point on hypersurface is "non-flat" if the product of all principal curvatures is non-zero. Remark 3, shows that for $n = 2$, the metrics $\mathcal{G}$ and $\overline{G}$ are not isometric, since they have different minimal surfaces. In particular, the only $\overline{G}$-minimal Lagrangian surfaces in $\mathbb{L}(\mathbb{S}^3(c))$ are the set of geodesics normal to a given geodesic, which is flat (see [1,3]).

Remark 3 says that every Lagrangian submanifold has closed Maslov 1-form, but $\mathcal{G}$ is not Einstein, answering the question posed previously in the negative.

Let (M, ω) be a symplectic manifold and $f : \Sigma \to M$ a Lagrangian submanifold. A vector field X along Σ is said to be *Hamiltonian* if the one-form $f^*(X \rfloor \omega)$ is exact. A smooth variation $F : \Sigma \times [0, T) \to M$ of Σ with $F(x, 0) = f(x)$ is called a *Hamiltonian deformation* if $\frac{\partial F}{\partial t}|_{t=0}$ is a Hamiltonian vector field.

If a (para-) Kähler structure (J, g, ω) is given on M, then a normal variation F of the Lagrangian submanifold Σ is Hamiltonian if

$$\frac{\partial F}{\partial t}|_{t=0} = J\nabla u,$$

where J is the (para-) complex structure and ∇u is the gradient of the smooth function u defined on Σ. A *Hamiltonian minimal submanifold* is a Lagrangian submanifold that is also a critical point of the volume functional with respect to Hamiltonian variations. The first variation formula of the volume functional implies that a Hamiltonian minimal submanifold is characterised by the equation $divJH = 0$. Here H denotes the mean curvature vector of Σ and div is the divergence operator with respect to the induced metric (for more details, see [11] and [12]). In [6] and [10] it is proven that smooth one-parameter deformations of a submanifold in $\mathbb{S}^{n+1}(c))$ induce Hamiltonian deformations of the corresponding Gauss map in $(\mathbb{L}(\mathbb{S}^{n+1}(c)), \Omega)$. We then have the following:

Remark 4. Let $\phi : \Sigma \to \mathbb{S}^{n+1}(c)$ be a non-flat hypersurface in $(\mathbb{S}^{n+1}(c), g)$. Then the Gauss map $\Phi : \Sigma \to \mathbb{L}(\mathbb{S}^{n+1}(c))$ is a Hamiltonian minimal submanifold of $(\mathbb{L}(\mathbb{S}^{n+1}(c)), \mathcal{G}, \Omega)$ if and only if ϕ is a critical point of the functional

$$\mathcal{F}(\phi) = \int_\Sigma \sqrt{|K|}\, dV,$$

where K and dV denote, respectively, the Gaussian curvature of ϕ and the volume element of the induced metric $\phi^* g$.

The author would like to thank Brendan Guilfoyle for interesting comments and observations about the early version of this manuscript.

2. A canonical para Kähler structure

Let $\mathbb{S}^{n+1}(c)$ be a real space form of constant sectional curvature $c \in \{-1, 1\}$. That is, let $\mathbb{H}^{n+1} = \mathbb{S}^{n+1}(-1)$ is the hyperbolic $(n + 1)$-dimensional space defined by:

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+2} \,|\, \langle x, x\rangle_{-1} = 1,\, x_0 > 0\},$$

where $\langle x, x\rangle_{-1} := x_0^2 - x_1^2 - \cdots - x_{n+1}^2$. Moreover $\mathbb{S}^{n+1} = \mathbb{S}^{n+1}(1)$ is the $(n+1)$-dimensional sphere defined by:

$$\mathbb{S}^{n+1} = \{x \in \mathbb{R}^{n+2} \mid \langle x, x\rangle_1 = 1\},$$

where $\langle x, x\rangle_1 := x_0^2 + x_1^2 + \cdots + x_{n+1}^2$.

The space of oriented geodesics in $\mathbb{S}^{n+1}(c)$ will be identified with the

$$\mathbb{L}(\mathbb{S}^{n+1}(c)) = \{x \wedge y \in \Lambda^2(\mathbb{R}^{n+2}) \,|\, y \in T_x\mathbb{S}^{n+1}(c),\, \langle y, y\rangle_c = c\}.$$

Every tangent vector in $T_{x\wedge y}\mathbb{L}(\mathbb{S}^{n+1}(c))$ can be written as:

$$x \wedge X + y \wedge Y,$$

where $X, Y \in (x\wedge y)^{\perp}$ are in $\mathbb{R}^{n+2}$. $\mathbb{L}(\mathbb{S}^{n+1}(c))$ is equipped with the Riemannian metric $G_0 = \iota^* \langle\langle ., .\rangle\rangle_c$ where,

$$\iota : \mathbb{L}(\mathbb{H}^n) \hookrightarrow \Lambda^2(\mathbb{R}^{n+1}) : x \wedge y \mapsto x \wedge y,$$

and $\langle\langle ., .\rangle\rangle_c$ is the flat metric in $\Lambda^2(\mathbb{R}^{n+1})$:

$$\langle\langle x_1 \wedge y_1, x_2 \wedge y_2\rangle\rangle_c = \langle x_1, x_2\rangle_c < y_1, y_2\rangle_c - \langle x_1, y_2\rangle_c\langle x_2, y_1\rangle_c.$$

For the hyperbolic case ($c = -1$), fixing a point $p \in \mathbb{R}^{n+2}$, every oriented geodesic $\gamma = \gamma(t)$, with t being its arc-length, can be identified with the pair $(\gamma(t_0), \gamma'(t_0))$, where $\gamma(t_0)$ is the closest point of γ to p and, $\gamma'(t_0)$ is its velocity. When p is the origin, it is not hard to see that $\langle\gamma(t_0), \gamma'(t_0)\rangle_1 = 0$.

