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Certain results on almost contact pseudo-metric manifolds

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Abstract. We study the geometry of almost contact pseudo-metric manifolds in terms of tensor fields $h := \frac{1}{2}\pounds_\xi\varphi$ and $\ell := R(\cdot,\xi)\xi$, emphasizing analogies and differences with respect to the contact metric case. Certain identities involving ξ-sectional curvatures are obtained. We establish necessary and sufficient condition for a nondegenerate almost CR structure $(\mathcal{H}(M), J, \theta)$ corresponding to almost contact pseudo-metric manifold M to be CR manifold. Finally, we prove that a contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ is Sasakian pseudo-metric if and only if the corresponding nondegenerate almost CR structure $(\mathcal{H}(M), J)$ is integrable and J is parallel along ξ with respect to the Bott partial connection.

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1. Introduction

In 1969, Takahashi [22] initiated the study of contact structures associated with pseudo-Riemannian metrics. Afterwards, a number of authors studied such structures mainly focusing on a special case, namely Sasakian pseudo-metric manifolds. The case of contact Lorentzian structures (η, g), where η is a contact 1-form and g a Lorentzian metric associated to it, has a particular relevance for physics and was considered in [12] and [4]. A systematic study of almost contact pseudo-metric manifolds was undertaken by Calvaruso and Perrone [7] in 2010, introducing all the technical apparatus which is needed for further investigations, and such manifolds have been extensively studied under several points of view in [1–3, 6, 9, 10, 15–17, 24], and references cited therein.

The operators $h := \frac{1}{2}\pounds_\xi\varphi$ and $\ell := R(\cdot,\xi)\xi$ play fundamental roles in the study of geometry of contact pseudo-metric manifolds. For contact metric manifolds, Sharma [21] obtained the following beautiful results (Theorem 1.1 in [21]):

(a) a contact metric manifold is K-contact if and only if h is a Codazzi tensor;
(b) a contact metric manifold is K-contact if and only if τ, the tensor metrically equivalent to the strain tensor $\pounds_\xi g$ of M along ξ, is a Codazzi tensor;
(c) the sectional curvatures of all plane sections containing ξ vanish if and only if the tensor ℓ is parallel.

The proof of these results exploit, in an essential way, the fact that in the contact Riemannian case, the self-adjoint operator h vanishes if $h^2 = 0$. But in the contact pseudo-metric case the condition $h^2 = 0$ does not necessarily imply that $h = 0$ (see [16]). So the corresponding results fail for general contact pseudo-metric structures.

Under these circumstances, becomes interesting to explore more the geometry of contact pseudo-metric manifolds. The paper is organized as follows. In Sect. 2, we give the basics of almost contact pseudo-metric manifolds. In Sect. 3, we study contact pseudo-metric manifold M with h satisfying Codazzi condition and we prove that M is Sasakian pseudo-metric manifold if and only if the Eq. (2.10) is satisfied and h is a Codazzi tensor. In Sect. 4, we investigate the Codazzi condition for the operator τ, and we obtain a necessary and sufficient condition for τ to be a Codazzi tensor on contact pseudo-metric manifold. Moreover, if τ is a Codazzi tensor, then $h^2 = 0$ and the Ricci operator Q satisfies $Q\xi = 2\varepsilon n\xi$, and we prove that M is a Sasakian pseudo-metric manifold if and only if the Eq. (2.10) is satisfied and τ is a Codazzi tensor. In Sect. 5, we obtain certain identities involving ξ-sectional curvatures of contact pseudo-metric manifolds. It is proved that the parallelism of the tensor ℓ together with the condition $\nabla_\xi h = 0$ on a contact pseudo-metric manifold implies that all ξ-sectional curvatures vanish. At the end, we investigate the nondegenerate almost CR structure $(\mathcal{H}(M), J, \theta)$ corresponding to almost contact pseudo-metric manifold M, and establish a necessary and sufficient condition for an almost contact pseudo-metric manifold to be a CR manifold. Finally, we show that a contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ is Sasakian pseudo-metric if and only if the corresponding nondegenerate almost CR structure $(\mathcal{H}(M), J)$ is integrable and J is parallel along ξ with respect to the Bott partial connection. We note that many of results of this paper are the corresponding results to those obtained in [14, 25] on almost paracontact metric manifolds.

