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The Hajja–Martini inequality in a weak absolute geometry

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Dedicated to Helmut Karzel on the occasion of his 90th birthday.

Abstract. Solving a problem left open in Hajja and Martini (Mitt. Math. Ges. Hamburg 33:135–159, 2013), we prove, inside a weak plane absolute geometry, that, for every point P in the plane of a triangle ABC there exists a point Q inside or on the sides of ABC which satisfies:

$$AQ \leq AP,\ BQ \leq BP,\ CQ \leq CP. \tag{1}$$

If P lies outside of the triangle ABC, then Q can be chosen to both lie inside the triangle ABC and such that the inequalities in (1) are strict. We will also provide an algorithm to construct such a point Q.

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Keywords. Absolute plane geometry, geometric inequalities.

1. Introduction

Searching for results that bear some similarity to Propositions 20 and 21 of Book I of Euclid's *Elements*, Hajja and Martini arrive in [1, Theorem 12] at the following theorem, whose validity they prove in the real Euclidean plane

Theorem 1. *Let P be a point in the plane of a triangle ABC. Then there exists a point Q inside or on the boundary of ABC that satisfies* (1).

Aware of the discrepancy between the statement of the theorem, whose notions belong to Hilbert's absolute geometry (whose axioms are the plane axioms of incidence, order, and congruence of groups I, II, and III of Hilbert's *Grundlagen der Geometrie*), which is where one expects a proof to be carried through, and

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the methods of proof used, the authors ask: “Its fanciful proof, using Zorn’s lemma and the Bolzano-Weierstrass theorem, raises the question whether such a heavy machinery is indeed inevitable” [1, p. 13]. Moreover, since they can only prove the existence of the point Q, they also ask “whether there is a procedure (an algorithm) to construct the point Q” [1, p. 14].

It is the purpose of this paper to provide an elementary proof, within a very weak plane absolute geometry (all of whose axioms can be deduced inside Hilbert’s plane absolute geometry), of Theorem 1, to provide an algorithm for constructing Q, and to prove that, for any point P that lies outside of triangle ABC (for which we will also write $\triangle ABC$), there exists a point Q in the interior of triangle ABC, for which all of the inequalities in (1) are strict. To do this, we will first present the axiom system Σ for a very weak absolute geometry, as well as an additional axiom to be added to Σ to form an axiom system Σ', giving rise to what one would still refer to as a very weak absolute geometry, followed by the proofs of some basic geometric truths valid in Σ and Σ', as preparation for the main results mentioned above.

2. The axiom system

In the spirit of *reverse geometry* (see [2]), we will set up an axiom system, consisting precisely of those assumptions needed to ensure that the notions of Theorem 1 make sense, and in which our proof can be carried through.

The language is a two-sorted one, with variables for *points*, denoted by upper-case letters, and *lines*, denoted by lower-case letters. There are two binary relation symbols, $\in$, for *point-line incidence*, with $P \in g$ to be read as “P is incident with g,” and $\perp$, for *line orthogonality*, with $g \perp h$ to be read “line g is orthogonal to line h,” there is a ternary relation ζ, with point arguments, with $\zeta(A, B, C)$ to be read as “B lies strictly between A and C,” there is a five-place operation ξ, and there are six binary operation symbols, φ, σ, μ, μ_0, π, and π_0. Here ξ has four point variables and a line variable as arguments and a point variable and value, with $\xi(A, M, B, C, g)$ to be read, in case $\zeta(A, M, B)$, $M \in g$, and $C \notin g$ holds, as “the intersection point of g with one of the segments AC or BC,” an arbitrary point, should one of $\zeta(A, M, B)$, $M \in g$, and $C \notin g$ not be the case, φ has two point variables as arguments and a line variable as value, with $\varphi(A, B)$ standing, for $A \neq B$, for the “line joining A and B,” an arbitrary line, otherwise, σ has point variables as arguments and a point variable as value, $\sigma(A, B)$ to be read, for $A \neq B$, as “the reflection of B in A,” arbitrary, otherwise, μ has point variables as arguments and a line variable as value, with $\mu(A, B)$ to be read, for $A \neq B$, as “the perpendicular bisector of AB,” arbitrary, otherwise, μ_0 has point variables both as arguments and as value, with $\mu_0(A, B)$ to be read, for $A \neq B$, as “the midpoint of AB,” an arbitrary point, otherwise, π has a point and a line variable as arguments and a line variable as value, with $\pi(P, g)$ to be read as “the perpendicular through P to g,” and π_0 has a point and a line variable as arguments and a point variable as value, with $\pi_0(P, g)$ to be read as "the foot of the perpendicular from P to g." Pieri's ternary notion of congruence of two segments with a common endpoint, ι, and that of segment inequality of two segments sharing an endpoint, λ, are defined notions in this set-up. $\iota(A, B, C)$ should be read as AB is congruent to AC, and $\lambda(A, B, C)$ as $AB < AC$. Their definitions are

$$\iota(A, B, C) \Leftrightarrow (B = C) \vee \mu(B, C) = \pi(A, \varphi(B, C)) \tag{2}$$

$$\lambda(A, B, C) \Leftrightarrow B \neq C \wedge A \notin \mu(B, C) \wedge \zeta(A, \xi(B, \mu_0(B, C), C, A, \mu(B, C)), C) \tag{3}$$

These two definitions are motivated by the follwoing view of the perpendicular bisector $\mu(A, B)$ of a non-degenerate segment AB: the points on it are equidistant from its endpoints, and thus offer a means of defining congruence for two segments sharing an endpoint, while the points in the two halfplanes determined by $\mu(A, B)$ are precisely those points for which the distances from the endpoints are different. Those points P in the halfplane determined by $\mu(A, B)$ in which A lies are such that $PA < PB$, while those points P in the halfplane determined by $\mu(A, B)$ in which B lies are such that $PB < PA$. In other words: AB is congruent to AC if the perpendicular bisector of BC goes through A, and $AB < AC$ if the perpendicular bisector of BC intersects the open segment AC.

