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# A characterization of the known finite Minkowski planes in terms of Klein–Kroll types with respect to $\mathcal{G}$-translations

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**Abstract.** Klein and Kroll classified Minkowski planes with respect to subgroups of Minkowski translations. In this paper we investigate finite Minkowski planes with respect to groups of automorphisms of Klein–Kroll type at least D with respect to $\mathcal{G}$-translations. We show that type E is not possible as the type of a finite Minikowski plane and that type F characterizes the known finite Minkowski planes among finite Minkowski planes.

**Mathematics Subject Classification.** 51B20, 51E25.

**Keywords.** Minkowski plane, Minkowski translation, Klein–Kroll type.

## 1. Introduction

Klein and Kroll [14] obtained a classification of groups of automorphisms of Minkowski planes with respect to linearly transitive subgroups of Minkowski translations contained in them. This is similar to the Lenz classification for projective planes. Many results have been obtained on the so-called Klein types of Minkowski planes with respect to Minkowski homotheties, see [11,13,15–17,27,28], and various restrictions on the groups of automorphisms of the 23 feasible Klein types with respect to homotheties were obtained. In particular, 15 of the 23 types were excluded as the Klein types of the full automorphism groups of finite Minkowski planes. However, Minkowski translations have not been investigated further in finite planes. In this paper we begin to close this gap. We show that the known finite Minkowski planes are characterized by Klein–Kroll type F with respect to $\mathcal{G}$-translations, see Sect. 2 for a description of these planes and definitions. Furthermore it is shown that type E cannot occur as the Klein–Kroll type of a finite Minkowski plane.

In Sect. 2 we recall the basic definitions of Minkowski planes and some results for finite Minkowski planes. We also give a summary of the classification by Klein and Kroll of Minkowski planes with respect to $\mathcal{G}$-translations. In Sect. 3

we investigate finite Minkowski planes of type at least D. This leads to the exclusion of type E and characterization of type F.

## 2. Minkowski planes and $\mathcal{G}$-translations

A *Minkowski plane* $\mathcal{M} = (P, \mathcal{C}, \mathcal{G}_1 \cup \mathcal{G}_2)$ is an incidence structure consisting of a point set $P$, a circle set $\mathcal{C}$, elements of which are non-empty subsets of $P$, and two different partitions $\mathcal{G}_1$ and $\mathcal{G}_2$ of $P$. Members of $\mathcal{G}_1 \cup \mathcal{G}_2$ are called generators or parallel classes, and two points are called parallel if they belong to the same member of $\mathcal{G}_1 \cup \mathcal{G}_2$. The generator in $\mathcal{G}_i$ that contains $p$ is denoted by $[p]_i$, and two circles $C$ and $D$ through a point $p$ are said to touch each other at $p$ if $C \cap D = \{p\}$ or $C = D$. Furthermore, the following axioms are satisfied:

- three pairwise non-parallel points can be joined by a unique circle (joining);
- the circles which touch a fixed circle $K$ at $p \in K$ partition $P \backslash ([p]_1 \cup [p]_2)$ (touching);
- each generator meets each circle in a unique point (parallel projection);
- each generator in $\mathcal{G}_1$ intersects each generator in $\mathcal{G}_2$ in a unique point;
- there is a circle that contains at least three points (richness);

compare [9] or [23]. Minkowski planes have been treated from a purely algebraic point of view in [1]. An immediate consequence of the above axioms is that for each point $p$ of $\mathcal{M}$ the incidence structure $\mathcal{A}_p = (A_p, \mathcal{L}_p)$ whose point set $A_p$ consists of all points of $\mathcal{M}$ that are not parallel to $p$ and whose line set $\mathcal{L}_p$ consists of all restrictions to $A_p$ of circles of $\mathcal{M}$ passing through $p$ and of all generators not passing through $p$ is an affine plane, called the *derived affine plane at p*. This affine plane extends to a projective plane $\mathcal{P}_p$, which we call the *derived projective plane at p*.

