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The Thomsen–Bachmann correspondence in metric geometry I

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Abstract. We study in this two part paper the Thomsen–Bachmann correspondence between metric geometries and groups which is often summarized by the phrase ‘Geometry can be formulated in the group of motions’. We show that (1) the correspondence can be precisely stated in a framework of first-order logic, (2) the correspondence, which was established by Thomsen and Bachmann for Euclidean and for plane absolute geometry, holds also for Hjelmslev geometries, Cayley–Klein geometries, isotropic and equiform geometries, and (3) these geometries and the theory of their group of motions are not only mutually interpretable but also bi-interpretable. Hence a reflection-geometric axiomatization of a class of motion groups corresponds to an elementary axiomatization of the underlying geometry and provides with the calculus of reflections a powerful proof method. In the first part of the paper we introduce the fundamental logical and geometric notions and show that the Thomsen–Bachmann correspondence can be rephrased in first-order logic by ‘The theory of the group of motions is a conservative extension of the underlying geometric theory’.

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1. Introduction

There are deep connections between groups and geometries. Klein [17] studies the relationship between geometry and transformation groups in his *Erlangen Programme* (published 1872). He describes his point of view by

> Let there be given a manifold and in it a group of transformations; it is our task to investigate those properties of a figure belonging

> to the manifold that are not changed by the transformations of the group.[1]

and proposes a *unification of all geometries based on the idea that geometry should be thought of as a transformation group rather than a space* (in the words of A'Campo and Papadopoulos [1]).

Hilbert [13] and Hjelmslev [14] were the first geometers who used reflections in points and lines, and the associated group-theoretical calculus as a *mathematical tool* for a *Begründung* (and algebraic characterization) of Euclidean and hyperbolic geometry.

Thomsen [34] and Bachmann [2] carried this idea a step further. They studied the relationship between geometries and their groups of motions and discovered that in classical Euclidean and non-Euclidean geometries points and lines correspond to elements of the group of motions (the reflections in points and lines) and geometric relations (such as incidence and orthogonality) to group-theoretical relations. This *Thomsen–Bachmann correspondence* is summarized by Bachmann as follows: *"The group of motions contains an image of the properties of the plane"* and *"Geometry can be formulated in the group of motions"* (see [8, p. 129 and p. 134]).[2]

In [4, §2,3–§2,5] it is shown that the Thomsen–Bachmann correspondence holds if in the underlying geometric structure the existence and uniqueness of joining lines and perpendiculars hold and if to every line there exists a reflection in that line (which is, under these assumptions, uniquely determined).

The Thomsen–Bachmann correspondence is open to questions of the following kind:

(a) What is the *natural generality* of the correspondence? The Thomsen–Bachmann correspondence holds, for example, in all real n-dimensional Cayley–Klein geometries—in spite of the fact that two distinct points may have several or no joining line at all (see H. Struve and R. Struve [29]).[3]
(b) What is the *range* of the correspondence? Which geometrical relations, for example, correspond to group-theoretical equations and vice versa?
(c) What is the *precise meaning* of the correspondence? Since the group of motions and the underlying geometric structure are formulated in different languages, they are not isomorphic structures. In what sense are they "different representations of the same theory" (as Lingenberg noticed in the preface of [18])?
(d) Group-theoretical axiom systems are—at first sight—axiomatizations of motion groups. Do they also provide *elementary axiomatizations*[4] of the

[1] Translation by Behnke et al. [8, p. 462].
[2] See also the preface to the first edition of [4] and [4, §2,4].
[3] Bachmann raised this question in [3]: "I am not so much interested in natural geometry, but I am interested in the natural generality of the calculus of reflections".
[4] In the sense of Tarski's article *What is elementary geometry?* [33].

> underlying geometric structures (first-order axiomatizations of the associated 'group-planes')? Or are group-theoretical characterizations "mainly a technical tool, since elementary axiomatizations ... will not assume line-reflections as undefined notions" (see Schnabel [26, p. 183])?

For answering questions of this kind one needs a language and a logic to deduce consequences. As Pambuccian [21] points out, based on the work of Skolem, Hilbert and Ackermann, Gödel, and Tarski, a consensus had been reached by the end of the first half of the 20th century that "if we are interested in producing an axiomatic system, we can only use first-order logic" (in Skolem's words, cp. [12, p. 472]). Following this approach we use in this article only first-order logic.