For the reader's reading convenience, we will write henceforth $AB \equiv AC$ for $\iota(A, B, C)$ and $AB < AC$ for $\lambda(A, B, C)$. By $AB \leq AC$ we will mean $AB \equiv AC \vee AB < AC$. The axioms are

A1. $A \neq B \rightarrow A \in \varphi(A, B) \wedge B \in \varphi(A, B)$

A2. $A \in g \wedge B \in g \rightarrow (A = B \vee g = \varphi(A, B))$

A3. $a \perp b \rightarrow b \perp a$

A4. $\pi_0(P, g) \in g \wedge \pi_0(P, g) \in \pi(P, g)$

A5. $A \in \pi(A, g) \wedge \pi(A, g) \perp g$

A6. $A \in h \wedge h \perp g \rightarrow h = \pi(A, g)$

A7. $\zeta(A, B, C) \rightarrow A \neq B \wedge A \neq C \wedge B \in \varphi(A, C)$

A8. $A \in g \wedge B \in g \wedge C \in g \wedge A \neq B \wedge B \neq C \wedge C \neq A$
$\rightarrow \zeta(A, B, C) \vee \zeta(B, C, A) \vee \zeta(C, A, B)$

A9. $\zeta(A, B, C) \rightarrow \zeta(C, B, A)$

A10. $\zeta(A, B, C) \rightarrow \neg\zeta(A, C, B)$

A11. $D \in g \wedge \zeta(A, D, B) \wedge C \notin g \rightarrow \xi(A, D, B, C, g) \in g$
$\wedge(\zeta(A, \xi(A, D, B, C, g), C) \vee \zeta(B, \xi(A, D, B, C, g), C))$

A12. $A \neq B \rightarrow \mu(B, A) = \mu(A, B)$

A13. $A \neq B \rightarrow \mu_0(A, B) \in \mu(A, B)$

A14. $A \neq B \rightarrow \pi(\mu_0(A, B), \varphi(A, B)) = \mu(A, B)$

A15. $A \neq B \rightarrow \zeta(A, \mu_0(A, B), B)$

A16. $A \neq B \rightarrow \mu_0(\sigma(B, A), A) = B$

A17. $AB < AC \wedge AC < AD \rightarrow AB < AD$

A18. $\zeta(A, B, C) \rightarrow \zeta(A, \mu_0(A, B), \mu_0(A, C))$

A1 and A2 are the trivial incidence axioms, stating that there exists a unique line joining two distinct points; A3–A6 are the trivial orthogonality axioms, stating that orthogonality is a symmetric relation (A3), that the line through P that is orthogonal to g intersects g (A4), that there is a perpendicular through every point to every line (A5), that there is no more than one perpendicular from a point to a line (A6); A7–A10 are some linear order axioms, A11 is the Pasch axiom, in which ABC is not required to be a non-degenerate triangle, given that the line g, once it entered through the side AB will have to exit even a degenerate triangle through one of the open segments AC or BC, if it does not go through C. A12–A15 introduce μ and μ_0, about which we are told that $\mu_0(A, B)$ is a point lying between A and B, and that the line $\mu(A, B)$ (which coincides with $\mu(B, A)$) is perpendicular in $\mu_0(A, B)$ to the line joining A with B. Axiom A15 also tells us that the order on any line determined by two points is dense. A16 states that $\sigma(B, A)$ is the reflection of A in B, in the sense that B is the midpoint of $\sigma(B, A)$ and A. A18 states that the half of a smaller segment is smaller than the half of a larger segment, if the smaller segment shares an endpoint with the larger one and is contained in it. A17 states that the relation $<$ is transitive.

Usually, axiom systems also contain an axiom stating that three non-collinear points exist, that is

A19. $(\exists ABC)\, A \neq B \wedge A \neq C \wedge \varphi(A, B) \neq \varphi(A, C)$

but we will not need such an axiom to prove Theorem 1, for if no three collinear points exist, Theorem 1 is vacuously true. Let $\Sigma = \{$A1–A17$\}$ and let $\Sigma' = \{$A1–A18$\}$. It is plain that all the axioms of Σ' hold in Hilbert's plane absolute geometry.

The model presented in [3, pp. 83–84] shows that, even in the presence of the Euclidean parallel postulate, A17 does not follow from the other axioms of Σ'.

When (1) holds with strict inequalities, we will write $(1)_<$. We will say that the points X and Y satisfy (1) if substituting X for Q and Y for P in (1) renders the inequalities in (1) valid.

3. Some basic facts about $\Sigma \cup \{A19\}$ and $\Sigma' \cup \{A19\}$

3.1. Facts about $\Sigma \cup \{A19\}$

We will state and prove some of the basic facts we will need in the sequel that are true in Σ in the presence of the assumption that there are three non-collinear points, that is of A19, which, although not part of Σ, is part of the hypothesis of Theorem 1, and thus can be assumed when proving it.

First, notice that, by A15 and A16, we have

$$A \neq B \to \zeta(\sigma(B, A), B, A) \tag{4}$$

Thus, in the presence of A19, the order axioms A7–A11, together with A15 and (4), form an axiom system for what is referred to in [4] as *ordered planes*. This means that all the universal axioms that we expect the betweenness relation to satisfy on a line hold, as well as that the Crossbar Theorem (see [5, p. 116]) holds. In addition, from now on, notions of ordered planes such as half-plane, angle, interior of a triangle, Pasch's Theorem, etc. are freely used without explicit definition, and we will say that a result holds "by the axioms of ordered geometry" or "by ordered geometry" to mean that the result is a consequence of several axioms of ordered geometry.

