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The Thomsen–Bachmann correspondence in metric geometry II

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Abstract. We continue the investigations of the Thomsen–Bachmann correspondence between metric geometries and groups, which is often summarized by the phrase 'Geometry can be formulated in the group of motions'. In the first part (H. Struve and R. Struve in J Geom, 2019. https://doi.org/10.1007/s00022-018-0465-8) of this paper it was shown that the Thomsen–Bachmann correspondence can be precisely stated in a framework of first-order logic. We now prove that the correspondence, which was established by Thomsen and Bachmann for Euclidean and for plane absolute geometry, holds also for Hjelmslev geometries, Cayley–Klein geometries, isotropic and equiform geometries, and that these geometries and the theory of their group of motions are mutually faithfully interpretable (and bi-interpretable, but not definitionally equivalent). Hence a reflection-geometric axiomatization of a class of motion groups corresponds to an elementary axiomatization of the underlying geometry and provides with the calculus of reflections a powerful proof method.

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1. Introduction

We continue the investigations in [36] of the Thomsen–Bachmann correspondence between metric geometries and groups, which is often summarized by the phrase 'Geometry can be formulated in the group of motions'.[1]

In [36] we introduced the terminology of model theory, which is necessary to formulate our results, and defined the geometric notions of a *symmetric space*[2] and of a *reflection group*. These results will be used without further ado.

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[1]See [7, p. 129 and p. 134] or the preface to the first edition of [4] and [4, §2,4].

[2]Or *Thomsen–Bachmann symmetric spaces* to indicate that these structures do not coincide with the so-called *Riemannian symmetric spaces* in differential geometry.

We then studied the relationship between symmetric spaces and reflection groups and showed that the theory $\mathcal{T}'$ of reflection groups is a *conservative extension* of the theory $\mathcal{T}$ of symmetric spaces (i.e., there is an interpretation δ of $\mathcal{T}$ in $\mathcal{T}'$ such that a sentence φ is a theorem of $\mathcal{T}$ if and only if the interpretation of φ is a theorem of $\mathcal{T}'$; see [36, Theorem 3.14]). This result describes conceptually the fundamental idea of the Thomsen–Bachmann correspondence, that the calculus of reflections provides a proof method for geometric theorems.

The conservative extension $\mathcal{T}'$ of $\mathcal{T}$ induces a faithful *translation* from the language of $\mathcal{T}$ to the language of $\mathcal{T}'$ (see [36, Theorem 3.15]). This theorem can be regarded as a precise formulation of the phrase 'geometry can be formulated in the group of motions'.

The paper [36] closes with a vast generalization (see [36, Theorem 3.16]): The Thomsen–Bachmann correspondence can be established not only for symmetric spaces but also for any definitional extension of a symmetric space, encompassing much more than the Euclidean and the classical non-Euclidean geometries.

In this article we will show that for a wide range of metric geometries an even stronger correspondence holds. In Sect. 2 we study the correspondence between plane absolute geometry and the associated motion groups (the so-called *Bachmann groups*).[3] We show that the theory of Bachmann groups and the theory of symmetric spaces, which satisfy very basic axioms (essentially, the existence and uniqueness of joining lines, the existence of perpendiculars, the local transport of angles and segments, and a dimension axiom) are bi-interpretable and hence mutually faithfully interpretable. In this sense they are different representations of the same theory.

In Sect. 3 we study the generalization of plane absolute geometry where the existence and uniqueness of perpendiculars hold, but not necessarily the existence and uniqueness of joining lines. These geometries and their groups of motions are studied in Hjelmslev's *Allgemeine Kongruenzlehre* [15]. We call the groups *AKL-groups* and show that the theory of AKL-groups and the associated theory of symmetric spaces are bi-interpretable (see Theorem 3.1). In this way the notion of a symmetric space allows an elementary axiomatization of Hjelmslev geometries.

In Sect. 4 we prove analogous results for plane isotropic and equiform geometry and for Cayley–Klein geometries (see Theorem 4.1 and Theorem 4.3). An axiomatization of their motion groups (see R. Struve [35]) leads in this way to an elementary axiomatization of the associated geometries.

In Sect. 5 the results of this article are summarized: The Thomsen–Bachmann correspondence between metric geometries and groups states that the theory

[3] This correspondence is studied by Bachmann [4, §2,3–§2,5] in the framework of second-order logic, where motions are defined as bijections of the collection of all points and lines.

of the group of motions is a conservative extension of the underlying geometric theory. For a wide range of metric geometries, including plane absolute geometry, isotropic geometry, equiform geometry, Hjelmslev geometries, and Cayley–Klein geometries an even stronger correspondence holds: the geometric theory and the theory of their motion groups are sententially equivalent (and bi-interpretable). In this sense they are different representations of the same theory. This may be called the *strong Thomsen–Bachmann correspondence*.

Theorem 5.3 states a general criterion for bi-interpretability of a theory of motion groups (formulated in the language of reflection groups) and the underlying geometry (formulated in the language of symmetric spaces). For bi-interpretability two conditions play an important role. The first condition refers to a property of the group G (namely, every element of G can be represented as the product of not more than n elements of $S \cup P$, for a fixed number n). The second condition refers to the *language* in which axioms are formulated (namely, if axioms are assumed in addition to the axioms of a reflection group, then they are formulated in the language of reflection groups, restricted to terms of the sorts S and P). These conditions are satisfied by all Bachmann groups, Hjelmslev groups and AKL groups and by the motion groups of the Cayley–Klein geometries.

As a supplement to the strong Thomsen–Bachmann correspondence one may ask whether bi-interpretability is the strongest relationship between the theory of motion groups and the underlying elementary geometry or whether the theories, which are bi-interpretable, are even definitionally equivalent. An application of the method of Padoa allows us to prove that this is not the case. Finally, the article closes with some remarks on the literature.

2. Plane absolute geometry

In this section we proof the following theorem:

Theorem 2.1. *The theory of Bachmann groups and the theory of symmetric spaces, which satisfy the following axioms, are bi-interpretable.*

E1′. *If* $a \perp b$ *then there exists* C *with* $C \mid a, b$.
E2′. *If* $A \mid b$ *then there exists* c *with* $c \mid A$ *and* $c \perp b$.
E3′. *If* $a \perp b$ *and* $a, b \mid C$ *then* $E\sigma_a\sigma_b = E\varrho_C$.
E4′. *For* A, B *there exists* c *with* $A, B \mid c$.
E5′. *If* $A, B \mid c, d$ *then* $A = B$ *or* $c = d$.
E6′. *If* $a, b, c \mid D$ *then there exists* d *with* $\measuredangle(b, c) \equiv \measuredangle(a, d)$.
E7′. *If* $A, B, C \mid d$ *then there exists* D *with* $\overline{BC} \equiv \overline{AD}$.
E8′. *There exist* a, b, C *with* $a \perp b$ *and* $C \nmid a, b$.