A finite Minkowski plane $\mathcal{M}$ is one which has only a finite number of points. In this case the order of $\mathcal{M}$ is the order of any of its derived affine (or projective) planes. If $\mathcal{M}$ has order $n$, then each generator and each circle have $n + 1$ points and $\mathcal{M}$ has $(n + 1)^2$ points altogether. Furthermore, the plane has $n(n^2 - 1)$ circles. A simple counting argument shows that in an incidence structure with these number of points, generators and circles the axiom of touching is automatically satisfied if the other axioms are.

Every Minkowski plane $\mathcal{M}$ can be described in the following way. The point set of $\mathcal{M}$ is $\overline{\mathbb{F}} \times \overline{\mathbb{F}}$, where $\overline{\mathbb{F}} = \mathbb{F} \cup \{\infty\}$ and $\mathbb{F}$ is a coordinatizing ternary field of a derived affine plane, generators are of the form $\{x_0\} \times \overline{\mathbb{F}}$ (elements of $\mathcal{G}_1$) and $\overline{\mathbb{F}} \times \{y_0\}$ (elements of $\mathcal{G}_2$) where $x_0, y_0 \in \overline{\mathbb{F}}$. Each circle $K$ of $\mathcal{M}$ is described by a function $f_K : \overline{\mathbb{F}} \to \overline{\mathbb{F}}$ as

$$K = \{(x, f_K(x)) \mid x \in \overline{\mathbb{F}}\}.$$

The axiom of parallel projection shows that each function $f_K$ is a permutation of $\overline{\mathbb{F}}$. The axiom of joining implies that the collection of all those permutations $f_K$ is a sharply 3-transitive set of permutations of $\overline{\mathbb{F}}$. Conversely, each such incidence structure constructed from a sharply 3-transitive set of permutations of $\overline{\mathbb{F}}$ is equivalent to a more general hyperbola structure or (B*)-geometry, that is, all axioms of a Minkowski plane are satisfied except the axiom of touching. Although every finite hyperbola structure is a Minkowski plane there are infinite hyperbola structurs that are not Minkowski planes, see for example [22]. The *miquelian Minkowski plane* over a field $\mathbb{F}$ is obtained in the above fashion when the sharply 3-transitive set of permutations of $\overline{\mathbb{F}}$ is the sharply 3-transitive group $\mathrm{PGL}(2, \mathbb{F})$ of all fractional linear maps over $\mathbb{F}$.

There are many models of Minkowski planes, see for example, [23, Section 4] for planes with point set $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$. However, there is only one family of finite non-miquelian Minkowski planes known. In the above setting of 3-transitive sets of permutations, these planes are described by the finite sharply 3-transitive sets

$$G(q, \alpha) = \mathrm{PSL}(2, q) \cup ((\mathrm{PGL}(2, q) \backslash \mathrm{PSL}(2, q))\alpha)$$

acting in the usual way on $\overline{\mathbb{F}}_q$ where $q$ is a prime power and $\alpha$ is an automorphism of the Galois field $\mathbb{F}_q$ of order $q$. We denote the corresponding Minkowski plane by $\mathcal{M}(q, \alpha)$. The sets $G(q, \alpha)$ of permutations of $\overline{\mathbb{F}}_q$ are groups if and only if $\alpha$ has order at most 2; see Zassenhaus [30] or Passman [19, Theorem 20,5] for a determination of the finite sharply 3-transitive groups. These latter finite Minkowski planes are coordinatized over nearfields and were characterized geometrically by Hartmann [7, Satz 1] and Percsy [21, Theorem B] by the existence of a reflection at each circle.

The following result on finite Minkowski planes, due to Heise [8] for even order and Chen and Kaerlein [4] and independently Payne and Thas [20, Section VII] for odd order, shows that many of the principles of construction for infinite Minkowski planes do not apply for finite planes. Consequently, the number of models of finite Minkowski planes of a given order is very limited.