In Sect. 2 the terminology of model theory is introduced, which is necessary to describe the relationship between theories. Theories are logically equivalent if they have the same logical consequences (and hence the same set of models). This presupposes that they are formulated in the same language. Geometric theories $\mathcal{T}$ and $\mathcal{T}'$ are often formulated in different languages. In this case there may be an interpretation of $\mathcal{T}$ in $\mathcal{T}'$. So, for example, hyperbolic geometry is interpretable in Euclidean geometry by the model of Klein. This interpretation is not faithful, since not every hyperbolic theorem is a Euclidean theorem. Theories in different languages describe the 'same' theory, if they are mutually faithfully interpretable, which is called *sentential equivalence*.

From a semantical point of view an interpretation of a theory $\mathcal{T}$ in a theory $\mathcal{T}'$ induces a functor F from the category of models of $\mathcal{T}'$ into the category of models of $\mathcal{T}$, which associates, in the above example, to every model $\mathfrak{M}$ of Euclidean geometry a model $\mathfrak{N}$ of hyperbolic geometry (the Klein model).[5]

If $\mathcal{T}$ and $\mathcal{T}'$ are mutually interpretable and if the associated functors are denoted by F and G then $\mathcal{T}$ and $\mathcal{T}'$ are *bi-interpretable*, if the composition of F and G associates to every model $\mathfrak{N}$ of $\mathcal{T}'$ a model $\mathcal{N}^{\mathrm{FG}}$ of $\mathcal{T}'$ which is isomorphic to $\mathfrak{N}$, and to every model $\mathfrak{M}$ of $\mathcal{T}$ a model $\mathcal{M}^{\mathrm{GF}}$ of $\mathcal{T}$ which is isomorphic to $\mathfrak{M}$. Bi-interpretable theories are mutually faithfully interpretable, but the converse does not hold.[6]

In Sect. 3.1 we define geometric structures which can be regarded as the most general structures for which the Thomsen–Bachmann correspondence can be established. They are expressed in a first-order language with points and lines as undefined notions and a ternary symmetry relation $\sigma(A, b, C)$ (to be interpreted as 'b is a line of symmetry of the points A and C) and the dual relation

[5] For categorical aspects we refer to Visser [36].

[6] Analogously to the well-known fact that isomorphic models are elementarily equivalent but not conversely (see Button et al. [10, p. 36] or Rothmaler [25, Prop. 6.1.3 and 8.1.1]).

$\varrho(a, B, c)$. Essentially, the axioms state the symmetry, invariance[7] and functionality[8] of the relations σ and ϱ. We call these structures *symmetric spaces.*[9] In these spaces the calculus of reflections is replaced by a calculus of relations (see Tarski [32]).

In Sect. 3.2 the corresponding groups G are defined with two invariant subsets S and P of involutory elements (to be interpreted as the sets of reflections in lines and points) and, essentially, one additional assumption, that G can be represented as a group of permutations of P resp. S. We call (G, S, P) a *reflection group.*

In Sect. 3.3 the relationship between symmetric spaces and reflection groups is studied and shown that the theory $\mathcal{T}'$ of reflection groups is a *conservative extension* of the theory $\mathcal{T}$ of symmetric spaces (i.e., there is an interpretation δ of $\mathcal{T}$ in $\mathcal{T}'$ such that a sentence φ is a theorem of $\mathcal{T}$ if and only if the interpretation of φ is a theorem of $\mathcal{T}'$; see Theorem 3.14). This result describes conceptually the fundamental idea of the Thomsen–Bachmann correspondence, that the calculus of reflections provides a proof method for geometric theorems.

The conservative extension $\mathcal{T}'$ of $\mathcal{T}$ induces a faithful *translation* from the language of $\mathcal{T}$ to the language of $\mathcal{T}'$ (see Theorem 3.15). This theorem can be regarded as a precise formulation of the phrase 'geometry can be formulated in the group of motions'.

In Sect. 3.4 we study the translation of geometric notions, such as incidence, orthogonality, segments, and angles, into the associated group-theoretical language and establish a *dictionary.* Theorem 3.16 provides a wide generalization: The Thomsen–Bachmann correspondence can be established not only for symmetric spaces but also for any definitional extensions of a symmetric space (defined with notions of the dictionary), as for example, Cayley–Klein geometries and circle geometries (see Remark 3.13).