The axioms make the follwing statements: **E1′** states that orthogonal lines have a point of intersection. **E2′** states that if A and b are incident then there exists a line through A which is perpendicular to b. Axiom **E3′** states an elementary relationship between the symmetry relations σ and ϱ. Let a be a line of symmetry of E and E' and b a line of symmetry of E' and E'' and C a common point of a and b. Then C is a center of symmetry of E and E'' if a and b or orthogonal lines. This statement can be regarded as an upper dimension axiom which limits the dimension of the symmetric space to two. **E4′** and **E5′** state the *existence and uniqueness of joining lines*. **E6′** states the *local transport of angles*, i.e., every oriented angle can be laid off upon a given line which passes through the vertex of the angle.[4] **E7′** is the dual axiom of **E6′** and states the *local transport of segments* (on a given line every oriented segment can be laid off upon a given point of the line). According to **E8′** there exist two orthogonal lines a and b and a point C which is not incident with a or b.

We denote the natural translations from $L_{\mathcal{A}}$ to $L_{\mathcal{B}}$ of **E1′**, **E2′**, ..., **E8′** by **E1**, **E2**, ..., **E8**. The axioms **E1′**, **E2′**, **E4′**, **E5′** and **E8′** coincide with their natural translations, if the incidence relation and the orthogonality relation are interpreted by the stroke-relation of $L_{\mathcal{B}}$. The translation of **E3′** is the statement "*If* $a\,|\,b$ *and* $a, b\,|\,C$ *then* $E^{ab} = E^{C}$" which is, by **B6**, equivalent to the following statement:

E3. *If* $a\,|\,b$ *and* $a, b\,|\,C$ *then* $abC = 1$.

The axioms **E6′** and **E7′** correspond in $L_{\mathcal{B}}$ to the following three-reflections theorems:

E6. *If* $a, b, c\,|\,D$ *then there exists* d *with* $abc = d$.

E7. *If* $A, B, C\,|\,d$ *then there exists* D *with* $ABC = D$.

A Bachmann group (G, S, P) is defined in [4, §3,2] as a group G which is generated by a set S of involutions of G. This assumption cannot be formulated in a first-order language but it can be substituted by the following statement (since every element of G is representable as the product of two elements of $S \cup P$; see [4, §3,7]):

E0. *If* $\alpha \in G$ *then there are* a, b *with* $\alpha = ab$ *or* A, b *with* $\alpha = Ab$.

This allows us, to give a first-order axiomatization of Bachmann groups.

Theorem 2.2. *The Bachmann groups are the reflection groups which satisfy* **E0**, ..., **E8**.

Proof. According to [4, §3,2] a triplet (G, S, P) is a Bachmann group if G is a group, which is generated by an invariant subset S of involutions of G, which satisfies the axioms **E4**, **E5**, **E6** and **E8** and the following variant of **E7**:

(†) *If* $a, b, c\,|\,d$ *then there exists* e *with* $abc = e$.

[4] Hilbert [13] assumes with axiom III.4 the *global transport of angles*. Hartshorne [12, §2.9] calls Hilbert's axiom a "transporter of angles" which acts as a substitute for Euclid's constructions with a compass.

under the assumption that $P = I(S^2)$ denotes the set of involutions of G which can be represented as the product of two elements of S.[5]

If (G, S, P) is a Bachmann group then (G, S, P) is a reflection group and satisfies **E0**–**E8** according to well-known theorems (**B1**–**B5** hold since G is a group and since S and P are invariant subsets of involutions of G; **B6** holds according to [4, §3,7 Satz 19]; **B7**, **B8** and **B9** hold by [4, §3,11]; **E0** holds according to [4, §3,7]; **E1** holds since $P = I(S^2)$; **E2** holds according to [4, §3,4 Satz 4]; **E3** is an immediate implication of [4, §3,4 Satz 1]; **E4**, **E5** and **E6** hold since they are axioms of a Bachmann group; **E7** holds by [4, §3,9 Satz 24]; **E8** holds according to [4, §3,11]).

Now, conversely, let (G, S, P) be a reflection group which satisfies **E0**–**E8**. Since (G, S, P) is a reflection group, G is a group, and S and P are invariant subsets of involutions of G. According to **E0** the group G is generated by $S \cup P$. We show $P = I(S^2)$. If $a \,|\, b$ then, by **E1**, there exists C with $C \,|\, a, b$ and according to **E3** it is $ab = C$ and hence $I(S^2) \subseteq P$. For a proof of $P \subseteq I(S^2)$ let A be an arbitrary point. According to **E8** and **E4** there exists a line b through A and, by **E2**, a line c with $c \,|\, A$ and $c \,|\, b$. According to **E3** it is $A = bc$ and hence $P \subseteq I(S^2)$.

We show that (G, S, P) satisfies the axioms of a Bachmann group, i.e., **E4**, **E5**, **E6** and **E8** and the statement (†), as a variant of **E7**. The axioms **E4**, **E5**, **E6** and **E8** hold according to our assumptions.

For a proof of (†) let $a, b, c \,|\, d$. Then $ad, bd, cd \in P$ and $ad, bd, cd \,|\, d$. By **E7** there exists E with $ad \cdot bd \cdot cd = E$ and hence $abc = Ed$. If $E \neq d$ then $E \,|\, d$ and $Ed \in S$ (since $E \in I(S^2)$ and **E6** holds). Hence $abc \in S$ and (†) holds. Suppose $E = d$. Then $abc = 1$ and $a \,|\, b \,|\, c$ and $ab, bc, ca \in P$ and $ab, bc, ca \,|\, d$ (the assumption, for example, $ab = d$ leads to a contradiction since $ab = c$, but $c \,|\, d$ implies $c \neq d$). According to **E7** it is $ab \cdot bc \cdot ca \in P$ and $1 \in P$. This is a contradiction since P is a set of involutions. Hence $E \neq d$ and (†) holds. □

Let (G, S, P) be a Bachmann group. For a given number n one can introduce an equivalence relation $\simeq_n$ on the set $\{(\alpha_1, \ldots, \alpha_k) \colon 1 \leq k \leq n \text{ and } \alpha_1, \ldots, \alpha_k \in S \cup P\}$ of k-tuples over $S \cup P$ by $(\alpha_1, \ldots, \alpha_k) \simeq_n (\beta_1, \ldots, \beta_m)$ if $\alpha_1 \cdots \alpha_k = \beta_1 \cdots \beta_m$. We denote the equivalence class of $(\alpha_1, \ldots, \alpha_k)$ by $\langle \alpha_1, \ldots, \alpha_k \rangle$ or, more precisely, by $\langle \alpha_1, \ldots, \alpha_k \rangle_n$.