2. Preliminaries

In this section, we briefly recall some general definitions and basic properties of almost contact pseudo-metric manifolds. For more information and details, we recommend the reference [7].

A $(2n+1)$-dimensional smooth connected manifold M is said to be an *almost contact manifold* if there exists on M a $(1,1)$ tensor field φ, a vector field ξ, and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{2.1}$$

It is known that the first relation along with any one of the remaining three relations in (2.1) imply the remaining two relations. Also, for an almost contact structure, the rank of φ is $2n$. For more details, we refer to [5].

If an almost contact manifold is endowed with a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \tag{2.2}$$

where $\varepsilon = \pm 1$, for all $X, Y \in TM$, then $(M, \varphi, \xi, \eta, g)$ is called an *almost contact pseudo-metric manifold*. The relation (2.2) is equivalent to

$$\eta(X) = \varepsilon g(X, \xi) \text{ along with } g(\varphi X, Y) = -g(X, \varphi Y). \tag{2.3}$$

In particular, in an almost contact pseudo-metric manifold, it follows that $g(\xi, \xi) = \varepsilon$ and so, the characteristic vector field ξ is a unit vector field, which is either space-like or time-like, but cannot be light-like.

The *fundamental 2-form* of an almost contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ is defined by

$$\Phi(X, Y) = g(X, \varphi Y),$$

which satisfies $\eta \wedge \Phi^n \neq 0$. An almost contact pseudo-metric manifold is said to be a *contact pseudo-metric manifold* if $d\eta = \Phi$, where

$$d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y])).$$

The curvature operator R is given by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

This sign convention of R is opposite to the one used in [7,9,15–17]. The Ricci operator Q is determined by

$$S(X, Y) = g(QX, Y).$$

In an almost contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ there always exists a special kind of local pseudo-orthonormal basis $\{e_i, \varphi e_i, \xi\}_{i=1}^n$, called a local φ-basis.

In a contact pseudo-metric manifold, the $(1, 1)$ tensor $h = \frac{1}{2}\pounds_\xi\varphi$ is self-adjoint and satisfies

$$h\xi = 0, \quad \varphi h + h\varphi = 0, \quad \mathrm{tr}(h) = \mathrm{tr}(\varphi h) = 0.$$

Further, one has the following formulas:

$$\nabla_X \xi = -\varepsilon\varphi X - \varphi h X, \tag{2.4}$$

$$(\pounds_\xi g)(X, Y) = 2g(h\varphi X, Y), \tag{2.5}$$

$$(\nabla_\xi h)X = \varphi X - h^2\varphi X + \varphi R(\xi, X)\xi, \tag{2.6}$$

$$R(\xi, X)\xi - \varphi R(\xi, \varphi X)\xi = 2(h^2 + \varphi^2)X, \tag{2.7}$$

$$\mathrm{tr}\, \nabla\varphi = 2n\xi. \tag{2.8}$$

A contact pseudo-metric manifold M is said to be a *K-contact pseudo-metric manifold* if ξ is a Killing vector field (or equivalently, $h = 0$), and is said to be a *Sasakian pseudo-metric manifold* if the almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

is integrable, where $X \in TM$, t is the coordinate on $\mathbb{R}$ and f is a C^∞ function on $M \times \mathbb{R}$. It is well known that a contact pseudo-metric manifold M is a Sasakian pseudo-metric manifold if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \varepsilon\eta(Y)X \tag{2.9}$$

for all $X, Y \in TM$. A Sasakian pseudo-metric manifold is always K-contact pseudo-metric. A 3-dimensional K-contact pseudo-metric manifold becomes a Sasakian pseudo-metric manifold, which may not be true in higher dimensions. Further on a Sasakian pseudo-metric manifold we have

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.10}$$

In contact metric case, the condition (2.10) implies that the manifold is Sasakian, which is not true in contact pseudo-metric case [15]. However, we have the following:

Lemma 2.1 [15]. *Let M be a K-contact pseudo-metric manifold. Then M is a Sasakian pseudo-metric manifold if and only if the curvature tensor R satisfies* (2.10).