In this article, when we write the oriented geodesic γ as the oriented plane $x \wedge y$ we mean that $< x, y >_1 = 0$.

Proposition 1. *The following embedding is well defined:*

$$f : \mathbb{L}(\mathbb{H}^{n+1}) \longrightarrow T\mathbb{H}^{n+1} : x \wedge y \mapsto (x, -y). \tag{2}$$

Proof. Indeed, let $z \wedge w \in \mathbb{L}(\mathbb{H}^{n+1})$ be such that $z \wedge w = x \wedge y$, where $< x, y >_1 = < z, w >_1 = 0$. Then

$$z = x\cosh t + y \sinh t, \qquad w = x \sinh t + y\cosh t, \tag{3}$$

for some real t. Note that $\langle x, y\rangle_1 = 0$ and thus we have that $y_0 = 0$. The fact that $\langle y, y\rangle_{-1} = -1$ implies $\langle y, y\rangle_1 = 1$.

From $\langle z, w\rangle_1 = 0$, we then have,

$$(|x|_1^2 + |y|_1^2)\sinh t\cosh t + \langle x, y\rangle_1 (\cosh^2 t + \sinh^2 t) = 0,$$

which yields,

$$(|x|_1^2 + 1)\sinh t\cosh t = 0,$$

Thus, $t = 0$ and substituting this in (3), we finally get $(x, -y) = (z, -w)$, which means that $f(x \wedge y) = f(z \wedge w)$. □

We now use the embedding f to define a new geometric structure on $\mathbb{L}(\mathbb{H}^{n+1})$. To do this, consider the neutral metric $\mathcal{G}_0$ on $T\mathbb{H}^{n+1}$:

$$\mathcal{G}_0(\bar{X},\bar{Y}) = g(\Pi\bar{X}, K\bar{Y}) + g(K\bar{X}, \Pi\bar{Y}),$$

where $\bar{X} \simeq (\Pi\bar{X}, K\bar{X})$, $\bar{Y} \simeq (\Pi\bar{Y}, K\bar{Y})$ in $TT\mathbb{H}^{n+1} = T\mathbb{H}^{n+1} \oplus T\mathbb{H}^{n+1}$, and g is the metric $\langle .,. \rangle_{-1}$ induced by the inclusion map $i : \mathbb{H}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$. For more details about this metric, see [5] and [8].

Let $\mathcal{G}$ be the metric $\mathcal{G}_0$ induced by f on $\mathbb{L}(\mathbb{H}^{n+1})$, i.e. $\mathcal{G} = f^*\mathcal{G}_0$. It can be shown that

$$\mathcal{G}(x \wedge X_1 + y \wedge Y_1, x \wedge X_2 + y \wedge Y_2) = g(X_1, Y_2) + g(X_2, Y_1). \tag{4}$$

Proposition 2. *The metrics $\mathcal{G}$ and G_0 are projectively equivalent.*

Proof. Let $x, y, e_1, \ldots e_n$ be an orthonormal frame of $\mathbb{R}^{n+2}$, and define the vector $E_1, \ldots E_{2n}$ in $T_{x\wedge y}\mathbb{L}(\mathbb{H}^{n+1})$ by:

$$E_i = x \wedge e_i, \qquad E_{n+i} = y \wedge e_i,$$

where $i = 1, \ldots, n$. If ∇ is the Levi–Civita connection of G_0, one can show that $\nabla_{E_i} E_j = 0$.

An almost complex structure $\mathcal{J}_0$ in $\mathbb{L}(\mathbb{H}^{n+1})$ can be defined by

$$\mathcal{J}_0(x \wedge X + y \wedge Y) = -y \wedge X + x \wedge Y.$$

Then $\nabla_{E_i}\mathcal{J}_0 = \mathcal{J}_0\nabla_{E_i}$, which shows that $\mathcal{J}_0$ is ∇-parallel and therefore integrable. We also have that T is symmetric with respect to G_0, i.e.

$$G_0(\mathcal{J}_0\bar{X}, \bar{Y}) = G_0(\bar{X}, \mathcal{J}_0\bar{Y}),$$

for any $\bar{X}, \bar{Y} \in T_{x\wedge y}\mathbb{L}(\mathbb{H}^{n+1})$. Namely,

$$\begin{aligned} G_0(\mathcal{J}_0(x \wedge X_1 + y \wedge Y_1), x \wedge X_2 + y \wedge Y_2) &= G_0(-y \wedge X_1 + x \wedge Y_1, x \wedge X_2 + y \wedge Y_2) \\ &= g(X_1, Y_2) + g(X_2, Y_1), \end{aligned} \tag{5}$$

which implies

$$\begin{aligned} &G_0(\mathcal{J}_0(x \wedge X_1 + y \wedge Y_1), x \wedge X_2 + y \wedge Y_2) \\ &\quad = G_0(x \wedge X_1 + y \wedge Y_1, \mathcal{J}_0(x \wedge X_2 + y \wedge Y_2)). \end{aligned}$$