An easy consequence of (4) and A7 is

$$A \neq B \to \sigma(A, B) \neq B \tag{5}$$

Notice also that, in the presence of A19, every line can be written as $\varphi(A, B)$ for some $A \neq B$. This implies that there are no isotropic lines, i.e., that

$$g \not\perp g \tag{6}$$

For suppose some $g = \varphi(A, B)$, with $A \neq B$, were such that $g \perp g$. By A19, we know that there exists C with $C \notin g$. Then, by A5, $C \in \pi(C, g)$ and $\pi(C, g) \perp g$, and, by A4, $\pi_0(C, g) \in g$ and $\pi_0(C, g) \in \pi(C, g)$. Now, $\pi_0(C, g) \in g$ and $g \perp g$ imply, by A6, $g = \pi(\pi_0(C, g), g)$. Also by A6 we have that $\pi_0(C, g) \in \pi(C, g)$ and $\pi(C, g) \perp g$ imply $\pi(C, g) = \pi(\pi_0(C, g), g)$. This means that $\pi(C, g) = g$. However, this cannot be, for $C \in \pi(C, g)$ while $C \notin g$.

We now turn to the proof that

$$\mu_0(A, B) = \mu_0(B, A) \tag{7}$$

For $A = B$ there is nothing to prove, so we assume $A \neq B$. By A13 and A12 we have that $\mu_0(A, B) \in \mu(A, B)$ and $\mu_0(B, A) \in \mu(A, B)$. By A15, A7, A9, and the fact that $\varphi(A, B) = \varphi(B, A)$, we get $\mu_0(A, B) \in \varphi(A, B)$ and $\mu_0(B, A) \in \varphi(A, B)$. By A14 and A12, $\pi(\mu_0(A, B), \varphi(A, B)) = \pi(\mu_0(B, A), \varphi(A, B)) = \mu(A, B)$. If $\mu_0(A, B) \neq \mu_0(B, A)$, then, by A5, $\mu_0(A, B) \in \mu(A, B)$ and $\mu_0(B, A) \in \mu(A, B)$, so, by A1 and A2, $\varphi(\mu_0(A, B), \mu_0(B, A)) = \mu(A, B)$. However, we also have $\varphi(\mu_0(A, B), \mu_0(B, A)) = \varphi(A, B)$, so $\mu(A, B) = \varphi(A, B)$, and since, by A14 and A5, we have $\mu(A, B) \perp \varphi(A, B)$, we get $\varphi(A, B) \perp \varphi(A, B)$, contradicting (6).

An easy consequence of the definition of segment inequality $<$ and of the uniqueness of the perpendicular from a point to a line is the fact that the hypotenuse of a right triangle is greater than the leg, i.e., that

$$O \neq A \wedge O \neq B \wedge \varphi(O,A) \perp \varphi(O,B) \rightarrow AO < AB \tag{8}$$

The perpendicular bisector of the segment OB, $\mu(O,B)$, intersects, by the Pasch axiom A11—since it cannot pass through A, as that would imply that there are two distinct perpendiculars from A to $\varphi(O,B)$, contradicting A6—one of the sides OA or AB. However, it cannot intersect OA for the same reason that, from that intersection point there would be two distinct perpendiculars to $\varphi(O,B)$, contradicting A6. Thus $\mu(O,B)$ intersects the side AB, and thus, by (3), we have $AO < AB$.

A17 implies that the foot of the altitude to the hypotenuse in a right triangle lies between the endpoints of the hypotenuse, i.e., that

$$O \neq A \wedge O \neq B \wedge \varphi(O,A) \perp \varphi(O,B) \rightarrow \zeta(A, \pi_0(O, \varphi(A,B)), B) \tag{9}$$

For suppose $D = \pi_0(O, \varphi(A,B))$ lies outside of the segment AB (that $D \neq A$ and $D \neq B$ follows from A2–A6), say $\zeta(D,A,B)$. According to (8), $BD < BO$. Since $\zeta(D,A,B)$, we also have $BA < BD$, so, by A17, $BA < BO$. However, by (8), $BO < BA$, a contradiction, for, by Pasch's Theorem, a line (in this case $\mu(O,A)$) cannot intersect all three sides of a triangle (in this case $\triangle OAB$).

With the axioms from Σ, we can prove, as in [6, p. 486], that (9) is equivalent with the statement **RR**: "A right angle cannot be included inside another right angle with the same vertex." Thus (9), by means of **RR**, allows for a meaningful introduction of the concept of an acute angle and of that of an obtuse angle. We say that $\widehat{AOB}$ is *acute* (and write $\alpha(AOB)$) if $\overrightarrow{OB}$, lies inside the angle formed by $\overrightarrow{OA}$ and the ray emanating from O, which is part of $\pi(O, \varphi(O,A))$, and which lies in the same half-plane determined by $\varphi(O,A)$ as B. By **RR**, in the definition of the notion of an acute angle one can interchange A and B, so that the definition does not depend on the side of the angle on which one raises the perpendicular. We say that $\widehat{AOB}$ is *obtuse* (and write $\omega(AOB)$) if the ray emanating from O, which is part of $\pi(O, \varphi(O,A))$, and which lies in the same half-plane determined by $\varphi(O,A)$ as B, lies inside $\widehat{AOB}$. Again, by **RR**, we can switch A and B in the definition and get the same notion. An easy consequence of the definition of $<$ and the independence of the definition of an obtuse angle on the side on which one raises the perpendicular is the fact that

$$\omega(AOB) \rightarrow AO < AB \wedge BO < BA \tag{10}$$

We also have: If P is a point on the side AB of a triangle ABO then $AP < AO$ or $BP < BO$. Formally

$$O \neq A \wedge O \neq B \wedge \varphi(O, A) \neq \varphi(O, B) \wedge \zeta(A, P, B) \\ \rightarrow (AP < AO \vee BP < BO) \tag{11}$$