**Theorem 1.** *1. A finite Minkowski plane of even order is miquelian.*
*2. A finite Minkowski plane of odd order is miquelian if at least one of its derived affine planes is desarguesian.*

Note that a circle $C$ not passing through the distinguished point $p$ induces an oval in the derived projective plane $\mathcal{P}_p$ at $p$ by removing the two points $[p]_1 \cap C$ and $[p]_2 \cap C$ and adding in $\mathcal{P}_p$ the points $\omega_1$ and $\omega_2$ at infinity of the lines that come from the generators $[[p]_2 \cap C]_1$ and $[[p]_1 \cap C]_2$, respectively. The above theorem combined with the classification of finite projective planes of small orders and their ovals was used to obtain a complete description of finite Minkowski planes of orders up to 9; see [24].

**Theorem 2.** *A finite Minkowski plane of order at most 8 is miquelian. Up to isomorphisms, there are precisely two finite Minkowski planes of order 9, the miquelian plane $\mathcal{M}(9, id)$ and $\mathcal{M}(9, \alpha)$ where $\alpha$ is the unique automorphism $x \mapsto x^3$ of $\mathbb{F}_9$ of order 2. There is no finite Minkowski plane of order 6 or 10.*

An *automorphism* of a Minkowski plane is a permutation of the point set such that generators are mapped to generators and circles are mapped to circles. The collection of all automorphisms of a Minkowski plane $\mathcal{M}$ forms a group with respect to composition, the automorphism group $\mathrm{Aut}(\mathcal{M})$ of $\mathcal{M}$. A proper automorphism is one which preserves each $\mathcal{G}_i$; such an automorphism induces a permutation of $\mathcal{G}_1$ and of $\mathcal{G}_2$. The collection of all automorphisms of $\mathcal{M}$ that fix each generator in $\mathcal{G}_i$ is a normal subgroup of $\mathrm{Aut}(\mathcal{M})$, called the *kernel* $T_i$ of $\mathcal{M}$. A *central automorphism* of a Minkowski plane is an automorphism that fixes at least one point and a central collineation is induced in the derived projective plane at this fixed point.

In this paper we are solely interested in central automorphisms of $\mathcal{M}$ that induce translations in derived planes at each fixed point. In particular, we are investigating Klein-Kroll types with respect to what we shall call $\mathcal{G}$-translations (indicating that the set of fixed points is a generator, except in case of the identity). More precisely, let $G \in \mathcal{G}_i$ be a generator of a Minkowski plane $\mathcal{M}$. A *$G$-translation* of $\mathcal{M}$ is an automorphism of $\mathcal{M}$ that either fixes precisely the points of $G$ or is the identity. Note that in this case each $G$-translation is a proper automorphism and belongs to the kernel $T_{3-i}$. A group of $G$-translations of $\mathcal{M}$ is called *$G$-transitive*, if it acts transitively on each generator $H \in \mathcal{G}_{3-i}$ without the point of intersection with $G$. We say that a group of automorphisms of $\mathcal{M}$ is $G$-transitive if it contains a $G$-transitive subgroup of $G$-translations.

With respect to $\mathcal{G}$-translations Klein and Kroll obtained six types of groups of automorphisms of Minkowski planes, in fact, the more general hyperbola structures, see [14, Theorem 3.4].

**Theorem 3.** *If $\mathcal{Z} = \mathcal{Z}(\Gamma)$ denotes the set of all generators $G$ for which a group $\Gamma$ of automorphisms of a hyperbola structure is $G$-transitive, then exactly one of the following statements is valid for $\mathcal{Z}$:*

*A.* $\mathcal{Z} = \emptyset$;
*B.* $|\mathcal{Z}| = 1$;
*C.* $\mathcal{Z} = \{[p]_1, [p]_2\}$ *for some point* $p$;
*D.* $\mathcal{Z} = \mathcal{G}_1$ *or* $\mathcal{Z} = \mathcal{G}_2$;
*E.* $\mathcal{Z} = \mathcal{G}_1 \cup \{G_2\}$ *or* $\mathcal{Z} = \mathcal{G}_2 \cup \{G_1\}$ *where* $G_i \in \mathcal{G}_i$;
*F.* $\mathcal{Z} = \mathcal{G}_1 \cup \mathcal{G}_2$.