In a Bachmann group every element of G is representable as the product of not more than two elements of $S \cup P$ (see [4, §3,7]). Hence, for a given number $n \geq 2$, every element $\alpha \in G$ can be represented by $\langle \alpha_1, \ldots, \alpha_k \rangle$ with $\alpha_1, \ldots, \alpha_k \in S \cup P$ and $\alpha_1 \cdots \alpha_k = \alpha$. The elements of P are represented by the equivalence classes $\langle A \rangle$ with $A \in P$ and the elements of S by the equivalence classes $\langle a \rangle$ with $a \in S$. A product of elements of G corresponds in this representation of G to the product of the associated equivalence classes

[5] The existence axiom **D** for Bachmann groups (see [4, §3,2]) is equivalent with **E8** according to Struve, H. [33, Chap. I, Proposition 11].

$\langle\alpha_1\cdots\alpha_k\rangle\circ\langle\beta_1\cdots\beta_m\rangle = \langle\delta_1\cdots\delta_j\rangle$ with $(\alpha_1\cdots\alpha_k)\cdot(\beta_1\cdots\beta_m) = \delta_1\cdots\delta_j$. Obviously, the set of equivalence classes, endowed with this operation, is isomorphic to (G, S, P). Following the terminology of Button [9, Definition 5.2] we say that (G, S, P) is represented as a *quotient structure* of (G, S, P). We formulate this result as a theorem.

Theorem 2.3. *A Bachmann group (G, S, P) is isomorphic to the quotient structure of (G, S, P) with respect to the equivalence relation $\simeq_n$ on the set of k-tuples over $S \cup P$ with $1 \leq k \leq n$ (for a given number $n \geq 2$).*

Proof of Theorem 2.1. The theory $\mathcal{T}$ of symmetric spaces, which satisfy the axioms **E1′–E8′**, and the theory $\mathcal{T}'$ of Bachmann groups are bi-interpretable if there exists an interpretation δ of $\mathcal{T}$ in $\mathcal{T}'$ and an interpretation ε of $\mathcal{T}'$ in $\mathcal{T}$ such that $\mathfrak{M} \cong \mathfrak{M}^{\delta\varepsilon}$ and $\mathfrak{N} \cong \mathfrak{N}^{\varepsilon\delta}$ (for all models $\mathfrak{M}$ of $\mathcal{T}'$ and for all models $\mathfrak{N}$ of $\mathcal{T}$).

Let δ be the interpretation of $L_{\mathcal{A}}$ in $L_{\mathcal{B}}$ which is defined by the equations (3.1) of [36], i.e., the elements of S are called 'lines' and the elements of P are called 'points' and a line b is a 'line of symmetry' with respect to points A and C if $A^b = C$ and a point B is a 'center of symmetry' of lines a and c if $a^B = c$. According to Theorem 3.14 and Theorem 3.16 of [36] the interpretation δ is an interpretation of $\mathcal{T}$ in $\mathcal{T}'$.

For a definition of the interpretation ε of $L_{\mathcal{B}}$ in $L_{\mathcal{A}}$ we define (for a given number n) an equivalence relation $\approx_n$ on the set $\{(\alpha_1, \ldots, \alpha_k) : 1 \leq k \leq n$ and $\alpha_1, \ldots, \alpha_k \in \mathcal{L} \cup \mathcal{P}\}$ of k-tuples over $\mathcal{L} \cup \mathcal{P}$ by $(\alpha_1, \ldots, \alpha_k) \approx_n (\beta_1, \ldots, \beta_m)$ if $\alpha_1 \cdots \alpha_k = \beta_1 \cdots \beta_m$ where "$\cdot$" denotes the product which is defined by the equations (3.2)* of [36]. In other words, two ordered tuples of elements of $\mathcal{L} \cup \mathcal{P}$ are considered as 'equivalent' if the compositions of the associated relations σ, resp. ϱ, operate identically on $\mathcal{P}$. We denote the equivalence class of $(\alpha_1, \ldots, \alpha_k)$ by $[\alpha_1, \ldots, \alpha_k]$.

Let ε be the interpretation of $L_{\mathcal{B}}$ in $L_{\mathcal{A}}$ where the elements of G are interpreted by the equivalence classes of $\approx_2$ and the elements of P by the equivalence classes $[A]$ with $A \in \mathcal{P}$ and the elements of S by the equivalence classes $[a]$ with $a \in \mathcal{L}$ and the group operation on G by the product $[\alpha_1, \ldots, \alpha_k] \star [\beta_1, \ldots, \beta_m] = [\delta_1, \ldots, \delta_j]$ with $(\alpha_1 \cdots \alpha_k) \cdot (\beta_1 \cdots \beta_m) = \delta_1 \cdots \delta_j$.

According to Theorem 3.16 of [36], the theory $\mathcal{T}'$ is a conservative extension of $\mathcal{T}$. Hence a sentence φ, which is formulated in the language of symmetric spaces, is a theorem of $\mathcal{T}$, if the natural translation $\tau(\varphi)$ is a theorem of $\mathcal{T}'$.

Thus, Theorem 2.3 and Theorem 3.16 of [36] imply that ε is an interpretation of $\mathcal{T}'$ in $\mathcal{T}$, i.e., the axioms of a Bachmann group, given in Theorem 2.2, are derivable from the axioms of $\mathcal{T}$.

If $\mathfrak{M} = (G, S, P)$ is a model of $\mathcal{T}'$ then $\mathfrak{M} \cong \mathfrak{M}^{\delta\varepsilon}$ since the set of lines and the set of points of $\mathfrak{M}$ are bijectively mapped onto the set of lines and the set of points of $\mathfrak{M}^{\delta\varepsilon}$, and since the group operations are preserved.

If $\mathfrak{N} = (\mathcal{L}, \mathcal{P}, \sigma, \varrho)$ is a model of $\mathcal{T}$ then $\mathfrak{N} \cong \mathfrak{N}^{\varepsilon\delta}$ since the set of lines and the set of points of $\mathfrak{N}$ are bijectively mapped onto the set of lines and the set of points of $\mathfrak{N}^{\varepsilon\delta}$, and since the symmetry relations σ and ϱ are preserved.