Consider the following Lemma:

Lemma 1 [1]. *Let (N, G) be a pseudo-Riemannian manifold with Levi–Civita connection D and T a symmetric, D-parallel $(1,1)$ tensor. Then the Levi–Civita connection of the pseudo-Riemannian metric $G' = G(.,T.)$ is D.*

From (5), we have

$$\mathcal{G} = G_0(.,\mathcal{J}_0.).$$

The proposition then follows. □

Considering the $(n+1)$-dimensional real space form $\mathbb{S}^{n+1}(c)$ and defining the almost (para-)complex structure $\mathcal{J}_0$ by

$$\mathcal{J}_0(x\wedge X+y\wedge Y)=cy\wedge X+x\wedge Y,$$

we now define the metric $\mathcal{G}$ by

$$\mathcal{G}=G_0(.,\mathcal{J}_0.),$$

which is given by (4). It is easily seen that, $\mathcal{G}$ is ∇ and $\mathcal{J}_0$ symmetric and therefore $\mathcal{G}$ and G_0 share the same Levi–Civita connection.

The following theorem explores the curvature of $\mathcal{G}$:

Theorem 1. *The metric $\mathcal{G}$ on $\mathbb{L}(\mathbb{S}^{n+1}(c))$, defined by*

$$\mathcal{G}(x\wedge X_1+y\wedge Y_1,x\wedge X_2+y\wedge Y_2)=g(X_1,Y_2)+g(X_2,Y_1),$$

where $x\wedge X_1+y\wedge Y_1$, $x\wedge X_2+y\wedge Y_2\in T_{x\wedge y}\mathbb{L}(\mathbb{S}^{n+1}(c))$, is scalar flat and non-Einstein. Furthermore, $\mathcal{G}$ is locally conformally flat if and only if $n=2$.

Proof. Consider the frame E_i used previously, where again $i=1,\ldots,n$, then $\mathcal{J}_0E_i=cE_{n+i}$ and $\mathcal{J}_0E_{n+i}=E_i$. Let $\mathcal{R}$ and $\mathcal{R}ic$ be the Riemann curvature and Ricci tensor respectively of $\mathcal{G}$. Since the metrics $\mathcal{G}$ and G_0 have the same Levi–Civita connection then $\mathcal{R}=R$, where R is the Riemann curvature tensor of G_0. Then,

$$\begin{aligned}\mathcal{G}(\mathcal{R}(.,.).,.)&=G_0(\mathcal{R}(.,.).,\mathcal{J}_0.)\\&=G_0(R(.,.).,\mathcal{J}_0.)\end{aligned}$$

For $i,j=1,\ldots,n$ we have,

$$\mathcal{G}_{i,n+j}=c\delta_{ij},\quad \mathcal{G}_{ij}=\mathcal{G}_{n+i,n+j}=0,$$

and therefore the inverse matrix has coefficients

$$\mathcal{G}^{i,n+j}=c\delta_{ij},\quad \mathcal{G}^{ij}=\mathcal{G}^{n+i,n+j}=0.$$

Using the fact that $G_0^{ij}=cG_0^{n+i,n+j}=\delta_{ij}$ and $G_0^{i,n+j}=0$, we then have

$$\begin{aligned}\mathcal{R}ic(X,Y)&=\sum_{i=1}^{n}\mathcal{G}^{i,n+i}\left(\mathcal{G}(\mathcal{R}(X,E_i)Y,E_{n+i})+\mathcal{G}(\mathcal{R}(X,E_{n+i})Y,E_i)\right)\\&=\sum_{i=1}^{n}\left(G_0(R(X,E_i)Y,\mathcal{J}_0E_{n+i})+G_0(R(X,E_{n+i})Y,\mathcal{J}_0E_i)\right)\\&=\sum_{i=1}^{n}\Big(\langle\langle R(X,E_i)Y,E_i\rangle\rangle_c+c\,\langle\langle R(X,E_{n+i})Y,E_{n+i}\rangle\rangle_c\Big)\\&=\sum_{i=1}^{n}\Big(G_0^{ii}\,\langle\langle R(X,E_i)Y,E_i\rangle\rangle_c+G_0^{n+i,n+i}\,\langle\langle R(X,E_{n+i})Y,E_{n+i}\rangle\rangle_c\Big)\\&=Ric(X,Y)\end{aligned}$$

Now, G_0 is an Einstein metric with scalar curvature $S = 2cn^2$ (for more details, see [1]), so

$$Ric = \frac{S}{2n} G_0 = cn\, G_0.$$

That means,

$$\begin{aligned}\mathcal{R}ic(X,Y) &= cn\, G_0(X,Y)\\ &= cn\, \langle\langle X,Y\rangle\rangle_c\,,\end{aligned}$$

and thus, $\mathcal{G}$ is non-Einstein.