We say that a Minkowski plane is of type X if its full automorphism group is of type X. As noted in [14, Section 3], if a Minkowski plane $\mathcal{M}$ admits an improper automorphism (that is, an automorphism that interchanges $\mathcal{G}_1$ and $\mathcal{G}_2$), then $\mathcal{M}$ can only be of type A, C or F.

*Remark 1.* If a finite Minkowski plane $\mathcal{M}$ has a point $p$ such that the automorphism group of $\mathcal{M}$ is $[p]_1$- and $[p]_2$-transitive, then the derived affine plane of $\mathcal{M}$ at $p$ is a translation plane (compare [10, Theorem 4.19]), and thus $\mathcal{M}$ has order a prime power. In particular, this is the case when $\mathcal{M}$ contains a group of automorphisms of type C, E or F.

There are models of infinite (2-dimensional compact) Minkowski planes of types A, B, C, D and F; see [23, 4.5.4]. The following Proposition shows that all known finite Minkowski planes are of type F.

**Proposition 1.** *A finite Minkowski plane $\mathcal{M}(q,\alpha)$ is of type F.*

This follows from [25, Theorem 3.2] and the fact that each automorphism of a finite field of odd order is order-preserving in the sense of [25].
One can also directly verify the above proposition by observing that for each $a \in \overline{\mathbb{F}}_q$ there is a subgroup $T_a \leq \mathrm{PSL}(2,q)$ (a conjugate of the group of translations $x \mapsto x+t$ where $t \in \mathbb{F}_q$ and $\infty + t = \infty$) that acts regularly on $\overline{\mathbb{F}}_q \backslash \{a\}$. Then each of the maps $(x,y) \mapsto (\tau(x), y)$ and $(x,y) \mapsto (x, \tau(y))$ where $\tau \in T_a$, $a \in \overline{\mathbb{F}}_q$, is a $\mathcal{G}$-translation of $\mathcal{M}(q,\alpha)$. (Note that $\alpha\mathrm{PSL}(2,q) = \mathrm{PSL}(2,q)\alpha$.) More precisely, these maps are $[(a,\infty)]_1$- and $[(\infty,a)]_2$-translations, respectively. Furthermore, varying $t \in T_a$ one sees that $\mathcal{M}(q,\alpha)$ is $[(a,\infty)]_1$- and $[(\infty,a)]_2$-transitive. Thus the Minkowski plane is $G$-transitive for each $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ and thus of type F.

## 3. Finite Minkowski planes of type at least D

We begin with an observation that applies to all (not necessarily finite) Minkowski planes.

**Lemma 1.** *Let $\mathcal{M}$ be a Minkowski plane whose automorphism group is $G$-transitive for each $G \in \mathcal{G}_i$. Then the group $\Delta$ generated by all $G$-translations for $G \in \mathcal{G}_i$ acts 2-transitively on each generator not in $\mathcal{G}_i$. Furthermore, $\Delta$ is contained in the kernel $T_{3-i}$ and the stabilizer of three points no two of which are on the same generator in $\mathcal{G}_i$ is trivial.*