Hence $\mathcal{T}$ and $\mathcal{T}'$ are bi-interpretable. □

3. Hjelmslev geometries

In plane absolute geometry any two points have a unique joining line. Hjelmslev [14] admitted the existence of different lines which have several points—but not all points—in common. This originated in his interest in "natural geometry" where, as he pointed out, such lines occur.

In his "Allgemeine Kongruenzlehre" [15] he studies these geometries and their group of motions, which we call *AKL-groups*. They are reflection groups (G, S, P) which satisfy the axiom system of a Bachmann group, but substitute the existence and uniqueness of joining lines by the existence and uniqueness of perpendiculars (see [15, 1. Mitteilung, p. 5]). More precisely, **E4** and **E5** are replaced by

E9. For A, b there exists c with $A, b \,|\, c$.
E10. If $A, b \,|\, c, d$ then $A = b$ or $c = d$.

These axioms correspond in $L_{\mathcal{A}}$ to the following statements

E9′. For A, b there exists c with $A \,|\, c$ and $b \perp c$.
E10′. If $A \,|\, c, d$ and $b \perp c, d$ then $\pi(A, b)$ or $c = d$.

With this definition of AKL-groups the following holds.

Theorem 3.1. *The theory of AKL-groups and the theory of symmetric spaces, which satisfy* **E1′**, **E2′**, **E3′**, **E6′**, **E7′**, **E8′**, **E9′**, **E10′** *are bi-interpretable.*

We will prove this theorem in the framework of *Hjelmslev groups*, a notion which was introduced by Bachmann [3] (see also [6]), who was interested in the natural generality of the calculus of reflections. For this reason he replaced in the axiom system for Bachmann groups the existence and uniqueness of joining lines (**E4** and **E5**) by the existence and uniqueness of perpendiculars (**E9** and **E10**) and the existence axiom **E8** by a slightly weaker axiom, which states the existence of two orthogonal lines.

Every AKL-group is a Hjelmslev group, but the converse statement does not hold, since a Hjelmslev group (G, S, P) may contain two distinct lines $a, b \in S$ which are incident with the same points $A \in P$ (i.e., $A \,|\, a$ if and only if $A \,|\, b$; this implies $A^a = A^b = A$ for all $A \in P$ with $A \,|\, a$), whereas Hjelmslev assumes in [15, 1. Mitteilung, p. 5], that there is one and only one motion $\alpha \in G$ with $\alpha \neq 1$, which leaves all points of a given line a fixed, i.e., with $A^\alpha = A$ for all $A \in P$ with $A \,|\, a$. For examples of Hjelmslev groups, which contain distinct lines which carry the same points, we refer to Bachmann [3, §3.3] and [6,

§14.6]. In a reflection group this phenomenon cannot occur, since axiom **B8** holds, which states that for a, b with $a \neq b$ there exists C with $C \mid a$ and $C \nmid b$ (see [36, Sect. 3.2]).

Theorem 3.2. *For a Hjelmslev group* (G, S, P) *are equivalent:*

(a) (G, S, P) *is a reflection group.*
(b) (G, S, P) *satisfies* **B8** *and* **E8**.

Proof. (a) $\Rightarrow$ (b): Let (G, S, P) be a Hjelmslev group, which is a reflection group. **B8** is satisfied since (G, S, P) is a reflection group. **E8** is an immediate consequence of the existence axioms **B7**, **B8** and **B9** and the existence of perpendiculars.

(b) $\Rightarrow$ (a): Let (G, S, P) be a Hjelmslev group, which satisfies **B8** and **E8**. According to the definition of a Hjelmslev group the axioms **B1**–**B5** of a reflection group hold. By **B8** two distinct lines are not incident with the same points. Hence, if $A^\alpha = A$ for all $A \in P$ then $\alpha = 1$ (according to [6, Theorem 3.10 and Theorem 9.30]). If $b^\alpha = b$ for all $b \in S$ then $\alpha \in Z(G)$ (since G is generated by S) and since there exist three lines, which have neither a common point nor a common line, it is $\alpha = 1$ (by [6, Theorem 3.29]) and **B6** holds. Axiom **B7** is a consequence of **E8**. Axiom **B8** holds according to our assumptions. For a proof of **B9** let us assume that A and B are two distinct points with the property that every line through A is incident with B. Since orthogonal lines e, f through A have a unique point of intersection, it is $A = B$, which is a contradiction to our assumption $A \neq B$. Hence **B9** holds and (G, S, P) is a reflection group. □

Theorem 3.3. *The AKL-groups are the Hjelmslev groups with* **B8** *and* **E8**.

Proof. If (G, S, P) is an AKL-group then (G, S, P) satisfies the variant of the axiom system of Bachmann groups where **E4** and **E5** are substituted by **E9** and **E10**. Hence (G, S, P) is a Hjelmslev group with **E8**. Axiom **B8** holds since every AKL-group is a reflection group.

If, conversely, (G, S, P) is a Hjelmslev group with **B8** and **E8**, then (G, S, P) is a reflection group (by Theorem 3.2) and hence a AKL-group. □

For a proof of Theorem 3.1 we proceed in the same steps as in the last section: We give a first-order axiomatization of AKL-groups (see Theorem 3.4), show that every AKL-group (G, S, P) can be represented as a quotient structure of (G, S, P) (see Theorem 3.5), and prove Theorem 3.1.

Hjelmslev groups (G, S, P) are defined in [6, §1.3] by an axiom system which starts with the assumption that G is a group which is generated by a set S of involutions of G. This cannot be formulated in a first-order language but it can be substituted by the following statement (since every element of G is representable as the product of three elements of $S \cup P$; see [6, §3.2]):

E0'. *If $\alpha \in G$ then there are a, b, c with $\alpha = abc$ or a, B, c with $\alpha = aBc$.*

This allows a first-order axiomatization of AKL-groups.

Theorem 3.4. *The AKL groups are the reflection groups which satisfy* **E0'** *and* **E1**–**E3** *and* **E6**–**E10**.