3. The Codazzi condition for h

A self-adjoint tensor A of type $(1, 1)$ on a pseudo-Riemannian manifold is known to be a Codazzi tensor if

$$(\nabla_X A)Y = (\nabla_Y A)X \tag{3.1}$$

for all $X, Y \in TM$. Now, we prove the following:

Theorem 3.1. *Let M be a contact pseudo-metric manifold. Then the following statements are true:*

(i) *If h is a Codazzi tensor, then $h^2 = 0$.*
(ii) *M is a Sasakian pseudo-metric manifold if and only if M satisfies* (2.10) *and h is a Codazzi tensor.*

Proof. (i) Suppose that h is a Codazzi tensor, that is,

$$(\nabla_X h)Y = (\nabla_Y h)X, \qquad X, Y \in TM.$$

For $Y = \xi$, using (2.4) in the above equation, we obtain

$$(\nabla_\xi h)X = -\varepsilon\varphi hX - h^2\varphi X.$$

In view of (2.6), the above equation turns into

$$\varphi R(\xi, X)\xi = -\varepsilon\varphi hX - \varphi X. \tag{3.2}$$

Operating φ on both sides of (3.2), it follows that

$$R(\xi, X)\xi = \varphi^2 X - \varepsilon hX. \tag{3.3}$$

Making use of (3.3) in (2.7), shows that $h^2 = 0$.

(ii) If M is a Sasakian pseudo-metric manifold, then $h = 0$ and M satisfies (2.10); and the result is trivial. Conversely, suppose that (2.10) is true and h is a Codazzi tensor. From (2.10), we obtain that

$$R(\xi, X)\xi = \varphi^2 X, \qquad X \in TM. \tag{3.4}$$

Equations (3.3) and (3.4) imply that $h = 0$, that is, M is a K-contact pseudo-metric manifold. Thus, the result follows from Lemma 2.1. □

Remark 3.2. In a contact metric manifold, if h is a Codazzi tensor, then $h = 0$, that is, the manifold becomes K-contact manifold [21]. In the Riemannian case, as $h^2 = 0$ implies $h = 0$, Theorem 3.1 (a) holds in a stronger form, that is, M is K-contact if and only if h is a Codazzi tensor. But, in the case of M being contact pseudo-metric, the condition $h^2 = 0$ does not imply that $h = 0$, because h may not be diagonalizable (see [16]). Note that the result (ii) of Theorem 3.1 is stronger than the Lemma 2.1 which was proved in [15].

In a contact Lorentzian manifold, just like the case of contact metric manifold, the condition $h^2 = 0$ implies $h = 0$ (see [6]). Hence, we immediately have the following

Corollary 3.3. *Let M be a contact Lorentzian manifold. If h is a Codazzi tensor, then $h = 0$, that is, M is K-contact Lorentzian manifold.*

4. The Codazzi condition for τ

We denote by τ, the tensor metrically equivalent to the strain tensor $\pounds_\xi g$ along ξ, that is,

$$g(\tau X, Y) = (\pounds_\xi g)(X, Y)$$

for all $X, Y \in TM$. As pointed out in the introduction, in a contact metric manifold, if τ satisfies the Codazzi condition, then $h = 0$, that is, the manifold is a K-contact manifold. This fact need not be true in the case of contact pseudo-metric manifolds. So, it is quite interesting to study contact pseudo-metric manifolds, which satisfy the Codazzi condition for τ. Now we prove the following:

Lemma 4.1. *In a contact pseudo-metric manifold, τ is a Codazzi tensor if and only if the curvature tensor R satisfies*

$$R(\xi, X)Y = \varepsilon(\nabla_X \varphi)Y. \tag{4.1}$$

Proof. Treating $\nabla\xi$ as a tensor of type $(1,1)$, that is $\nabla\xi : X \mapsto \nabla_X\xi$, one can see that

$$R(X,Y)\xi = (\nabla_X\nabla\xi)Y - (\nabla_Y\nabla\xi)X,$$

which together with (2.4) gives

$$R(X,Y)\xi = -\varepsilon(\nabla_X\varphi)Y - (\nabla_X\varphi h)Y + \varepsilon(\nabla_Y\varphi)X + (\nabla_Y\varphi h)Y. \tag{4.2}$$