*Proof.* Let $\mathcal{M}$ be a Minkowski plane such that the automorphism group of $\mathcal{M}$ is $G$-transitive for each $G \in \mathcal{G}_2$. We consider the group $\Delta$ generated by all $G$-translations for $G \in \mathcal{G}_2$. Since each $G$-translation belongs to the kernel $T_1$ so does $\Delta$. Clearly the $G$-transitivity for each $G \in \mathcal{G}_2$ implies that $\Delta$ acts 2-transitively on each generator in $\mathcal{G}_1$. Note that if $\delta \in T_1$ fixes three points no two of which are on the same generator in $\mathcal{G}_2$ then $\delta$ fixes pointwise three generators in $\mathcal{G}_2$. Therefore $\delta$ fixes every circle and so must be the identity. □

In case of finite Minkowski planes there are only four possibilities for the group $\Delta$ from the previous lemma, see [5, 4.3.27]. The list of these groups, given in Theorem 4 below, also follows from the classification of finite 2-transitive effective groups; see, for example, the list of such groups with a simple socle given in [2] and those with elementary abelian socle given in [12].

**Theorem 4.** *Let $\Pi$ be a 2-transitive permutation group of degree $n+1$ such that only the identity fixes more than two points. Then one of the following is true:*

1. *$\Pi$ is sharply 2-transitive (and so is isomorphic to the group of all permutations $x \mapsto xa+b$, where $a \neq 0$, of a finite nearfield of order $n+1$).*
2. *$\Pi$ is isomorphic to an affine semilinear group $\mathrm{A\Gamma L}(1,n+1)$ where $n = 2^q - 1$ and $q$ is a prime.*
3. *$\Pi$ contains $\mathrm{PSL}(2,n)$ as a normal subgroup of index at most 2.*
4. *$\Pi$ is isomorphic to a Suzuki group $\mathrm{Sz}(2^{2r+1})$ where $n = 2^{2(2r+1)} \geq 64$.*

Note that in the first two cases $n+1$ is a prime power and in the latter two cases $n$ is a prime power. Also the first two cases are the only ones where $\varPi$ contains a regular normal subgroup.

*Remark 2.* When applying the above Theorem to finite Minkowski planes we are interested in the case where $n$, the order of the plane, is odd. In case 1 of Theorem 4 the degree $n+1$ then is a power of 2. Furthermore, either $n$ is not a prime power or $n$ is a Mersenne prime $n = 2^k - 1$ where $k$ necessarily is a prime, compare [19, Lemma 19.3]. From the classification of finite nearfields; see [6] and [31], one then sees that the nearfield is regular and, in fact, a field. Hence the 2-transitive group $\varPi$ is isomorphic to the affine linear group $\mathrm{AGL}(1, n+1)$ in this case.

**Proposition 2.** *Let $\mathcal{M}$ be a finite Minkowski plane whose automorphism group is $G$-transitive for each $G \in \mathcal{G}_i$. If the group $\varDelta$ generated by all $G$-translations for $G \in \mathcal{G}_i$ is non-solvable, then the order of $\mathcal{M}$ is a prime power $q$ and $\mathcal{M}$ is isomorphic to a plane $\mathcal{M}(q, \alpha)$.*

*Proof.* A miquelian Minkowski plane of order $q$ is isomorphic to $\mathcal{M}(q, id)$. Hence the proposition is true for finite Minkowski planes of even order. Let $\mathcal{M}$ be a finite Minkowski plane of odd order $n$ and assume that the automorphism group of $\mathcal{M}$ is $G$-transitive for each $G \in \mathcal{G}_2$. We consider the group $\varDelta$ generated by all $G$-translations for $G \in \mathcal{G}_2$. From Lemma 1 we know that $\varDelta$ is contained in the kernel $T_1$ and that $\varDelta$ acts 2-transitively on each generator in $\mathcal{G}_1$ such that the stabilizer of three points on a generator in $\mathcal{G}_1$ is the identity.
By Theorem 4 and because $n$ is odd and $\varDelta$ is non-solvable we see that $\varDelta$ contains a subroup $\varSigma$ isomorphic to $\mathrm{PSL}(2, n)$. In particular, $n$ must be a power of an odd prime. Moreover, $\varSigma$ has two orbits on $\mathcal{C}$.
We introduce coordinates in $\mathcal{M}$ such that the automorphisms in $\varSigma$ are given by $(x, y) \mapsto (x, \sigma(y))$ where $\sigma \in \mathrm{PSL}(2, n)$, and such that the diagonal $D = \{(x, x) \mid x \in \overline{\mathbb{F}}\}$ is a circle where $\mathbb{F} = \mathbb{F}_n$ is the Galois field of order $n$. Then the graphs