If $\mathcal{S}$ denotes the scalar curvature of $\mathcal{G}$ then,

$$\begin{aligned}\mathcal{S} &= \sum_{a,b=1}^{2n} \mathcal{G}^{ab}\mathcal{R}ic(E_a,E_b)\\ &= 2\sum_{i=1}^{n}\mathcal{G}^{i,n+i}\mathcal{R}ic(E_i,E_{n+i})\\ &= 2c^2n\sum_{i}^{n}\langle\langle E_i,E_{n+i}\rangle\rangle_c\\ &= 0.\end{aligned}$$

We now proceed with the proof of the second part of the theorem. Since $\mathcal{G}$ is scalar flat, the Weyl tensor $\mathcal{W}$ is given by

$$\begin{aligned}\mathcal{W}(X,Y,Z,W) &= \mathcal{G}(\mathcal{R}(X,Y)Z,W) - \frac{1}{2(n-1)}\mathcal{R}ic\circ\mathcal{G}(X,Y,Z,W)\\ &= G(R(X,Y)Z,TW) - \frac{1}{2(n-1)}Ric\circ\mathcal{G}(X,Y,Z,W)\\ &= G(R(X,Y)Z,TW) - \frac{cn}{2n-2}G_0\circ\mathcal{G}(X,Y,Z,W).\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{W}(E_1,E_2,E_2,E_{n+1}) &= G(R(E_1,E_2)E_2,TE_{n+1}) - \frac{cn}{2n-2}G_0\circ\mathcal{G}(E_1,E_2,E_2,E_{n+1})\\ &= G(R(E_1,E_2)E_2,E_1) - \frac{cn}{2n-2}G_0(E_2,E_2)\mathcal{G}(E_1,E_{n+1})\\ &= 1-\frac{c^2n}{2n-2} = 1-\frac{n}{2n-2} = \frac{n-2}{2n-2},\end{aligned}$$

which is zero if and only if $n = 2$. Similarly, one can prove the same for the other coefficients of the Weyl tensor. $\square$

When $n = 2$, there is a complex structure on $\mathbb{L}(\mathbb{S}^3(c))$, defined as follows:

$$\mathcal{J}_0'(x\wedge X + y\wedge Y) = x\wedge J'X - y\wedge J'Y,$$

where J' is the complex structure in the plane $(x\wedge y)^\perp$ in $\mathbb{R}^4$. Then the metric $\overline{G} = G_0(.,\mathcal{J}_0\circ\mathcal{J}_0'.)$ on $\mathbb{L}(\mathbb{S}^3(c))$, is locally conformally flat, scalar flat and is invariant under the isometry group action in $\mathbb{L}(\mathbb{S}^3(c))$. The metric $\overline{G}$ has been studied by several authors (see for example, [1,2,4,7,9]).

Finally, for the hyperbolic case ($c=-1$), we show the following:

Remark 1. The isometric embedding $f:(\mathbb{L}(\mathbb{H}^{n+1}),\mathcal{G})\longrightarrow(T\mathbb{H}^{n+1},\mathcal{G}_0):x\wedge y\mapsto(x,-y)$ is minimal.

Proof. The derivative of f is given by:

$$df(x\wedge X+y\wedge Y)=(-Y,-X).$$

Note that $X=\nabla_Y y$ and if $\bar{D}$ denotes the Levi–Civita connection of G, we have

$$\begin{aligned}\bar{D}_{df(x\wedge X_1+y\wedge Y_1)}df(x\wedge X_1+y\wedge Y_1)&=\bar{D}_{(-Y_1,-X_1)}(-Y_2,-X_2)\\&=(D_{Y_1}Y_2,\ R(y,Y_1)Y_2+D_{Y_1}X_2)\\&=(D_{Y_1}Y_2,\ g(Y_1,Y_2)y+D_{Y_1}X_2)\\&=(D_{Y_1}Y_2,\ -g(X_1,X_2)y+D_{Y_1}X_2)\\&\quad+(0,(g(X_1,X_2)+g(Y_1,Y_2))y),\end{aligned}$$

which implies that the second fundamental form h_f is given by

$$h_f(x\wedge X_1+y\wedge Y_1,x\wedge X_2+y\wedge Y_2)=(0,(g(X_1,X_2)+g(Y_1,Y_2))y).$$

Recalling the basis $(E_1,\dots E_{2n})$ of $T_{x\wedge y}\mathbb{L}(\mathbb{S}^{n+1}(c))$, the mean curvature H_f of f is

$$H_f=\mathcal{G}^{mn}h_f(E_m,E_n),$$

so that

$$\begin{aligned}H_f&=\mathcal{G}^{i,n+i}h_f(E_i,E_{n+i})\\&=h_f(x\wedge e_i,y\wedge e_i)\\&=(0,(g(e_i,0)+g(0,e_i))y),\end{aligned}$$

which shows that f is minimal. □

Considering the almost para-complex structure $\mathcal{J}$ in $\mathbb{L}(\mathbb{S}^{n+1}(c))$:

$$\mathcal{J}(x\wedge X+y\wedge Y)=x\wedge X-y\wedge Y,$$

we then have:

(1) $\mathcal{J}$ is compatible with $\mathcal{G}$. Namely,

$$\begin{aligned}\mathcal{G}(\mathcal{J}(x\wedge X_1+y\wedge Y_1),\mathcal{J}(x\wedge X_2+y\wedge Y_2))&=\mathcal{G}(x\wedge X_1-y\wedge Y_1,x\wedge X_2-y\wedge Y_2)\\&=-g(X_1,Y_2)-g(X_2,Y_1)\\&=-\mathcal{G}(x\wedge X_1+y\wedge Y_1,x\wedge X_2+y\wedge Y_2).\end{aligned}$$

(2) $\mathcal{J}$ is integrable, i.e., $D\mathcal{J}=\mathcal{J}D$. In fact,

$$\mathcal{J}E_i=E_i,\qquad \mathcal{J}E_{n+i}=-E_{n+i},$$

and the claim follows from $D_{E_i}E_j=0$.