On the other hand, if τ is a Codazzi tensor, then from (2.5) we have

$$(\nabla_X h\varphi)Y = (\nabla_Y h\varphi)X.$$

Thus, (4.2) shows that τ is a Codazzi tensor if and only if

$$R(X,Y)\xi = \varepsilon\{(\nabla_Y\varphi)X - (\nabla_X\varphi)Y\}. \tag{4.3}$$

Now if τ is a Codazzi tensor, then by using Bianchi identity and (4.3), we get

$$\begin{aligned} R(\xi,X,Y,Z) &= \varepsilon\{g(X,(\nabla_X\varphi)Y) - g(Y,(\nabla_X\varphi)Z) + g(Z,(\nabla_X\varphi)Y) \\ &\quad - g(X,(\nabla_Z\varphi)Y)\} \\ &= -2\varepsilon g((\nabla_X\varphi)Z,Y) + R(Z,Y,\xi,X), \end{aligned}$$

and so

$$R(\xi,X,Y,Z) = -\varepsilon g((\nabla_X\varphi)Z,Y),$$

which gives (4.1).

Conversely, if (4.1) is true, then from Bianchi identity we have

$$\begin{aligned} R(X,Y,\xi,Z) = R(\xi,Z,X,Y) &= -R(Z,X,\xi,Y) - R(X,\xi,Z,Y) \\ &= -R(\xi,Y,Z,X) + R(\xi,X,Z,Y) \\ &= -\varepsilon\{g((\nabla_Y\varphi)Z,X) - g((\nabla_X\varphi)Z,Y)\}, \end{aligned}$$

which leads to (4.3), and hence τ is a Codazzi tensor. □

Now we prove the contact semi-Riemannian version of Theorem 3.3 of [14].

Theorem 4.2. *Let M be a contact pseudo-metric manifold. Then the following statements are true.*

(i) *If τ is a Codazzi tensor, then $h^2 = 0$ and the Ricci operator Q satisfies*

$$Q\xi = 2\varepsilon n\xi. \tag{4.4}$$

(ii) *M is Sasakian if and only if M satisfies* (2.10) *and τ is a Codazzi tensor.*

Proof. (i). If τ is a Codazzi tensor, then (4.1) gives

$$R(\xi,X)\xi = \varepsilon(\nabla_X\varphi)\xi = \varphi^2X - \varepsilon hX,$$

where we used (2.4). This implies

$$\varphi R(\xi,\varphi X)\xi = -\varphi^2X - \varepsilon hX,$$

and so

$$R(\xi,X)\xi - \varphi R(\xi,\varphi X)\xi = 2\varphi^2X. \tag{4.5}$$

Comparing (2.7) and (4.5), we obtain $h^2 = 0$.

Now, if $\{e_i\}_i^{2n+1}$ is any local pseudo-orthonormal basis, then considering (4.1) we get

$$\begin{aligned} S(X,\xi) &= \sum_{i=1}^{2n+1} \varepsilon_i R(e_i, X, \xi, e_i) = \varepsilon \sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i}\varphi)e_i, X) \\ &= \varepsilon g(\mathrm{tr}(\nabla\varphi), X), \end{aligned}$$

which by using (2.8) we have (4.4).

(ii). Suppose that M is a Sasakian pseudo-metric manifold, then M satisfies (2.10) and $h = 0$.

Conversely, suppose that M satisfies (2.10) and τ is a Codazzi tensor. Then (4.1) shows that

$$\begin{aligned} g((\nabla_X\varphi)Y, Z) &= \varepsilon R(\xi, X, Y, Z) = -\varepsilon R(Z, Y, \xi, X) \\ &= -\varepsilon\{\eta(Y)g(Z, X) - \varepsilon g(Z, \xi)g(X, Y)\} \end{aligned}$$

which gives (2.9). Hence M becomes a Sasakian pseudo-metric manifold. □

Corollary 4.3. *Let M be a contact Lorentzian manifold. If τ is a Codazzi tensor, then $h = 0$, that is, M is K-contact Lorentzian manifold.*

5. ξ-Sectional curvatures

The *ξ-sectional curvature* $K(\xi, X)$ of a contact pseudo-metric manifold is defined by

$$K(\xi, X) = \varepsilon\varepsilon_X g(R(\xi, X)X, \xi),$$

where X is a unit vector field such that $X \in \operatorname{Ker}\eta$ and $g(X, X) = \varepsilon_X = \pm 1$.