Proof. The AKL-groups (G, S, P) are the Hjelmslev groups with **B8** and **E8** (by Theorem 3.3), which are reflection groups according to Theorem 3.2. Now the proof of the theorem proceeds completely analogously to the proof of Theorem 2.2, if in that proof **E0**, **E4**, **E5** are substituted by **E0'**, **E9** and **E10**, respectively. □

Theorem 3.5. *An AKL-group (G, S, P) can be represented as the quotient structure of (G, S, P) with respect to the equivalence relation $\simeq_n$ on the set of k-tuples over $S \cup P$ with $1 \leq k \leq n$ (for a given number $n \geq 3$).*

Proof. For a proof we can refer to the proof of Theorem 2.3, which is completely analogously, if the equivalence relation in that proof is replaced by the equivalence relation of Theorem 3.5. □

Proof of Theorem 3.1. The AKL-groups (G, S, P) are the Hjelmslev groups with **B8** and **E8** (cp. Theorem 3.3), which are reflection groups according to Theorem 3.2. Now the proof of the theorem proceeds completely analogously to the proof of Theorem 2.1, if in that proof **E0**, **E4**, **E5** are substituted by **E0'**, **E9** and **E10**, respectively. □

4. Cayley–Klein geometries

Cayley [10] and Klein [18] discovered that Euclidean and non-Euclidean geometries can be introduced as geometries living inside of a projective space which is endowed with a projective metric. They recognized that from a projective point of view there are 3^n (in the n-dimensional case) 'autonomous' geometries which are in no way inferior or subservient to the Euclidean one. These geometries are commonly referred to as *Cayley–Klein geometries* (see Yaglom [39], Giering [11] and H. Struve and R. Struve [34]).

A common axiomatic characterization of the group of motions of all plane Cayley–Klein geometries[6] over fields of characteristic $\neq 2$ is given in R. Struve [35]. To this end the notion of a *Cayley–Klein group* is introduced, which generalizes the notion of a Bachmann group. The most important aspect is that the principle of duality holds: the dual of a Cayley–Klein group (which is obtained by interchanging 'points' and 'lines'; see the precise definition below) is a Cayley–Klein group, corresponding to the fact that the dual of a Cayley–Klein geometry is a Cayley–Klein geometry.

Let $\mathcal{C}$ denote the axiom system for Cayley–Klein groups (see [35, §3.1]). The group of motions of a plane Cayley–Klein geometry satisfies $\mathcal{C}$ and one of the

[6] For reasons of simplicity with the exception of the doubly-hyperbolic case.

following axioms, which are expressed in the language $L_{\mathcal{B}}^{+}$ (see Remark 3.17 of [36]):

I. For A, B there exists c with $A, B \,|\, c$.

I*. For a, b there exists C with $a, b \,|\, C$.

E. For A, b with $A \nmid b$ there is one and only one line through A which has no common point with b.

E*. For a, B with $a \nmid B$ there is one and only one point on a which has no joining line with B.

The axioms make the following statements: According to **I** any two points have a joining line. Dually, **I*** states that any two lines have a point of intersection. **E** states that through a point A, which is not incident with a line b, there is one and only one line which has no common point with b. This is the affine parallel axiom (see, e.g., Hartshorne [12]). Axiom **E*** is the dual statement of **E**, i.e., on a line a, which is not incident with a point B, there is one and only one point which has no joining line with B.

Theorem 4.1. *The Cayley–Klein groups* (G, S, P), *which satisfy one of the axioms* **I**, **I***, **E** *or* **E***, *are reflection groups and every element of* G *is representable as the product of not more than six elements of* $S \cup P$.

For a proof of this theorem we recall that a Cayley–Klein group is defined by the following axiom system $\mathcal{K}$ which consists of the Basic Assumption (that G is a group and S and P invariant subsets of involutions of G which generate G) and the following axioms

K0. If $a \,|\, b$ then $ab \in P$ and if $A \,|\, B$ then $AB \in S$.

K1. For every pair (A, b) there exists (a, B) with $a \,|\, A$ and $B \,|\, b$ and $Aa = bB$ and if $A \neq b$ then (a, B) is unique.

K2. If $A, B \,|\, c, d$ then $A = B$ or $c = d$.

K3. If $A, B, C \,|\, d$ then $ABC \in P$ and if $a, b, c \,|\, D$ then $abc \in S$.

K4. If $A \,|\, a$ and $B \,|\, b$ and $C \,|\, c$ and $Aa = Bb = Cc$ then $ABC \in P$ and $abc \in S$.

K5. There exists a quadrangle.

The axioms make the following statements (in the language $L_{\mathcal{B}}^{+}$; see Remark 3.17 of [36]): Axiom **K0** states that the product of two orthogonal line-reflections is a point-reflection and that the dual statement holds also. Axiom **K1** states the existence and uniqueness of parallel flags (which generalizes the existence and uniqueness of perpendiculars). **K2** is the axiom **E5** of Bachmann groups and states the uniqueness of joining lines. **K3** is the conjunction of the three-reflections axioms **E6** and **E7** of Bachmann groups. **K4** is a three-reflections axiom for parallel flags. According to **K5** there exists a quadrangle.

If (G, S, P) is a Cayley–Klein group, then we get by interchanging points and lines again a Cayley–Klein group (G, S', P') with $S' = P$ and $P' = S$ which we call the *dual Cayley–Klein group*.

Proof of Theorem 4.1. Let (G, S, P) be a Cayley–Klein group. We note that every line (resp. point) of a Cayley–Klein group is incident with at least two points (resp. lines) according to **K5** and **K1**. For a proof of the theorem we consider three cases.

If there exist a, b with $a\,|\,b$ then $P = I(S^2)$ and (G, S, P) satisfies the axioms of a Hjelmslev group (since **K1** and **K2** imply the existence and uniqueness of perpendiculars, and **K3** the three-reflections theorems **E6** and **E7**). Moreover, **E8** and **B8** hold according to **K5** and **K2**. Hence (G, S, P) is a reflection group (see Theorem 3.2). According to [6, §3.2] it is $G = (S \cup P)^3$.

Dually, if there exist A, B with $A \top B$ then in the dual Cayley–Klein group (G, S', P') with $S' = P$ and $P' = S$ there exist $a, b \in S'$ with $a \mid b$. Hence (G, S', P') is a reflection group and $G = (S' \cup P')^3$. This implies $G = (S \cup P)^3$. Since in the theory of reflection groups the principle of duality holds, (G, S, P) is a reflection group.

Now, let us assume that there are no $a, b \in S$ with $a\,|\,b$ and no $A, B \in P$ with $A \top B$. In this case (G, S, P) does not satisfy **I** or **I***, since the Cayley–Klein groups with axiom **I** (resp. **I***) are Bachmann groups (which contain $a, b \in S$ with $a\,|\,b$) resp. dual Bachmann groups (which contain $A, B \in P$ with $A \top B$; see [35, §4 and §5]).