Since $\mu_0(P, O)$ is a point in the interior of triangle ABO, and since $\mu(P, O)$ passes through $\mu_0(P, O)$ and $\mu(P, O) \neq \varphi(P, O)$ (by (6)), by the axioms for ordered geometry, $\mu(P, O)$ will have to intersect at least one of the open segments OA and OB. Thus one of $AP < AO$ and $BP < BO$ holds.

The variant of Proposition 21 of Book I of Euclid's *Elements* that we will need is:

If $P \neq C$ is a point inside or on the boundary of triangle ABC, then $PA < CA$ or $PB < CB$. (12)

Applying the Pasch axiom to triangles PBC and PAC with secant $\mu(P, C)$, we get that $\mu(P, C)$, unless it passes through one of A and B. and then, by the Crossbar Theorem, must intersect the side BC respectively the side AC of triangle ABC, intersects the sides PB or BC, as well as the sides AC or PA. The only way in which we would not have one of $PA < CA$ and $PB < CB$ would be if $\mu(P, C)$ were to intersect the open segments PB and PA in points E and F, respectively. By the Crossbar Theorem, ray $\overrightarrow{CP}$ intersects AB in a point D, and by the same Crossbar Theorem, ray $\overrightarrow{PD}$ intersects EF in a point G with $\zeta(P, G, D)$. Since we have, by A15, $\zeta(P, \mu_0(P, C), C)$, the lines $\varphi(E, F)$ and $\varphi(C, P)$ have two points in common: $\mu_0(P, C)$ and G, and thus must coincide, which cannot be the case, as neither E nor F are on $\varphi(P, C)$.

We can define the operation of line reflection ϱ_g in line g as a unary operation with point variables as both arguments and values, by $\varrho_g(P) = P$ if $P \in g$ and $\varrho_g(P) = \sigma(\pi_0(P, g), P)$ if $P \notin g$. There is no reason to think that ϱ_g is collinearity-preserving, to say nothing of orthogonality-preserving, so it is very likely a significantly weaker notion than the standard one. What we can prove about it is that

$$O \in g \rightarrow OP \equiv O\varrho_g(P) \tag{13}$$

If $P \in g$, then $\varrho_g(P) = P$, so (13) holds by (2). If $P \notin g$, then we will have to show that $\mu(P, \varrho_g(P)) = \pi(O, \varphi(P, \varrho_g(P))$. Notice that, since $P \neq \pi_0(P, g)$ (for, by A4, $\pi_0(P, g) \in g$, so if $\pi_0(P, g)$ were P, we would have $P \in g$), by (5), $\varrho_g(P) \neq P$. By (7) and A16, $\mu_0(P, \varrho_g(P)) = \mu_0(\varrho_g(P), P) = \mu_0(\sigma(\pi_0(P, g), P), P) = \pi_0(P, g)$. Let $m = \varphi(P, \varrho_g(P))$ and $h = \pi(\pi_0(P, g), m)$. By A4 and A5, $\pi_0(P, g) \in h$, $\pi_0(P, g) \in g$, and $h \perp m$. By A14, $h = \mu(P, \varrho_g(P))$. By (4), A1, A2, and A7, we have $\varphi(P, \pi_0(P, g)) = m$. By A1, A2, and A4, $\varphi(P, \pi_0(P, g)) = \pi(P, g)$, thus $m = \pi(P, g)$, so $m \perp g$, and, by A3, $g \perp m$. By A6, $h = g$. Thus, the equality we have to show, $\mu(P, \varrho_g(P)) = \pi(O, \varphi(P, \varrho_g(P))$, has become $g = \pi(O, m)$, which follows from A6. We incidentally also proved that

$$P \notin g \rightarrow \mu(P, \varrho_g(P)) = g \tag{14}$$

3.2. Facts about $\Sigma' \cup \{$A19$\}$

A16 tells us that σ and μ_0 are some sort of inverse operations, but it tells only half of that story. The other half

$$A \neq B \rightarrow \sigma(\mu_0(A, B), A) = B \tag{15}$$

can be proved in Σ'. Let $A \neq B$ and let $\sigma(\mu_0(A, B), A) = B'$. By (4), we have $\zeta(B', \mu_0(A, B), A)$. Since, by A15, we have $\zeta(A, \mu_0(A, B), B)$, and since we also have $\zeta(B', \mu_0(A, B), A)$, by the axioms of ordered geometry, we have that $B = B' \vee \zeta(A, B, B') \vee \zeta(A, B', B)$. By (7) and A16, $\mu_0(A, B') = \mu_0(B', A) = \mu_0(A, B)$. By A18, $\zeta(A, B, B') \rightarrow \zeta(A, \mu_0(A, B), \mu_0(A, B'))$, which contradicts A7, since $\mu_0(A, B') = \mu_0(A, B)$. The same contradiction follows if we assume $\zeta(A, B', B)$, so $B = B'$, which proves (15).

An easy consequence of (15) is that reflections in points are involutory transformations, i.e., that

$$O \neq A \rightarrow \sigma(O, \sigma(O, A)) = A \tag{16}$$

This can be seen by first noticing that, by A16, $\mu_0(\sigma(O, A), A) = O$. By (5) and (15), $\sigma(\mu_0(\sigma(O, A), A), \sigma(O, A)) = A$, which proves (16).