It is well known that a contact metric manifold is K-contact if and only if all ξ-sectional curvatures are equal to $+1$ (see [5]). We recall that for a K-contact pseudo-metric manifold, all ξ-sectional curvatures are equal to ε (see [15, Theorem 3.3]).

Now we prove the following result which is related to Theorem 3.5 of [14].

Theorem 5.1. *On a contact pseudo-metric manifold M, the ξ-sectional curvatures satisfy*

$$K(\xi, X) = \varepsilon\{1 - \varepsilon_X g(h^2X, X) - \varepsilon_X g((\nabla_\xi h)X, \varphi X)\}, \tag{5.1}$$

$$K(\xi, X) = K(\xi, \varphi X) - 2\varepsilon\varepsilon_X g((\nabla_\xi h)X, \varphi X) \tag{5.2}$$

for any unit vector $X \in Ker\eta$.

Proof. Using (2.6), we have

$$\begin{aligned} K(\xi, X) &= -\varepsilon\varepsilon_X R(\xi, X, \xi, X) \\ &= -\varepsilon\varepsilon_X g(-\varphi(\nabla_\xi h)X - X + h^2X, X) \\ &= \varepsilon\{\varepsilon_X g(\varphi(\nabla_\xi h)X, X) + {\varepsilon_X}^2 - \varepsilon_X g(h^2X, X)\}, \end{aligned}$$

which gives (5.1).

Now, plugging X by φX in (5.1) keeping $h\varphi = -\varphi h$ and $\nabla_\xi \varphi = 0$ in mind, we obtain

$$K(\xi, \varphi X) = \varepsilon\{1 - \varepsilon_X g(h^2 X, X) + \varepsilon_X g((\nabla_\xi h)X, \varphi X)\}. \tag{5.3}$$

Now, from (5.1) and (5.3), we get (5.2). □

As we discussed in introduction, due to the fact that $h^2 = 0$ does not imply $h = 0$ in a contact pseudo-metric manifold, the parallel condition of ℓ does not imply that ξ-sectional curvatures vanish. However, in the following we show that this is true with an additional assumption $\nabla_\xi h = 0$.

Theorem 5.2. *Let M be a contact pseudo-metric manifold with $\nabla_\xi h = 0$. Then the following are true.*

(i) *The tensor $h^2 = 0$ if and only if all ξ-sectional curvatures are equal to ε.*
(ii) *If $\nabla \ell = 0$, then all ξ-sectional curvatures vanish.*

Proof. (i). Taking the inner product of the unit vector field $X \in \operatorname{Ker} \eta$ with (2.7) yields the following formula for sectional curvatures:

$$K(\xi, X) + K(\xi, \varphi X) = 2\varepsilon\{1 - \varepsilon_X g(h^2 X, X)\}. \tag{5.4}$$

Now, since $\nabla_\xi h = 0$, (5.2) yields

$$K(\xi, X) = K(\xi, \varphi X) \tag{5.5}$$

for any unit vector $X \in \operatorname{Ker} \eta$. From (5.4) and (5.5) we see

$$K(\xi, X) = \varepsilon \text{ if and only if } g(h^2 X, X) = 0.$$

This concludes the proof of (i).

(ii). Applying by φ on both sides of (2.6) and using $\nabla_\xi h = 0$, it follows that

$$\ell X = -h^2 X + X - \eta(X)\xi, \tag{5.6}$$

for any $X \in TM$. Now, in view of $(\nabla_X \ell)\xi = 0$ and (5.6), we have

$$\varepsilon h^2 \varphi X - h^3 \varphi X - \varepsilon \varphi X + h \varphi X = 0. \tag{5.7}$$

If $X \in \operatorname{Ker} \eta$ is a unit vector field, then taking the inner product of φX with (5.7) leads to

$$\varepsilon g(h^2 X, X) + g(h^3 X, X) - \varepsilon g(X, X) - g(hX, X) = 0. \tag{5.8}$$

Now replacing X by φX in (5.7) and then taking inner product of X with the resulting equation gives

$$-\varepsilon g(h^2 X, X) + g(h^3 X, X) + \varepsilon g(X, X) - g(hX, X) = 0. \tag{5.9}$$