Hence we can assume that (G, S, P) satisfies **E*** (if **E** is satisfied, then the dual Cayley–Klein group satisfies **E***). Let J be the set of involutions of $S \cdot P$ and $S^* = S \cup J$. Then (G, S^*, P) satisfies the axioms of a Hjelmslev group, according to [35, Theorem 7.7].[7] By [6, §3.2 Satz 3.2] it is $G = (S^* \cup P)^3 \subseteq (S \cup J \cup P)^3 \subseteq (S \cup P)^6$ and every element of G representable as the product of not more than six elements of $S \cup P$.

It remains to show that (G, S, P) is a reflection group. Since (G, S, P) is a Cayley–Klein group, S and P are invariant subsets of involutions of a group G, which is generated by $S \cup P$. Hence the axioms **B1**–**B5** of a reflection group are satisfied. **B7** is a consequence of **K5**. Axiom **B8** and **B9** hold according to **K2** and since every line (resp. point) of a Cayley–Klein group is incident with at least two points (resp. lines).

For a proof of **B6** we recall that (G, S^*, P) is a Hjelmslev group. Let $A^\alpha = A$ for all $A \in P$. For every line $e \in S$ the elements e, e^α are incident with the same elements of P (according to [6, Theorem 3.10 and §9.5]). This implies $e = e^\alpha$ (by **K2**) for all $e \in S$, and $\alpha \in Z(G)$ (since G is generated by $S \cup P$) and $\alpha = 1$ (by [6, Theorem 3.29]). If, dually, $a^\alpha = a$ for all $a \in S$ then $E^\alpha = E$ for all $E \in P$ (since every point E is the unique point of intersection of two lines $e, f \in S$ according to **K1** and **K5**). □

Theorem 4.2. *A Cayley–Klein group* (G, S, P)*, which satisfies one of the axioms* **I**, **I***, **E**, **E*** *is isomorphic to the quotient structure of* (G, S, P) *with respect*

[7] In [35, Theorem 7.7] it is assumed that the given Cayley–Klein group satisfies **E** and **E***, but for the proof of the axioms of a Hjelmslev group—which are denoted in that proof by G0, G1, G2, G5 and G6—the Euclidean parallel axiom **E** is not used.

to the equivalence relation $\simeq_n$ on the set of k-tuples over $S \cup P$ with $1 \leq k \leq n$ (for a given number $n \geq 6$).

Proof. Let (G, S, P) be a Cayley–Klein group, which satisfies one of the axioms **I**, $\mathbf{I}^*$, **E** or $\mathbf{E}^*$. Then $G \subseteq (S \cup P)^6$ according to Theorem 4.1. Hence for a proof of Theorem 4.2 we can refer to the proof of Theorem 2.3, which is completely analogously, if the equivalence relation in that proof is replaced by the equivalence relation of Theorem 4.2. □

Theorem 4.3. *The theory of Cayley–Klein groups, which satisfy one of the axioms* **I**, $\mathbf{I}^*$, **E**, $\mathbf{E}^*$*, and the theory of symmetric spaces, which satisfy one of these axioms and the following ones, are bi-interpretable.*

C1. *If $a \perp b$ then there exists C with $C \,|\, a, b$.*

C2. *If $A \top B$ then there exists c with $c \,|\, A, B$.*

C3. *If $a \perp b$ and $a, b \,|\, C$ then $E\sigma_a\sigma_b = E\varrho_C$.*

C4. *If $A \top B$ and $A, B \,|\, c$ then $e\varrho_A\varrho_B = e\sigma_c$.*

C5. *To (A, b) there exists (a, B) with $a \,|\, A$ and $B \,|\, b$ and $(A, a) \parallel (b, B)$, and if $A \neq b$ then (a, B) is unique.*

C6. *If $A, B \,|\, c, d$ then $A = B$ or $c = d$.*

C7. *If $a, b, c \,|\, D$ then there exists d with $\measuredangle(b, c) \equiv \measuredangle(a, d)$.*

C8. *If $A, B, C \,|\, d$ then there exists D with $\overline{BC} \equiv \overline{AD}$.*

C9. *If $(A, a) \parallel (B, b) \parallel (C, c)$ then there exists (D, d) with $(D, d) \parallel (A, a)$ and $\overline{BC} \equiv \overline{AD}$ and $\overline{bc} \equiv \overline{ad}$.*

C10. *There exists a quadrangle.*

The axioms make the following statements: **C1** states that orthogonal lines have a point of intersection. **C2** states that polar points have a joining line. **C3** is an upper dimension axiom which limits the dimension of the symmetric space to two. If a is a line of symmetry of E and E' and b a line of symmetry of E' and E'' and C a common point of a and b then C is a center of symmetry of E and E'' if a and b are orthogonal lines. **C4** is the dual statement of **C3**. Axiom **C5** states that to a point A and a line b there exist a line a and a point B such that (A, a) and (b, B) are parallel flags and if $A \neq b$ then a and B are unique. **C6** states the uniqueness of joining lines. **C7** states the *local transport of angles*, i.e., every oriented angle can be laid off upon a given line which passes through the vertex of the angle. **C8** states the local transport of segments. **C9** states that a class $\mathcal{F}$ of parallel flags contains with three flags (A, a), (B, b) and (C, c) a fourth flag (D, d) with $\overline{BC} \equiv \overline{AD}$ and $\overline{bc} \equiv \overline{ad}$. According to **C10** there exists a quadrangle.

Proof of Theorem 4.3. A Cayley–Klein group, which satisfies one of the axioms $\mathbf{I}, \mathbf{I}^*, \mathbf{E}$ or $\mathbf{E}^*$ is a reflection group (according to Theorem 4.1) and can be represented as a quotient structure as shown in Theorem 4.2. Now the proof of Theorem 4.3 proceeds completely analogously to the proof of Theorem 2.1. □

Table 1 *An axiomatic characterization of Cayley–Klein geometries*

	$\mathbf{I}^*$	$\mathbf{E}$	$\mathbf{H}$
$\mathbf{I}$	elliptic plane	Euclidean plane	hyperbolic plane
$\mathbf{E}^*$	co-Euclidean plane	Galilean plane	co-Minkowskian plane
$\mathbf{H}^*$	cohyperbolic plane	Minkowskian plane	–

The axiomatic characterization of the group of motions of the eight Cayley–Klein geometries is summarized in Table 1 (see [35]).

The axiom **H** is formulated in the language $L_{\mathcal{B}}^{+}$ and states that through a point A, which is not incident with a line b, there are two and only two lines which have neither a common point nor a common line with b. The axiom $\mathbf{H}^*$ is the dual statement of **H**.