Define the symplectic 2-form Ω in $\mathbb{L}(\mathbb{S}^{n+1})$ by

$$\Omega = \mathcal{G}(\mathcal{J}.,.).$$

In particular,

$$\Omega(x \wedge X_1 + y \wedge Y_1, x \wedge X_2 + y \wedge Y_2) = g(X_1, Y_2) - g(X_2, Y_1).$$

Then the quadruple $(\mathbb{L}(\mathbb{S}^{n+1}(c)), \mathcal{G}, \mathcal{J}, \Omega)$ form a para-Kähler structure, so that the symplectic structure is the same as the symplectic structure defined by the (para-) Kähler structure $(\mathbb{L}(\mathbb{S}^{n+1}(c)), G_0, \mathcal{J}_0)$, since

$$\Omega = G_0(\mathcal{J}_0.,.).$$

The latter (para-) Kähler structure has been widely studied in [1], [4] and [7].

Every isometry $\phi : \mathbb{S}^{n+1}(c) \to \mathbb{S}^{n+1}(c)$, can be extended to a linear orthogonal transformation $\bar{\phi}$ in $\mathbb{R}^{n+2}$ restricted to $\mathbb{S}^{n+1}(c)$. This induces a mapping F in the space of oriented geodesics defined by

$$F(x \wedge y) = \phi(x) \wedge \bar{\phi}(y).$$

The derivative of F is

$$dF(x \wedge X + y \wedge Y) = \phi(x) \wedge d\bar{\phi}(X) + \bar{\phi}(y) \wedge d\phi(Y).$$

Using now the fact that $X, Y \in (x \wedge y)^{\perp}$ (see [1]), we have that $X \in T_x\mathbb{S}^{n+1}(c)$ and thus,

$$dF(x \wedge X + y \wedge Y) = \phi(x) \wedge d\phi(X) + \bar{\phi}(y) \wedge d\phi(Y).$$

We now have

$$\begin{aligned}
&\mathcal{G}(dF(x \wedge X_1 + y \wedge Y_1), dF(x \wedge X_2 + y \wedge Y_2)) \\
&\quad = \mathcal{G}(\phi(x) \wedge d\phi(X_1) + \bar{\phi}(y) \wedge d\phi(Y_1), \phi(x) \wedge d\phi(X_2) + \bar{\phi}(y) \wedge d\phi(Y_2)) \\
&\quad = g(d\phi(X_1), d\phi(Y_2)) + g(d\phi(X_2), d\phi(Y_1)) \\
&\quad = g(X_1, Y_2) + g(X_2, Y_1) \\
&\quad = \mathcal{G}(x \wedge X_1 + y \wedge Y_1, x \wedge X_2 + y \wedge Y_2),
\end{aligned}$$

which shows the following:

Proposition 3. *The metric $\mathcal{G}$ is invariant under the action of the isometry group of $(\mathbb{S}^{n+1}(c), g)$ in the space of oriented geodesics $\mathbb{L}(\mathbb{S}^{n+1}(c))$.*

3. Geodesics

We now study geodesics in $(\mathbb{L}(\mathbb{S}^{n+1}(c)), \mathcal{G})$. We start with the following proposition:

Proposition 4. *If the curve $\gamma(t) = x(t) \wedge y(t)$ is a $\mathcal{G}$-geodesic $\mathbb{L}(\mathbb{S}^{n+1}(c))$, then the vector field $y = y(t)$ is orthogonal to the curve $x = x(t)$ in $\mathbb{S}^{n+1}(c)$.*

Proof. We prove the proposition for $c = 1$, the proof is similar for $c = -1$. Denote the flat connection of $\Lambda^2\mathbb{R}^{n+2}$ by $\overline{\nabla}$ and the Levi–Civita connection of g by by $\overline{D}$. Then

$$\overline{\nabla}_{\dot\gamma}\dot\gamma = \overline{D}_{\dot x}\dot x \wedge y + x \wedge (\overline{D}^2_{\dot x} y + \langle \dot x, y\rangle\, \dot x) - x \wedge y + 2\dot x \wedge \overline{D}_{\dot x} y.$$

If ∇ is the Levi–Civita connection of $\mathcal{G}$, we then have:

$$\nabla_{\dot\gamma}\dot\gamma = \overline{D}_{\dot x}\dot x \wedge y + x \wedge (\overline{D}^2_{\dot x} y + \langle \dot x, y\rangle\, \dot x).$$

Suppose γ is a $\mathcal{G}$-geodesic. Then

$$\overline{D}_{\dot x}\dot x = ay \quad \text{and} \quad \overline{D}^2_{\dot x} y + \langle \dot x, y\rangle\, \dot x = by,$$

for some functions $a = a(t)$, $b = b(t)$ along the curve $x = x(t)$. Assuming t is the arc-length of the curve x, it follows that

$$0 = g(\overline{D}_{\dot x}\dot x, \dot x) = ag(\dot x, y).$$

If $a \neq 0$ in some open interval, then obviously we have that $g(\dot x, y) = 0$. Assuming $a = 0$ in an open interval, we have that x is a geodesic in that interval. Note that $\dot x, y$ are linearly independent, since otherwise it can be shown that $y = \pm\dot x$ and therefore the curve $\gamma(t) = \pm x \wedge \dot x$ is not regular.