$$C_\sigma = \{(x, \sigma(x)) \mid x \in \overline{\mathbb{F}}\}$$

of members $\sigma$ in $\mathrm{PSL}(2, n)$ all describe circles in $\mathcal{M}$. The other orbit of $\varSigma$ comes from a circle $C_f = \{(x, f(x)) \mid x \in \overline{\mathbb{F}}\}$ where without loss of generality we may assume that $f$ is a permutation of $\overline{\mathbb{F}}$ that fixes $\infty$ and 0. The circles in this orbit are then of the form

$$C_{\sigma f} = \{(x, \sigma(f(x))) \mid x \in \overline{\mathbb{F}}\}$$

where $\sigma \in \mathrm{PSL}(2, n)$.
Since $n$ is odd, the set $\mathbb{F}^{(2)}$ of the non-zero squares is a subgroup of the multiplicative group $\mathbb{F}\backslash\{0\}$ of index 2, that is, $\mathbb{F}$ is a half-ordered field in the sense of [25]. For $a \in \mathbb{F}\backslash\{0\}$ we write $a > 0$ if $a \in \mathbb{F}^{(2)}$ and $a < 0$ otherwise. Furthermore, $\mathrm{PSL}(2, n)$ consists of order-preserving permutations of $\overline{\mathbb{F}}$, that is,

in the notation from [25] one has for all pairwise distinct $x_1, x_2, x_3 \in \overline{\mathbb{F}}$ that $\varepsilon(\sigma(x_1), \sigma(x_2), \sigma(x_3))\varepsilon(x_1, x_2, x_3)^{-1} > 0$ where

$$\varepsilon(x_1, x_2, x_3) = \begin{cases} (x_1 - x_2)(x_2 - x_3)(x_3 - x_1), & \text{if } x_1, x_2, x_3 \neq \infty \\ x_3 - x_2, & \text{if } x_1 = \infty \\ x_1 - x_3, & \text{if } x_2 = \infty \\ x_2 - x_1, & \text{if } x_3 = \infty \end{cases}.$$

The circles $C_\sigma$ are the same as in the miquelian Minkowski plane $\mathcal{M}(n, \mathrm{id})$ of order $n$. In this miquelian plane one has that positive triples of points, that is, points $(x_i, y_i)$, $i = 1, 2, 3$, such that $\varepsilon(y_1, y_2, y_3)\varepsilon(x_1, x_2, x_3)^{-1} > 0$, are on circles described by permutations in $\mathrm{PSL}(2, n)$ and negative triples of points, that is, points $(x_i, y_i)$, $i = 1, 2, 3$, such that $\varepsilon(y_1, y_2, y_3)\varepsilon(x_1, x_2, x_3)^{-1} < 0$, are on circles described by permutations in $\mathrm{PGL}(2, n)\backslash\mathrm{PSL}(2, n)$. Hence the circles $C_\sigma$ cover all positive triples of points, and the circles $C_{\sigma f}$ must therefore cover all negative triples of points $(x_i, y_i)$. In particular, all triples of points $(x_i, f(x_i))$ of the circle $C_f$ are negative. Hence $f$ is order-reversing. But then $\mathcal{M}$ is isomorphic to a Minkowski plane $\mathcal{M}(\mathbb{F}, \sigma_0 f, id)$ in the notation from [25] where $\sigma_0$ is a fixed element of $\mathrm{PGL}(2, n)\backslash\mathrm{PSL}(2, n)$. (In [25] only order-preserving permutations where used to describe the Minkowski planes.) We choose $\sigma_0$ such that it also fixes $\infty$ and 0. Then $\sigma_0 f$ is an order-preserving permutation which fixes $\infty$ and 0. By [3] such a permutation is an automorphism$\alpha$ of $\mathbb{F}$. Hence $\mathcal{M}$ is of the form $\mathcal{M}(n, \alpha)$. □

*Remark 3.* The case where the group $\Delta$ is solvable leads to a Minkowski plane of prime order or one whose order is not a prime power, compare Remark 2.