Now subtracting (5.8) from (5.9) yields

$$g(h^2 X, X) = g(X, X) = \varepsilon_X \tag{5.10}$$

for any unit vector $X \in \operatorname{Ker} \eta$. Using (5.10) and $\nabla_\xi h = 0$ in (5.1) we conclude that $K(\xi, X) = 0$. □

Corollary 5.3. *A contact Lorentzian manifold is a K-contact Lorentzian manifold if and only if all ξ-sectional curvatures are equal to -1.*

6. Almost *CR* structures

First, we recall few notions of almost *CR* structures (see [11,16,18]). Let M be a $(2n+1)$-dimensional (connected) differentiable manifold. Let $\mathcal{H}(M)$ be a smooth real subbundle of rank $2n$ of the tangent bundle TM (also called Levi distribution), and $J : \mathcal{H}(M) \to \mathcal{H}(M)$ be a smooth bundle isomorphism such that $J^2 = -I$. Then the pair $(\mathcal{H}(M), J)$ is called an almost *CR* structure on M. An almost *CR* structure is called a *CR* structure if it is integrable, that is, the following two conditions are satisfied

$$[JX, Y] + [X, JX] \in \mathcal{H}(M), \tag{6.1}$$

$$J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y] \tag{6.2}$$

for all $X, Y \in \mathcal{H}(M)$.

On an almost *CR* manifold $(M, \mathcal{H}(M), J)$, we define a 1-form θ such that $\operatorname{Ker}\theta = \mathcal{H}(M)$, and such a differential 1-form θ is called a pseudo-Hermitian structure on M. Then on $\mathcal{H}(M)$, the Levi form L_θ is defined by

$$L_\theta(X, Y) = d\theta(X, JY)$$

for all $X, Y \in \mathcal{H}(M)$. Furthermore, we define a $(0, 2)$-tensor field on $\mathcal{H}(M)$ by

$$\alpha(X, Y) = (\nabla_X \theta)(JY) + (\nabla_{JX}\theta)(Y)$$

for all $X, Y \in \mathcal{H}(M)$.

Then we have the following:

Proposition 6.1. *For an almost CR structure $(\mathcal{H}(M), J, \theta)$, the following statements are equivalent*:

- (i) *L_θ is Hermitian, that is, $L_\theta(JX, JY) = L_\theta(X, Y)$;*
- (ii) *L_θ is symmetric, that is, $L_\theta(X, Y) = L_\theta(Y, X)$;*
- (iii) *$[JX, Y] + [X, JY] \in \mathcal{H}(M)$;*
- (iv) *α is symmetric, that is, $\alpha(X, Y) = \alpha(Y, X)$.*

Proof. It is immediate that (i)⇔(ii) and (ii)⇔(iii) follows from the fact that

$$d\theta(X, Y) = -\frac{1}{2}\theta([X, Y])$$

for all $X, Y \in \mathcal{H}(M)$. On the other hand, as in general

$$d\theta(X, Y) = \frac{1}{2}((\nabla_X \theta)Y - (\nabla_Y \theta)X),$$

the condition (ii) is equivalent to

$$(\nabla_X \theta)(JY) + (\nabla_{JX}\theta)Y = (\nabla_Y \theta)(JX) + (\nabla_{JY}\theta)X,$$

and so (ii)⇔(iv). □

The authors came to know from the reviewer that, the equivalence of (ii) with (iv) is related to Lemma 1 of [25], and the equivalence of (i), (ii) and (iii) can be found, for example, in [19] (see Proposition 6, p.17).

An almost pseudo-Hermitian *CR* structure $(\mathcal{H}(M), J, \theta)$ is said to be nondegenerate if the Levi form L_θ is a nondegenerate Hermitian form, and so the 1-form θ is a contact form.