Next, we will show that the reflection in a point preserves the betweenness relation, i.e., that

$$\zeta(O, A, B) \rightarrow \zeta(O, \sigma(O, A), \sigma(O, B)) \tag{17}$$

Let $A' = \sigma(O, A)$ and $B' = \sigma(O, B)$. By (4) and A9, we have $\zeta(A, O, A')$ and $\zeta(B, O, B')$. Since we also have $\zeta(O, A, B)$, by the axioms of ordered geometry (in the presence of A19), we have $A' = B'$ or $\zeta(O, A', B')$ or $\zeta(O, B', A')$. Now $A' = B'$ is impossible, for, by A16, $\mu_0(A', A) = \mu_0(B', B) = O$. By (15), $\sigma(\mu_0(A', A), A') = A$ and $\sigma(\mu_0(B', B), B') = B$. By (7), $\mu_0(A', A) = \mu_0(A, A') = O$ and $\mu_0(B', B) = \mu_0(B, B') = O$, so $\sigma(\mu_0(A', A), A') = \sigma(O, A') = A$ and $\sigma(\mu_0(B', B), B') = \sigma(O, B') = B$. If $A' = B'$, then $A = \sigma(O, A') = \sigma(O, B') = B$, contradicting A7 (given that, by A9, we have $\zeta(B, A, O)$). Suppose now $\zeta(O, B', A')$. By the axioms of ordered geometry, we have $\zeta(B, B', A')$ and $\zeta(A', A, B)$, and, by A18, $\zeta(B, \mu_0(B, B'), \mu_0(B, A'))$ and $\zeta(A', \mu_0(A', A), \mu_0(A', B))$. Since $\mu_0(A', A) = \mu_0(B, B') = O$, this means that, bearing in mind that, by (7), $\mu_0(A', B) = \mu_0(B, A')$, $\zeta(B, O, \mu_0(A', B))$ and $\zeta(A', O, \mu_0(A', B))$. By ordered geometry, the latter implies, together with $\zeta(A', \mu_0(A', B), B)$ (by A15), $\zeta(O, \mu_0(A', B), B)$, which, after applying A9, contradicts A10.

4. Theorem 1 holds in Σ

Lemma 1. *Let P be a point outside of triangle ABC and let P and C lie on different sides of $\varphi(A, B)$. Then $\pi_0(P, \varphi(A, B))$ and P satisfy* $(1)_<$.

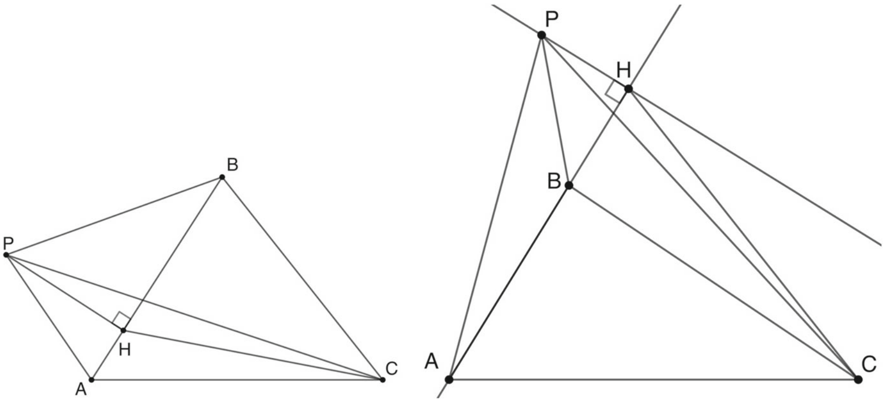

FIGURE 1 H and P satisfy $(1)_<$

Proof. In our proof it will not matter whether $H = \pi_0(P, \varphi(A, B))$ belongs or does not belong to the segment AB (see Fig. 1).

First, we will compare AP with AH and BP with BH. In $\triangle PHA$ and $\triangle PHB$—cone of which may be degenerate if $H = A$ or $H = B$—where AH, BH are legs and AP, BP are hypotenuses, so, by (8), we get $AH < AP$ and $BH < BP$, regardless of whether one of the triangles is non-degenerate or not. Now consider $\triangle PHC$. Since $\widehat{PHC}$ includes right angle $\widehat{PHB}$ or $\widehat{PHA}$, it is, by our definition, obtuse, so, by (10), $CH < CP$. □

The following two theorems represent a strengthening of [1, Theorem 11, p. 13], which states, inside Euclidean geometry, that "P lies outside the (closed) triangle ABC if and only if there exists $Q \neq P$ satisfying (1)."

Theorem 2. *For any point P inside or on the boundary of triangle ABC, there is no point Q, different from P, such that Q and P satisfy* (1).

Proof. If $P \neq Q$, the validity of (1) is equivalent with the statement that the closed triangle ABC is contained in the closed halfplane determined by $\mu(Q, P)$ which contains Q. If P is inside or on the boundary of $\triangle ABC$, then, for (1) to hold, P would have to lie in the closed halfplane determined by $\mu(Q, P)$ which contains Q. Yet this would mean that both P and Q lie in the same closed halfplane determined by $\mu(Q, P)$, which is not possible, for, by A13–A15, A5, and (6), P and Q lie on different sides of $\mu(Q, P)$.

Theorem 3. *For every point P outside of triangle ABC there exists a point Q inside or on the boundary of triangle ABC, such that Q and P satisfy* $(1)_<$.