Let $x, \dot x, y, e_1, \ldots e_{n-1}$ be a frame of $\mathbb{R}^{n+2}$ such that $g(e_i, e_j) = \delta_{ij}$ and set $c_0 = g(\overline{D}_{\dot x} y, \dot x)$ with $c_k = g(\overline{D}_{\dot x} y, e_k)$.

Now, $g(\overline{D}_{\dot x} y, \overline{D}_{\dot x} y) = \sum_{k=0} c_k^2 = -g(\overline{D}^2_{\dot x} y, y)$, and therefore

$$\sum_{k=0} c_k^2 = -b + g(\dot x, y)^2.$$

On the other hand

$$\begin{aligned} by - g(\dot x, y)\dot x = \overline{D}^2_{\dot x} y = \overline{D}_{\dot x}(\overline{D}_{\dot x} y) &= \overline{D}_{\dot x}(c_0 \dot x + \sum_{k=1} c_k e_k) \\ &= \dot c_0 \dot x + \sum_k (\dot c_k e_k + c_k \dot e_k) = \dot c_0 \dot x + \sum_k (\dot c_k e_k + c_k \dot e_k) \\ &= \dot c_0 \dot x - (\sum_{k=1} c_k^2) y + \Lambda(e_1, \ldots, e_{n-1}), \end{aligned}$$

where, $\Lambda \in \operatorname{span}\{e_1, \ldots, e_{n-1}\}$. Then

$$\dot c_0 = -g(\dot x, y), \quad b = -\sum_{k=1} c_k^2,$$

and $\Lambda = 0$. In particular, for every $k = 1, \ldots, n-1$, we have

$$\dot c_k + \sum_{i \neq k} g(e_i, \dot e_k) c_i = 0.$$

Thus

$$\sum_{k=1}^{n-1} c_k \dot c_k = \sum_{k,i=1}^{n-1} c_i c_k g(e_i, \dot e_k) = 0,$$

which implies that $\sum_{k=1}^{n-1} c_k^2 = \text{constant}$. This means, $b = -\sum_{k=1}^{n-1} c_k^2$ is constant and by definition we have

$$b = g(\overline{D}_{\dot{x}}^2 y + \langle \dot{x}, y\rangle\, \dot{x}, y) = -g(\overline{D}_{\dot{x}} y, \overline{D}_{\dot{x}} y) + \langle \dot{x}, y\rangle^2 .$$

Using now the fact that b is constant, we have

$$\dot{b} = 4g(\dot{x}, y) g(\dot{x}, \overline{D}_{\dot{x}} y) = 2\frac{d}{dt}\Big(g(\dot{x}, y)^2\Big) = 0.$$

It follows that $g(\dot{x}, y)$ is constant and therefore $g(\dot{x}, \overline{D}_{\dot{x}} y) = 0$. Moreover, $c_0 = 0$ since $c_0 = g(\overline{D}_{\dot{x}} y, \dot{x})$. But $0 = \dot{c}_0 = -g(\dot{x}, y)$ and the proposition follows. □

Every curve $\gamma = \gamma(t) = x(t) \wedge y(t)$ in $\mathbb{L}(\mathbb{S}^{n+1}(c))$, corresponds to a ruled surface in $\mathbb{S}^{n+1}(c)$ and such a surface, can be parametrised by

$$X(t, \theta) = x(t) \cos c(\theta) + y(t) \sin c(\theta), \tag{6}$$

where,

$$\cos c(\theta) = \begin{cases} \cos(\theta), & c = 1 \\ \cosh(\theta), & c = -1 \end{cases}$$

For $n = 2$, we show the following:

Remark 2. A curve γ in $(\mathbb{L}(\mathbb{S}^3(c)), \mathcal{G})$ is a geodesic if and only if the corresponding ruled surface in $\mathbb{S}^3(c)$ is minimal.