Let $(M, \mathcal{H}(M), J, \theta)$ be a nondegenerate pseudo-Hermitian almost *CR* manifold. We extend the complex structure J to an endomorphism φ of the tangent bundle TM in such a way that $\theta = J$ on $\mathcal{H}(M)$ and $\varphi\xi = 0$, where ξ is the Reeb vector field of θ. Then the Webster metric g_θ, which is a pseudo-Riemannian metric, is defined by

$$g_\theta(X, Y) = L_\theta(X, Y), \quad g_\theta(X, \xi) = 0, \quad g_\theta(\xi, \xi) = \varepsilon$$

for all $X, Y \in \mathcal{H}(M)$. In this case, $(\varphi, \xi, \eta = -\theta, g = g_\theta)$ defines a contact pseudo-metric structure on M. Conversely, if (φ, ξ, η, g) is a contact pseudo-metric structure, then $(\mathcal{H}(M), J, \theta)$, where $\mathcal{H}(M) = \operatorname{Ker}\eta$, $\theta = -\eta$, and $J = \varphi_{|\mathcal{H}(M)}$, defines a nondegenerate almost *CR* structure on M. Thus, we have:

Proposition 6.2 ([16]). *The notion of nondegenerate almost CR structure $(\mathcal{H}(M), J, \theta)$ is equivalent to the notion of contact pseudo-metric structure (φ, ξ, η, g).*

Now, we prove the following result which is related to Theorem 1 of [25].

Theorem 6.3. *An almost contact pseudo-metric manifold M is CR manifold if and only if*

$$(\nabla_X J)Y - (\nabla_{JX} J)JY = \alpha(X, Y)\xi \tag{6.3}$$

for all $X, Y \in \mathcal{H}(M)$.

Proof. Applying J to (6.2) gives

$$(\nabla_Y J)X - (\nabla_X J)Y = J(\nabla_{JX} J)Y - J(\nabla_{JY} J)X$$

for all $X, Y \in \mathcal{H}(M)$. Since $J(\nabla_{JX} J)Y = -(\nabla_{JX} J)JY$, the above equation becomes

$$(\nabla_Y J)X - (\nabla_X J)Y = (\nabla_{JY} J)JX - (\nabla_{JX} J)JY \tag{6.4}$$

for all $X, Y \in \mathcal{H}(M)$. If we define a (0, 3)-tensor field A on $\mathcal{H}(M)$ as

$$A(X, Y, Z) = g((\nabla_{JX} J)JY - (\nabla_X J)Y, Z) \tag{6.5}$$

for all $X, Y \in \mathcal{H}(M)$, then from (6.4) one obtain

$$A(X, Y, Z) = A(Y, X, Z). \tag{6.6}$$

Next, a simple computation shows that

$$\begin{aligned} &A(X,Y,Z)+A(X,Z,Y)\\ &\quad = g((\nabla_{JX}J)JY-(\nabla_X J)Y,Z)+g((\nabla_{JX}J)JZ-(\nabla_X J)Z,Y)\\ &\quad = -g((\nabla_{JX}J)Z,JY)+g((\nabla_{JX}J)JZ,Y)\\ &\quad = -g(\nabla_{JX}JZ,JY)-g((\nabla_{JX}Z),J^2Y)\\ &\qquad +g(\nabla_{JX}J^2Z,Y)-g(J(\nabla_{JX}JZ),Y)\\ &\quad = 0, \end{aligned}$$

where the skew-symmetry of J and ∇J are used. This together with (6.6) gives the following:

$$\begin{aligned} A(X,Y,Z) &= -A(X,Z,Y) = -A(Z,X,Y) = A(Z,Y,X)\\ &= A(Y,Z,X) = -A(Y,X,Z) = -A(X,Y,Z). \end{aligned}$$

Hence it follows that $A=0$, and so (6.5) implies

$$(\nabla_{JX}J)JY-(\nabla_X J)Y=\gamma(X,Y)\xi \tag{6.7}$$

for all $X,Y\in\mathcal{H}(M)$, for certain (0,2)-tensor field γ on $\mathcal{H}(M)$. It remains to show that $\gamma=\alpha$. From (6.7), it follows that

$$\begin{aligned} \gamma(X,Y) &= \varepsilon g((\nabla_{JX}J)JY-(\nabla_X J)Y,\xi)\\ &= \varepsilon\{-g((\nabla_{JX}J)\xi,JY)+g((\nabla_X J)\xi,Y)\}\\ &= \varepsilon\{g(\nabla_{JX}\xi,Y)-g(J\nabla_X\xi,Y)\}\\ &= (\nabla_{JX}\theta)Y+(\nabla_X\theta)JY\\ &= \alpha(X,Y). \end{aligned}$$