Proof. By the axioms of ordered geometry, one of the vertices of $\triangle ABC$ and P lie on opposite sides of the line determined by the other two vertices of

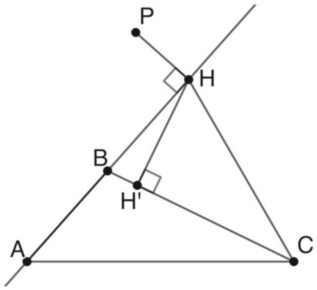

Figure 2 Both $\widehat{HBC}$ and $\widehat{HCB}$ are acute

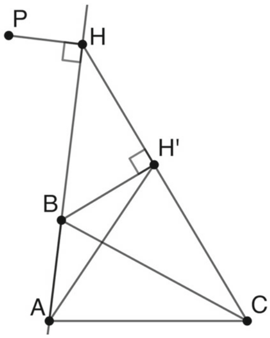

Figure 3 $\widehat{HBC}$ is not acute

$\triangle ABC$. We may assume that vertex is C, so the line determined by the other two vertices is $\varphi(A, B)$.

Let $H = \pi_0(P, \varphi(A, B))$. There are two possibilities: (i) H belongs or (ii) H does not belong to the segment AB. In case (i), we are done, since, by Lemma 1, the points H and P satisfy $(1)_<$, and H lies on the boundary of $\triangle ABC$.

In case (ii), we still get from Lemma 1 that H and P satisfy $(1)_<$. Considering $\triangle HBC$, we notice that there are three possible cases.

1. Both $\widehat{HBC}$ and $\widehat{HCB}$ are acute. Let $H' = \pi_0(H, \varphi(B, C))$ (see Fig. 2).

   We have $\zeta(B, H', C)$ (otherwise we get a contradiction by using the Crossbar Theorem and A6)) and by Lemma 1 we deduce that H' and H satisfy $1_<$. We also have that H and P satisfy $(1)_<$, so, by A17, H' and P satisfy $(1)_<$.
2. $\widehat{HBC}$ is not acute. Let $H' = \pi_0(B, \varphi(H, C))$ (see Fig. 3). By (9), the Crossbar Theorem and the linear properties of betweenness, H' belongs to the segment HC.

   Now we are going to prove that H' and H satisfy $(1)_<$. Given that $\zeta(H, H', C)$, by (3), we have $CH' < CH$. Since BH is the hypotenuse in

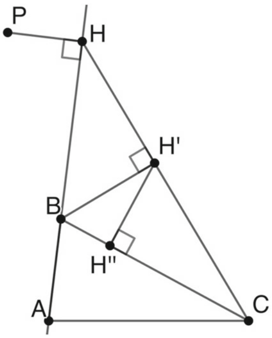

FIGURE 4 H'' and P satisfy $(1)_<$

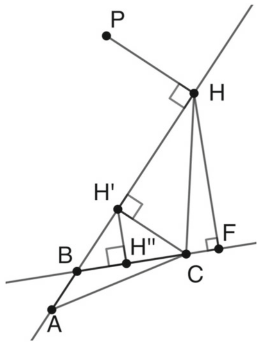

FIGURE 5 H'' and P satisfy $(1)_<$

$\triangle HH'B$, by (8), $BH' < BH$. Given $\zeta(A, B, H)$ and the definition of H', $\widehat{HH'A}$ is obtuse, implying $AH' < AH$ by (10), so we get that H' and H satisfy $(1)_<$. We know that H and P also satisfy $(1)_<$, so, by A17, H' and P also satisfy $(1)_<$.

Let $H'' = \pi_0(H', \varphi(B, C))$ (see Fig. 4). By (9), we have $\zeta(B, H'', C)$, so, by Lemma 1, H'' and H' satisfy $(1)_<$. Since H' and P satisfy $(1)_<$, by A17, H'' and P satisfy $(1)_<$.

3. $\widehat{HCB}$ is not acute. Let $H' = \pi_0(C, \varphi(B, H))$. Since $\widehat{HCB}$ is not acute, by the Crossbar Theorem, (9), and the linear order axioms, we have $\zeta(B, H', H)$. Thus $BH' < BH$ and $AH' < AH$. CH being the hypotenuse of $\triangle HH'C$, we have, by (8), $CH' < CH$, so we get that H' and H satisfy $(1)_<$. Since H and P also satisfy $(1)_<$, by A17, H' and P also satisfy $(1)_<$.

   Let $H'' = \pi_0(H', \varphi(B, C))$ (see Fig. 5). By (9), $\zeta(B, H'', C)$, and by Lemma 1 we have that H'' and H' satisfy $(1)_<$. Since H' and P also satisfy $(1)_<$, we have, by A17, that H'' and P also satisfy $(1)_<$.

□

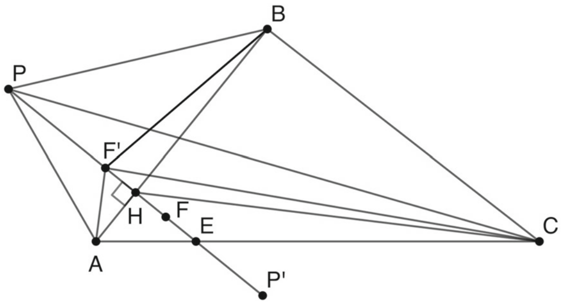

FIGURE 6 F is inside $\triangle ABC$ and F and P satisfy $(1)_<$

Thus, if ABC is a triangle and P is a point, then, by Theorem 3, if P lies outside of $\triangle ABC$, then there is a point Q inside or on the boundary of $\triangle ABC$, such that Q and P satisfy $(1)_<$, while, by Theorem 2, if P lies inside or on the boundary of $\triangle ABC$, then the only point Q such that Q and P satisfy (1) is $Q = P$ (and thus no Q and P can satisfy $(1)_<$).