Proof. We know that $\dot{x}, y$ are linearly independent and let, $\{x, \dot{x}, y, e_1\}$ be an orthonormal frame of $\mathbb{R}^4, \langle ., .\rangle)$ along the curve $x = x(t)$. The corresponding ruled surface, parametrised by (6), has normal vector fields N, where:

$$N(t, \theta) = e_1 - \frac{c_1 \sin c\theta}{|X_t|^2} X_t,$$

with $c_1 = \big\langle \overline{D}_{\dot{x}} y, e_1 \big\rangle$. Now

$$N_\theta = -\frac{c_1 \cos c\theta}{|X_t|^2} X_t, \quad N_t = \dot{e}_1 - \frac{\dot{c}_1 \sin c\theta}{|X_t|^2} X_t - \frac{c_1 \sin c\theta}{|X_t|^2} X_{tt}$$
$$+ \frac{c_1 \sin c\theta}{|X_t|^4} \langle X_{tt}, X_t\rangle\, X_t, ,$$

If h is the second fundamental form of X, we then have

$$h(X_t, X_t) = -\langle X_t, N_t\rangle = \dot{c}_1 \sin c\theta$$
$$h(X_t, X_\theta) = -\langle X_t, N_\theta\rangle = c_1 \cos c\theta$$
$$h(X_\theta, X_\theta) = -\langle X_\theta, N_\theta\rangle = 0.$$

If H is the mean curvature and t_{ij} the induced metric $X^* g$, we have that $t_{t\theta} = 0$. Therefore

$$H = \frac{1}{2} t^{ij} h(X_i, X_j) = \frac{1}{2} t^{tt} h(X_t, X_t) + \frac{1}{2} t^{\theta\theta} h(X_\theta, X_\theta).$$

Thus, using the previous proposition

$$H = \frac{\dot{c}_1 \sin c\theta}{2|X_t|^2} = 0,$$

since, c_1 is constant. □

It would be interesting to know whether Remark 2 can be extended to any dimension. We therefore conjecture the following:

Conjecture 1. *A curve γ in $(\mathbb{L}(\mathbb{S}^{n+1}(c)), \mathcal{G})$ is a geodesic if and only if the corresponding ruled surface in $\mathbb{S}^{n+1}(c)$ is minimal.*

4. Lagrangian submanifolds

Let $\phi : \Sigma^n \to \mathbb{S}^{n+1}(c)$, be an immersed, orientable hypersurface and N the unit normal vector field along Σ. The Gauss map Φ of ϕ, given in (1), defines a Lagrangian immersion in $\mathbb{L}(\mathbb{S}^{n+1})(c)$ with respect to the symplectic structure Ω [1]. It can be shown that any Lagrangian immersion in $\mathbb{L}(\mathbb{S}^{n+1}(c))$ is locally the Gauss map of a hypersurface in $\mathbb{S}^{n+1}(c))$ and hence is immersed by a mapping Φ. Identifying a vector field X in Σ with the derivative $d\phi(X)$, we have

$$\bar{X} = d\Phi(X) = X \wedge N + AX \wedge \phi,$$

where A denotes the shape operator of ϕ. Let $\overline{D}$ and D be the flat connections of $\mathbb{R}^{n+2}$ and $\Lambda^2\mathbb{R}^{n+2}$ respectively, then we get

$$D_{\bar{X}}\bar{Y} = (\overline{D}_X Y) \wedge N + (\overline{D}_X AY) \wedge \phi.$$

Since the Levi–Civita connection ∇ of $\mathcal{G}$ is the same as that of G_0, the second fundamental form $\bar{h}$ of Φ is:

$$\bar{h}(\bar{X}, \bar{Y}, \bar{Z}) = \mathcal{G}(\nabla_{\bar{X}}\bar{Y}, \mathcal{J}\bar{Z}).$$

Let $(e_1, \ldots, e_n)$ be an orthonormal frame of $(\Sigma, \phi^* g)$ such that $Ae_i = k_i e_i$, where A denotes the shape operator of ϕ. If we simply write the induced metric $\Phi^*\mathcal{G}$ as $\mathcal{G}$ then

$$\mathcal{G}(\bar{e}_i, \bar{e}_j) = 2\delta_{ij} k_i. \tag{7}$$

Away from flat points, i.e. $\Pi_{k=1}^n k_i \neq 0$, we have

$$\bar{h}(e_i, e_j, e_j) = -e_i(k_j),$$

and therefore, if $\mathbb{H}$ is the mean curvature of Φ, we obtain

$$\mathcal{G}(n\mathbb{H}, \mathcal{J}d\Phi(e_i)) = \sum_{i=1}^n \frac{\bar{h}(e_i, e_j, e_j)}{\mathcal{G}(e_j, e_j)} = -\sum_{i=1}^n \frac{e_i(k_j)}{2k_j} = e_i \log |k_1 \cdot \cdots \cdot k_n|^{-1/2}$$

Finally, we have that

$$\mathbb{H} = \frac{1}{n}\mathcal{J}\nabla \log |k_1 \cdot \cdots \cdot k_n|^{-1/2},$$

and thus, we obtain the following:

Remark 3. Every Lagrangian submanifold in $(\mathbb{L}(\mathbb{S}^{n+1}(c)), \mathcal{G}, \Omega)$ has closed Maslov 1-form. If Σ is a non-flat hypersurface of $\mathbb{S}^{n+1}(c)$ then it is of constant Gauss curvature if and only if the oriented geodesics normal to Σ form a minimal Lagrangian submanifold of $(\mathbb{L}(\mathbb{S}^{n+1}(c)), \mathcal{G}, \Omega)$.

A Lagrangian submanifold Σ is said to be *Hamiltonian minimal* if

$$\frac{d}{dt}\,\mathrm{vol} f_t(\Sigma)|_{t=0} = 0,$$

for all Hamiltonian deformations $\{f_t\}$ of Σ. Using the first variation formula, Σ is Hamiltonian minimal if

$$\delta a_H = 0,$$

where $a_H = \mathcal{G}(JH, .)$ is the Maslov 1-form and δ is the Hodge-dual of d on Σ [12].