- **H.** For A, b with $A \nmid b$ there are exactly two lines through A which have neither a common point nor a common perpendicular with b.
- $\mathbf{H}^*$**.** For a, B with $a \nmid B$ there are exactly two points on a, which are neither incident with a line through B, nor polar to a point which is polar to B.

The Cayley–Klein groups with axiom **I** are the Bachmann groups, i.e., the group of motions of plane absolute geometry. The Cayley–Klein groups with axiom $\mathbf{I}^*$ are the dual Bachmann groups. The Cayley–Klein groups with axiom **E** are the group of motions of *equiform geometry*. The Cayley–Klein groups with axiom $\mathbf{E}^*$ are the group of motions of *isotropic geometry* (see [35]).

The results of this section are summarized by the following theorem:

Theorem 4.4. *The theories of the group of motions of plane absolute geometry, dual absolute geometry, equiform geometry, isotropic geometry and of the Cayley–Klein geometries of Table 1 are bi-interpretable with the theories of their associated symmetric spaces.*

Proof. This is an immediate consequence of Theorem 4.3. □

Particularly, the axiomatizations of the group of motions lead to elementary axiomatizations of the underlying geometries.

5. The Thomsen–Bachmann correspondence

The Thomsen–Bachmann correspondence between geometries and their group of motions states—in the terminology of first-order logic—that the theory of the group of motions is a conservative extension of the underlying geometric theory (see Sect. 3.3 of [36]). The Thomsen–Bachmann correspondence captures in this way the fundamental idea that the calculus of reflections provides a proof method for geometric theorems.

For a wide range of metric geometries, including plane absolute geometry, isotropic geometry, equiform geometry, Hjelmslev geometries, and Cayley–Klein geometries an even stronger correspondence holds: the geometric theory and the theory of their motion groups are sententially equivalent (and bi-interpretable; see Sects. 2–4). In this sense they are different representations of the same theory. This may be called the *strong Thomsen–Bachmann correspondence*.

For bi-interpretability apparently two conditions play an important role (let n denote a fixed natural number):

$\mathbf{G_n}$. For $\alpha \in G$ there exist $\alpha_1, \ldots, \alpha_m \in S \cup P$ with $\alpha = \alpha_1 \cdots \alpha_m$ and $m \leq n$.

According to this condition every element of G can be represented as a product of not more than n elements of $S \cup P$ (for a fixed number n). This condition is well-known from *Cartan–Dieudonne theorems* (see, e.g., Snapper and Troyer [30]). We will use this condition as an axiom which we denote by $\mathbf{G_n}$ (the notation shall indicate that the elements of G can be represented by k-tuples of elements of $S \cup P$ with $k \leq n$).

Definition 5.1. A reflection group (G, S, P) is of *rank* n (or *n-reflective*)[8] if axiom $\mathbf{G_n}$ is satisfied for a fixed number n.

The second condition, which plays an important role for bi-interpretability, refers to the language in which the axioms are formulated.

(2) Axioms, which are assumed in addition to the Basic Assumption of a reflection group (**B1**—**B5**), are formulated in the language $L_{\mathcal{B}}$, restricted to terms of the sorts S and P.[9]

Condition (2) is satisfied by all Bachmann groups, Hjelmslev groups, AKL groups and Cayley–Klein groups (see Sects. 2–4).

On the other hand condition (2) is in no way necessary. So, for example, finite Hjelmslev groups (G, S, P) can be characterized by properties with respect to mobility and rigidity, such as 'To any two points there exists a motion which maps one point onto the other one' or 'The centralizer of a flag is a Klein four-group' (see Bachmann and Knüppel [5] and Bachmann [6, §6]).

We now show that the strong Thomsen–Bachmann correspondence holds for metric geometries of a wide generality.

Theorem 5.2. *Let* $\mathcal{B} \cup \{\mathbf{P}_1, \ldots, \mathbf{P}_n\}$ *be an axiom system which allows the derivation of* $\mathbf{G_n}$ *(for a natural number n) and let* $\mathbf{P}_1, \ldots, \mathbf{P}_n$ *be formulated in the laguage* $L_{\mathcal{B}}$*, restricted to terms of the sorts* S *and* P*.*

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[8] Cp. Bachmann [4, §3,7] who calls—following H. Wiener—a group "zweispiegelig" if every group element is the product of two involutions; please note that there is no first-order definition for reflection groups of finite rank.

[9] Please note that this is not a reduction of the language $L_{\mathcal{B}}$.

Then there exists an interpretation ε of $L_{\mathcal{B}}$ in $L_{\mathcal{A}}$ such that the theory of $\mathcal{B} \cup \{\mathbf{P}_1, \ldots, \mathbf{P}_n\}$ and the theory of $\mathcal{A} \cup \{\mathbf{Q}_1, \ldots, \mathbf{Q}_n\}$ are bi-interpretable, where $\mathbf{Q}_i$ denotes the interpretation of $\mathbf{P}_i$ by ε.

Proof. For a proof we can refer to the proof of Theorem 2.1 which is completely analogously. □

In other words, starting from the *geometry of involutory group elements* [4, §20,2] with a group G and two invariant subsets S and P of involutions of G, the following theorem holds.

Theorem 5.3. *Let $\mathcal{D}$ be an axiom system, which consists of the Basic Assumption (G is a group and S and P are two invariant subsets of involutions of G), and additional axioms $\mathbf{D}_1, \ldots, \mathbf{D}_n$.*

If the following conditions hold, then there exists a natural translation of $\mathcal{D}$ into an elementary axiom system $\mathcal{E}$, whose theory is sententially equivalent (and bi-interpretable) with the theory of $\mathcal{D}$:

(1) *The axioms $\mathbf{D}_1, \ldots, \mathbf{D}_n$ are formulated in the language $L_{\mathcal{B}}$, restricted to terms of the sorts S and P.*
(2) *The axioms allow the derivation of the following statements (where n denotes a fixed natural number):*
    (a) *Every $\alpha \in G$ is a product of not more than n elements of $S \cup P$.*
    (b) *G can be faithfully represented as a group of permutations of P, and, dually, as a group of permutations of S.*
    (c) *Every line (point) is determined by the set of incident points (lines).*
    (d) *There are at least two distinct lines and two distinct points.*

Proof. According to the Basic Assumption, the axioms **B1**–**B5** of a reflection group hold. According to the conditions (2), (a), (b) and (c) the axioms **B6**–**B9** are satisfied. Hence (G, S, P) is a reflection group which is of rank n (by condition (2), (a)). The theorem is a now a consequence of Theorem 5.2. □

The theorem shows that reflection-geometric axiom systems, which are at first sight axiomatizations of motion groups, also provide *elementary axiomatizations* of the underlying geometric structures and hence are much more than only a technical tool.