We thus have in Theorems 2 and 3 characterizations of the interior points and of the exterior points of a triangle, which can be summed up as follows:

Corollary 1. *If ABC is a triangle, then a point P lies inside or on the boundary of triangle ABC if and only of there is no point Q such that Q and P satisfy $(1)_<$; a point P lies outside of triangle ABC if and only if there exists a point Q such that Q and P satisfy $(1)_<$.*

Thus Theorem 12 of [1], which is our Theorem 1, holds in Σ. It is plain that, given a point P, the proof of Theorem 3 amounts to an algorithm, a flow-chart with questions to be answered with Yes and No, that leads to the point Q in at most four steps. Put differently, Q is one of 12 terms (4 if P and C are on different sides of $\varphi(A, B)$, and 4 for each of the other two possibilities) in the variables A, B, C, and P, using only the operation symbols φ and π_0.

5. A strenghtened version of Theorem 3 holds in Σ'

Lemma 2. *If P is a point outside of triangle ABC, such that P and C on different sides of $\varphi(A, B)$ and such that $\zeta(A, \pi_0(P, \varphi(A, B)), B)$, then there exists a point Q inside the triangle ABC, such that Q and P satisfy $(1)_<$.*

Proof. Let $P' = \varrho_{\varphi(A,B)}(P)$ and $H = \pi_0(P, \varphi(A, B))$. If P' is inside $\triangle ABC$, then, we let $E = P'$. If P' is on the boundary or outside of the triangle, then by the Pasch axiom, $\varphi(P, P')$, having intersected side AB of $\triangle ABC$ in H, must also intersect AC or BC. We will denote the intersection point by E (E may be P'). Without loss of generality we may assume that $\zeta(A, E, C)$ or $E = C$.

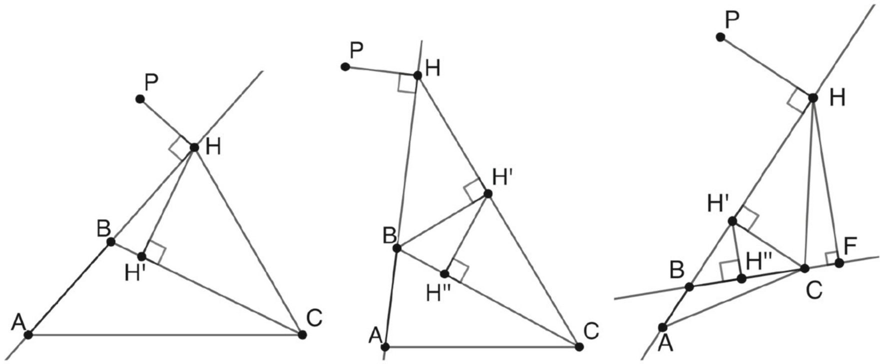

Figure 7 The cases in which H is outside of the closed segment AB

Let $F = \mu_0(H, E)$ (see Fig. 6). We will prove that Q can be chosen to be F. Let $F' = \sigma(H, F)$. By (17) and (16), we have $\zeta(H, F', P)$. Since $\zeta(F, H, F')$ (by (4) and A9) and $\zeta(H, F', P)$, we have, by the axioms of ordered geometry, $\zeta(F, F', P)$. By A18, we have $\zeta(F, \mu_0(F, F'), \mu_0(F, P))$. Since $\mu_0(F, F') = H$, we have $\zeta(F, H, \mu_0(F, P))$, so $\mu(F, P)$, having intersected the side FP of $\triangle FPA$, must, by the Pasch axiom, also intersect one of AP or AF. Since $\zeta(F, H, \mu_0(F, P))$ and $\mu_0(F, P) \in \mu(F, P)$, the line $\mu(F, P)$ cannot intersect AF, for if it did, it would have to intersect, by the Pasch axiom applied to $\triangle AHF$ with secant $\mu(F, P)$, its side AH, and from that intersection point there would be two perpendiculars to $\pi(P, \varphi(A, B))$, namely $\varphi(A, B)$ and $\mu(F, P)$, contradicting A6. Thus $\mu(F, P)$ intersects side AP, so $AF < AP$. Applying the Pasch axiom to $\triangle BPF$ with secant $\mu(F, P)$ we get, analogously, that $BF < BP$. If $E = C$, then $\zeta(C, F, P)$ and so $CF < CP$. Suppose now $\zeta(A, E, C)$. A halfline of $\pi(E, \pi(P, \varphi(A, B))$ has to lie inside $\widehat{PEC}$, for else it would, by the Crossbar Theorem, have to intersect the segment AH, and from that point of intersection there would be two perpendiculars to $\pi(P, \varphi(A, B))$, namely $\varphi(A, B)$ and $\pi(E, \pi(P, \varphi(A, B))$, contradicting A6. By the Crossbar Theorem, $\pi(E, \pi(P, \varphi(A, B))$ intersects segment PC, and, by the Pasch axiom, so does $\pi(F, \pi(P, \varphi(A, B))$, so we have $\omega(PFC)$. By (10), we have $CF < CP$. □

Theorem 4. *For every point P outside of triangle ABC there exists a point Q inside of triangle ABC, such that Q and P satisfy* $(1)_<$.

Proof. We can assume, without loss of generality, that P and C lie on opposite sides of $\varphi(A, B)$. Let $H = \pi_0(P, \varphi(A, B))$. There are three possibilities.

1. $\zeta(A, H, B)$. This is the case solved in Lemma 2.
2. $\zeta(A, B, H)$ or $\zeta(H, A, B)$.