Remark 4. Let $\phi : \Sigma \to \mathbb{S}^{n+1}(c)$ be a non-flat hypersurface in $(\mathbb{S}^{n+1}(c), g)$. Then the Gauss map $\Phi : \Sigma \to \mathbb{L}(\mathbb{S}^{n+1}(c))$ is a Hamiltonian minimal submanifold of $(\mathbb{L}(\mathbb{S}^{n+1}(c)), \mathcal{G}, \Omega)$ if and only if ϕ is a critical point of the functional

$$\mathcal{F}(\phi) = \int_\Sigma \sqrt{|K|}\, dV,$$

where K and dV denote, respectively, the Gaussian curvature of ϕ and the volume element of the induced metric $\phi^* g$.

Proof. Let Φ be the Gauss map of a smooth immersion of ϕ of the n-dimensional manifold Σ in $\mathbb{S}^{n+1}(c)$ and let $(e_1, \ldots, e_n)$ be an orthonormal frame, with respect to the induced metric $\phi^* g$, such that

$$Ae_i = k_i e_i, \qquad i = 1, \ldots, n,$$

where A denotes the shape operator of ϕ.

Let $(\phi_t)_{t\in(-t_0,t_0)}$ be a smooth variation of ϕ and (Φ_t) be the corresponded variation of the Gauss map Φ. We extend all extrinsic geometric quantities such as the shape operator A, the principal directions e_i and the principal curvatures k_i to the 1-parameter family of immersions (ϕ_t). Using (7), the induced metric $\Phi_t^* G$ is given by

$$\Phi_t^* G = \mathrm{diag}\Big(2k_1, \ldots, 2k_n\Big).$$

For every sufficiently small $t > 0$, the volume of every Gauss map Φ_t, with respect to the metric G, is

$$\mathrm{Vol}(\Phi_t) = \int_\Sigma \sqrt{|\det \Phi_t^* G|} dV = 2^{n/2} \mathcal{F}(\phi_t). \tag{8}$$

If ϕ is a critical point of the functional $\mathcal{F}$, we have

$$\partial_t(\mathrm{Vol}(\Phi_t))|_{t=0} = 0,$$

for any Hamiltonian variation of Φ. Therefore, Φ is a Hamiltonian minimal submanifold with respect to the para-Kähler structure $(\mathcal{G}, \mathcal{J})$. The converse follows directly from (8).

□

Combining Remark 3 and Remark 4 and using the fact that every minimal Lagrangian submanifold is hamiltonian minimal, we also have the following:

Proposition 5. *A non-flat submanifold in* $\mathbb{S}^{n+1}(c)$ *of constant Gaussian curvature is a critical point of the functional* $\mathcal{F}$.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Anciaux, H.: *Space of geodesics of pseudo-Riemannian space forms and normal congruences of hypersurfaces*. Trans. Am. Math. Soc. **366**, 2699–2718 (2014)

[2] Alekseevsky, D.V., Guilfoyle, B., Klingenberg, W.: *On the geometry of spaces of oriented geodesics*. Ann. Glob. Anal. Geom. **40**, 389–409 (2011). (Erratum: Ann. Global Anal. Geom. **50**, 97–99 (2016))

[3] Georgiou, N.: *On minimal Lagrangian surfaces in the product of Riemannian two manifolds*. Tohoku Math. J. **67**, 137–152 (2015)

[4] Georgiou, N., Guilfoyle, B.: *On the space of oriented geodesics of hyperbolic 3-space*. Rocky Mt. J. Math. **40**, 1183–1219 (2010)

[5] Georgiou, N., Guilfoyle, B.: A new geometric structure on tangent bundles. Preprint arXiv:1806.05440

[6] Georgiou, N., Lobos, G.A.: *On Hamiltonian minimal submanifolds in the space of oriented geodesics in real space forms*. Arch. Math. **106**, 285–293 (2016)

[7] Guilfoyle, B., Klingenberg, W.: *An indefinite Kähler metric on the space of oriented lines*. J. Lond. Math. Soc. **72**, 497–509 (2005)

[8] Gudmundsson, S., Kappos, E.: *On the geometry of the tangent bundle with the Cheeger–Gromoll metric*. Tokyo J. Math. **14**, 407–717 (2002)

[9] Salvai, M.: *On the geometry of the space of oriented lines of hyperbolic space*. Glasg. Math. J. **49**, 357–366 (2007)

[10] Palmer, B.: *Buckling eigenvalues, Gauss maps and Lagrangian susbmanifolds*. Differ. Geom. Appl. **4**, 391–403 (1994)

[11] Oh, Y.G.: *Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds*. Invent. Math. **101**, 501–519 (1990)

[12] Oh, Y.-G.: *Volume Minimization of Lagrangian submanifolds under Hamiltonian deformations*. Math. Z. **212**, 175–192 (1993)

Nikos Georgiou
Department of Mathematics
Waterford Institute of Technology
Waterford Co. Waterford
Ireland
e-mail: ngeorgiou@wit.ie

Received: February 3, 2020.
Revised: August 23, 2020.
Accepted: September 10, 2020.