Conversely, suppose that (6.3) holds true. Then projecting (6.3) onto ξ, it follows that α is symmetric and is equivalent to (6.1). The symmetry of α together with (6.3) gives (6.4), which yields

$$-[JX,Y]-[X,JY]=J[JX,JY]-J[X,Y],$$

for all $X,Y\in\mathcal{H}(M)$, and so satisfies the Eq. (6.2). □

Let (M,φ,ξ,η,g) be an almost contact pseudo-metric manifold with $(\mathcal{H}(M),J)$ as the corresponding almost CR structure. For $Y\in TM$, we denote $Y_{|\mathcal{H}(M)}$ to the orthogonal projection on $\mathcal{H}(M)$. Then, the Bott partial connection $\breve{\nabla}$ on $\mathcal{H}(M)$ (along ξ) is the map $\breve{\nabla}: S(\xi)\times\mathcal{H}(M)\to\mathcal{H}(M)$ defined by

$$\breve{\nabla}_\xi X := (\pounds_\xi X)_{|\mathcal{H}(M)} = [\xi,X]_{|\mathcal{H}(M)}$$

for any $X\in\mathcal{H}(M)$ (see, [20, p. 18]), where $S(\xi)$ is the 1-dimensional linear subspace of TM generated by ξ.

Theorem 6.4. *Let (M,φ,ξ,η,g) be an almost contact pseudo-metric manifold, and ξ a geodesic vector field. Then $h=0$ if and only if $\breve{\nabla}_\xi J=0$.*

Proof. As $h\xi = 0$, we may observe that, $h = 0$ if and only if $hX = 0$ for any $X \in \mathcal{H}(M)$.

Now using $\nabla_\xi \xi = 0$, for any $X \in \mathcal{H}(M)$, we have

$$\eta([\xi, X]) = \varepsilon g(\xi, \nabla_\xi X - \nabla_X \xi) = 0,$$

which means $\pounds_\xi X \in \mathcal{H}(M)$. Thus, we get

$$\begin{aligned} 2hX &= \pounds_\xi(\varphi X) - \varphi(\pounds_\xi X) = \breve{\nabla}_\xi(\varphi X) - \varphi(\breve{\nabla}_\xi X) \\ &= \breve{\nabla}_\xi(JX) - J(\breve{\nabla}_\xi X) = (\breve{\nabla}_\xi J)X \end{aligned}$$

for any $X \in \mathcal{H}(M)$, completing the proof. □

Note that the proof of Theorem 6.4 is related to the formula (4.3) of [14]. For contact pseudo-metric manifold, the structure vector field is geodesic. So we have the following:

Corollary 6.5. *A contact pseudo-metric manifold is K-contact pseudo-metric if and only if $\breve{\nabla}_\xi J = 0$.*

For paracontact metric manifolds the result of the following Theorem was proved in Corollary 4.9 of [14].

Theorem 6.6. *A contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ is Sasakian pseudo-metric if and only if the corresponding nondegenerate almost CR structure $(\mathcal{H}(M), J)$ is integrable and $\breve{\nabla}_\xi J = 0$.*

Proof. First we observe that, following the same proof given in [23] for the Riemannian case, the integrable condition (that is, (6.1) and (6.2)) of the corresponding *CR* structure $(\mathcal{H}(M), J)$ is equivalent to

$$(\nabla_X \varphi)Y = -\{(\nabla_X \eta)\varphi Y\}\xi - \eta(X)\varphi(\nabla_X \xi), \tag{6.8}$$

where

$$(\nabla_X \eta)\varphi Y = -g(X, Y) + \varepsilon\eta(X)\eta(Y) - \varepsilon g(hX, Y),$$

and

$$\varphi(\nabla_X \xi) = \varepsilon X - \varepsilon\eta(X)\xi + hX.$$

Thus, (6.8) becomes

$$(\nabla_X \varphi)Y = g(X + \varepsilon hX, Y)\xi - \varepsilon\eta(Y)(X + \varepsilon hX). \tag{6.9}$$

If the contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ is Sasakian, then (6.9) satisfies with $h = 0$, and so corresponding nondegenerate almost *CR* structure $(\mathcal{H}(M), J)$ is integrable and $\breve{\nabla}_\xi J = 0$.

Conversely, as $\breve{\nabla}_\xi J = 0$ implies $h = 0$, Eq. (6.9) reduces to (2.9), and so the structure is Sasakian pseudo-metric. □

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