Looking at case (ii) (H does not lie on AB) of the proof of Theorem 3, we notice that in all cases (see Fig. 7) we find a point X with

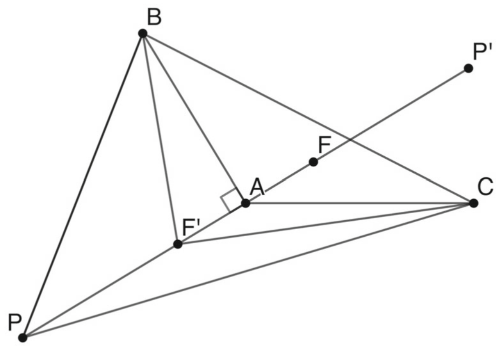

Figure 8 The open segment PP' contains a point F inside $\triangle ABC$

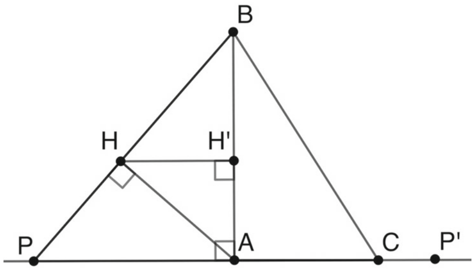

Figure 9 The open segment PP' contains points of the open segment AC

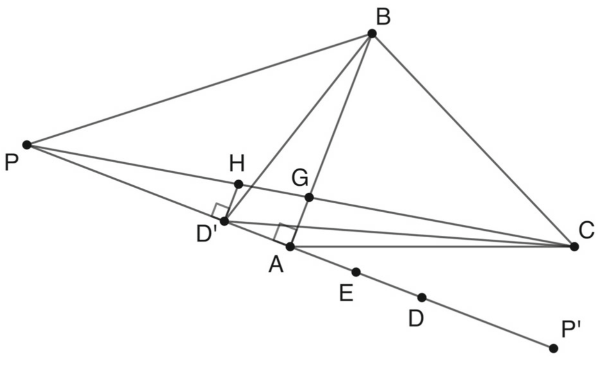

Figure 10 The intersection of the closed triangle ABC with the open segment PP' is A

$\zeta(B, \pi_0(X, \varphi(B, C)), C)$, such that X and P satisfy $(1)_<$, so we can use Lemma 2, with X playing the role of its P, to obtain a point Q in the interior of $\triangle ABC$, such that Q and X satisfy $(1)_<$. We can now use A17 to conclude that Q and P satisfy $(1)_<$.

3. $H = A$ or $H = B$. Without loss of generality we may assume that $H = A$. Let $P' = \varrho_{\varphi(A,B)}(P)$. We distinguish several cases.
   (a) The segment PP' contains at least one inner point of $\triangle ABC$. We denote by F an inner point of $\triangle ABC$ that belongs to the *open* segment PP' (see Fig. 8).

   The proof that $BF < BP$ and $CF < CP$ is the same as the proof that $AF < AP$ and $BF < BP$ in Lemma 2. That $AF < AP$ follows from the fact that $\zeta(F, F', P)$ and thus, by A18 and given that $\mu_0(F, F') = A$, $\zeta(F, A, \mu_0(F, P))$. We can thus choose Q to be F.

   (b) The segment PP' does not contain any inner point of $\triangle ABC$, but it contains a boundary point of $\triangle ABC$ which is different from A (it can only be a point from the side AC of $\triangle ABC$)

   Let $H = \pi_0(A, \varphi(B, P))$ and $H' = \pi_0(H, \varphi(A, B))$ (see Fig. 9). By (9), we have $\zeta(P, H, B)$ and $\zeta(A, H', B)$, so that H plays the role of P in Lemma 2, so there exists a Q inside $\triangle ABC$, such that Q and H satisfy $(1)_<$. We also have that $AH < AP$ by (8), $BH < BP$ since $\zeta(P, H, B)$, and $CH < CP$ by (10), since $\omega(CHP)$. By A17, Q and P satisfy $(1)_<$.

   (c) The segment PP' does not contain any interior or boundary point of $\triangle ABC$ besides A.

   Let $D = \mu_0(A, P')$, $D' = \mu_0(D, P)$, and $E = \mu_0(A, D)$. By the axioms of ordered geometry, we deduce from the fact that PP' does not contain any interior or boundary point of $\triangle ABC$ besides A that the open segment PC and $\overrightarrow{AB}$ intersect in a point G. By A18, $\zeta(P, D', A)$, and by A15 and the axioms of ordered geometry $\zeta(D, A, D')$. By the Pasch axiom and A6, $\pi(D', \varphi(P, A))$ intersects the open segment PG in H (see Fig. 10). By (3), $CD < CP$, $AD < AP$, and $BD < BP$. Analogously, we get $CE < CP$, $AE < AP$, and $BE < BP$.

   Notice that we cannot have $\varphi(P, P') \perp \varphi(A, C)$, for, if this were the case, then both $\varphi(A, C)$ and $\varphi(A, B)$ would be perpendiculars through A to $\varphi(P, P')$, contradicting A6. Nor does our hypothesis allow $\varphi(P, P') = \varphi(A, C)$. Thus, at most one of $\pi_0(D, \varphi(A, C))$ and $\pi_0(E, \varphi(A, C))$, can be C and none can be A. Thus, there is $X \in \{D, E\}$ for which $\pi_0(X, \varphi(A, C)) \notin \{A, C\}$. By cases 1 and 2 of this theorem, we have that there is a point Q in the interior of $\triangle ABC$, such that Q and X satisfy $(1)_<$. Since X and P also satisfy $(1)_<$, by A17, Q and P satisfy $(1)_<$.

□

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