One may ask whether bi-interpretability is the strongest relationship between an elementary geometric theory and the theory of the associated motion groups or whether the theories of Theorems 5.2 and 5.3 are even definitionally equivalent (see Remark 2.1 of [36]). Definitional equivalence explicates the thought that two theories are mere 'notational variants' of each other or synonymous (in de Bouvère's terminology [8]). So, for example, the theory of lattices can be developed with the $\leq$-relation as primitive notion or by taking the meet and join operations as primitives. Definitionally equivalent theories are always bi-interpretable, but not vice versa (see Button [9, §5.5]).

By an application of the method of Padoa (see Tarski [37]) we will prove that the theories of Theorem 5.2 (resp. Theorem 5.3) are bi-interpretable, but not definitionally equivalent. To this end let $\mathfrak{M} = (G, S, P)$ be the algebraic model of the group of motions of the real Euclidean plane with the orthogonality constant $k = 1$, which is given in [4, §13,1] by (3×3)—matrices. We show that in this model two distinct symmetry relations σ and σ' can be defined on $P \times S \times P$.

Let $\mathfrak{A}$ denote the real affine coordinate plane with the set $\mathcal{P}$ of points and the set $\mathcal{L}$ of lines. Points are pairs (x, y) of real numbers and lines are classes of proportional triplets $[u, v, w]$ of real numbers with $(u, v) \neq (0, 0)$ (see [4, §13,1]). We show in a first step that in $\mathfrak{A}$ two distinct symmetry relations σ and σ' can be defined on $\mathcal{P} \times \mathcal{L} \times \mathcal{P}$.

The group $\mathfrak{M}$ of motions induces on $\mathcal{P} \times \mathcal{L} \times \mathcal{P}$ a symmetry relation σ (in the standard way, i.e., if A is an element of $\mathcal{P}$ and b an element of $\mathcal{L}$ then $\sigma(\mathrm{A}, \mathrm{b}, \mathrm{A}')$ if A is mapped onto A′ by the reflection in b, which is given by the associated (3×3)—matrix of S; see [4, §13,1]). Dually, $\mathfrak{M}$ induces on $\mathcal{L} \times \mathcal{P} \times \mathcal{L}$ a symmetry relation ϱ.

We now define a symmetry relation σ' on $\mathcal{P} \times \mathcal{L} \times \mathcal{P}$ which is distinct from σ. To this end let $\mathfrak{M}' = (G', S', P')$ be the algebraic model of (3×3)—matrices of the group of motions of the real Euclidean plane with the orthogonality constant $k = 2$. The Bachmann groups $\mathfrak{M}$ and $\mathfrak{M}'$ are isomorphic (since the quotient of k and k' is a square).

$\mathfrak{M}'$ induces on $\mathcal{P} \times \mathcal{L} \times \mathcal{P}$ a symmetry relation σ' (in the same way as $\mathfrak{M}$ induces σ) and, dually, on $\mathcal{L} \times \mathcal{P} \times \mathcal{L}$ a symmetry relation ϱ'. We show that $\sigma \neq \sigma'$. A point $(x, y) \in \mathcal{P}$ is mapped by the reflection in a line $[u, v, w] \in \mathcal{L}$ onto a point (x^*, y^*) which satisfies the equation (8) of [4, §13,1]. Hence the point $(0, 0)$ is mapped by the reflection in the line $[1, 1, -1]$ onto the point $(1, 1)$, if the orthogonality constant $k = 1$. If $k = 2$ then $(0, 0)$ is mapped onto the point $(\frac{4}{3}, \frac{2}{3})$. Hence $\sigma \neq \sigma'$. The relations ϱ and ϱ' are identical (see [4, §13,1]).

The structures $(\mathcal{L}, \mathcal{P}, \sigma, \varrho)$ and $(\mathcal{L}, \mathcal{P}, \sigma', \varrho')$ are symmetric spaces (according to Theorem 2.1). Since the elements of $\mathcal{L}$ and $\mathcal{P}$ can be identified with the elements of S resp. P (by the equation (11) resp. (12) in [4, §13,1]) this shows that $\mathfrak{M}$ allows expansions $(G, S, P, \sigma, \varrho)$ and $(G, S, P, \sigma', \varrho')$ with distinct symmetry relations σ and σ'. According to Padoa's method this proves that the symmetry relation σ is not definable in $\mathfrak{M}$.

$\mathfrak{M}$ is a model of a Bachmann group and hence a model of a reflection group, of a Hjelmslev group and of a Cayley–Klein group. Thus our non-definability result holds for the corresponding theories (see Sects. 2–4 and [36]).

We close this article with some remarks on the *literature*. The relationship between elementary geometries and their group of motions was of interest for many geometers as, for example, Thomsen [38], Schmidt [28], Hjelmslev [15], Bachmann [1], Klingenberg [19], Schütte [29], Sperner [32], Karzel [17], Lingenberg [20] and Pambuccian [24]. For historical and bibliographic notes we refer to Bachmann [4, §2,3] and Karzel and Kroll [16].

Most geometers used the reflection-geometric approach as a “Zeichensprache” (in the words of Hjelmslev [15, 1. Mitt., §2]) by which points and lines are ‘identified’ with reflections in points and lines. Thomsen studies this relationship more explicitly in the case of Euclidean geometry and Bachmann generalizes Thomsen’s approach to plane absolute geometry. He proves the Thomsen–Bachmann correspondence (in the framework of second-order logic) for ‘metric planes’ which satisfy the axioms of the existence and uniqueness of joining lines and perpendiculars, and a ‘reflection axiom’, which states that to every line there exists a reflection in that line (see [2] and [4, §2,3 ff]). It is worthy to note that he shows in [4, p. 40] that the class of models of metric planes and the associated class of motion groups are—in modern terminology—second-order bi-interpretable.

First-order axiomatizations of Bachmann groups and the associated planes of absolute geometry are given, e.g., by Pambuccian [22–26], Müller [21] and Sörensen [31]. The relationship between groups and plane geometry is studied, particularly, by Pambuccian [24] and Prusińska and Szczerba [27].

Our results show, in addition, that (1) the Thomsen–Bachmann correspondence can be precisely stated in the framework of first-order logic, (2) the correspondence is not restricted to plane absolute geometry but holds also for Hjelmslev geometries, Cayley–Klein geometries, isotropic geometries and equiform geometries, and (3) the group of motions and the underlying geometries are sententially equivalent, which is a much stronger notion than mutually interpretability which is commonly referred to in the literature (see Button et. al. [9, §5.5]).

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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