**Theorem 5.** *Let $\mathcal{M}$ be a finite Minkowski plane whose automorphism group contains a group of type D. If at least one derived affine plane of $\mathcal{M}$ is an affine translation plane then $\mathcal{M}$ is isomorphic to a plane $\mathcal{M}(q, \alpha)$.*

*Proof.* By Heise's result (Theorem 1.1) and Proposition 2 we only need to consider the case of a finite Minkowski plane $\mathcal{M}$ of odd order $n$ and assume that the group $\Delta$ generated by all $G$-translations for $G \in \mathcal{G}_2$ is solvable. Theorem 4 then shows that $n = 2^m - 1$ where $m \geq 2$.
Since by assumption there is a point at which the derived affine plane of $\mathcal{M}$ is an affine translation plane, the order $n$ of $\mathcal{M}$ is a power of some prime $r$. Hence $n = r^e = 2^m - 1$ for some $e \geq 1$. It is well known, see [19, Lemma 19.3], that this equation implies $e = 1$ so that $n = r$ is a Mersenne prime. In particular, the order of $\mathcal{M}$ is an odd prime. But a translation plane of prime order is desarguesian; see [18, Theorem 1.13]. Therefore $\mathcal{M}$ is miquelian by Theorem 1.2, that is, $\mathcal{M} \cong \mathcal{M}(n, id)$. □

*Remark 4.* In the above Theorem we do not assume that each translation of the derived affine plane that is a translation plane is induced by an automorphism of the Minkowski plane. Indeed, in the proof we only used the fact that a translation plane has prime power order and that a translation plane of prime order is desarguesian.

If the Prime Power Conjecture for finite projective planes is true (see, for example, [29] for a survey on the Prime Power Conjecture for projective planes and various other geometries), the first step would be covered without the need for translation planes. It is also a longstanding conjecture that a projective plane of prime order is desarguesian. Hence, if these two conjectures are true, then Theorem 5 can be strenghtened to the following. A finite Minkowski plane whose automorphism group contains a group of type D is isomorphic to a plane $\mathcal{M}(q, \alpha)$. Moreover, this then implies that there is no finite Minkowski plane of Klein–Kroll type D.

**Corollary 1.** *Let $\mathcal{M}$ be a finite Minkowski plane whose automorphism group contains a group of type E. Then $\mathcal{M}$ is isomorphic to a plane $\mathcal{M}(q, \alpha)$.*
*In particular, there is no finite Minkowski plane of Klein–Kroll type E.*

*Proof.* In type E the automorphism group of $\mathcal{M}$ is $G$-transitive for all $G \in \mathcal{G}_2$ and there also is a generator $H \in \mathcal{G}_1$ for which the automorphism group of $\mathcal{M}$ is $H$-transitive. This implies that each derived affine plane at a point of $H$ is an affine translation plane; compare Remark 1. Thus the result follows from Theorem 5 and Proposition 1. □

**Corollary 2.** *A finite Minkowski plane is of Klein–Kroll type F if and only if it is isomorphic to a plane $\mathcal{M}(q, \alpha)$.*

*Remark 5.* It remains open whether or not there are Minkowski planes of Klein–Kroll type E. By Corollary 1 they have to be infinite. In [26, Corollary 4.3] it was shown that such a plane cannot be a topological, 2-dimensional, compact Minkowski plane.

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