In the second part of this article [31] we will show that for a wide range of metric geometries, including plane absolute geometry, isotropic geometry, equiform geometry, Hjelmslev geometries, and Cayley–Klein geometries an even stronger correspondence holds: the geometric theory and the theory of their motion groups are sententially equivalent (and bi-interpretable). In this sense they are different representations of the same theory. This may be called the *strong Thomsen–Bachmann correspondence.*

2. Preliminaries

In the literature on model theory various notions are defined in various ways. Our notation and terminology is that of Hodges [16], Pinter [23] and Button

[7] In the words of Schwan [27, § 3]: the mirror image of a symmetric object is symmetric.
[8] I.e., the ternary relations σ and ϱ induce binary functions.
[9] Or *Thomsen–Bachmann symmetric spaces* to indicate that these structures do not coincide with the so-called *Riemannian symmetric spaces* in differential geometry.

and Walsh [10], with a few exceptions as indicated. For example, we shall prefer to regard functions and constants as special relations and many-sorted languages as one-sorted languages with unary sort predicates.

2.1. Syntax

A many-sorted *first-order language* L is given by a *signature* Σ which defines the sort symbols, relation symbols, function symbols and constant symbols of L. Since constant symbols can be regarded as special function symbols and function symbols as special relation symbols, a signature Σ can be defined as a pair $\langle\{\sigma_1, \ldots, \sigma_m\}, \{\rho_1, \ldots, \rho_n\}\rangle$ of a set of sort symbols σ_i and a set of relation symbols ρ_k.

A many-sorted language can be converted into a one-sorted language by the addition of domain predicate symbols, one for each sort, and a corresponding modification of quantified formulas (see Monk [19] and Barrett and Halvorson [7]). Hence we can assume that the sort symbols of a signature Σ are unary relation symbols.

A *theory* $\mathcal{T}$ is a set of sentences which is closed under deduction,[10] i.e., contains with any set of sentences their consequences. A theory is (finitely) axiomatizable if there exists a finite set of sentences, called axioms, with the same consequences as $\mathcal{T}$. We denote the theory which is associated to an axiom system $\mathcal{A}$ by $\mathcal{T}_\mathcal{A}$ and write $\mathcal{T} \vdash \phi$ if a sentence ϕ is *derivable* from $\mathcal{T}$.

Theories can be formulated in different languages. Their relationship is most easily stated in terms of interpretations. An *interpretation* δ of a language L in a language L' is a set of sentences (one for each relation ρ of L which is not in L') of the following form:

$$(\forall x_1, \ldots, x_n)[\rho(x_1, \ldots, x_n) \leftrightarrow \phi(x_1, \ldots, x_n)] \tag{2.1}$$

where ϕ is a formula of L'. The sentence (2.1), which is formulated in the language $L \cup L'$, is called an *explicit definition* of ρ.[11]

We call an interpretation δ *sort-preserving* if every sort of L is interpreted as a sort of L' and if distinct sorts of L are interpreted as distinct sorts of L'.[12]

An interpretation δ induces a mapping $\varphi \to \varphi^\delta$ from formulas of $L \cup L'$ to formulas of L' which associates to φ the formula φ^δ which is obtained from φ by substituting the relations ρ of L with their defining formulas ϕ of L' [according to (2.1)]. The restriction of this mapping to formulas of L is a *translation* τ_δ from L to L'.

Let $\mathcal{T}$ be a theory of L and $\mathcal{T}'$ a theory of L' and δ an interpretation of L in L'. If $\mathcal{T}^\delta \subseteq \mathcal{T}'$ then δ is an *interpretation of* $\mathcal{T}$ *in* $\mathcal{T}'$ and we say that $\mathcal{T}$

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[10] This conditions implies that a theory $\mathcal{T}$ corresponds to the class of models of $\mathcal{T}$.

[11] For details with respect to associated *admissibility conditions*, which ensure the existence of constants and the properties of functions, see [16] or [23].

[12] In other words, the set of sorts of L can be assumed as a subset of the set of sorts of L'.

is *interpretable* in $\mathcal{T}'$, in symbols $\mathcal{T} \trianglelefteq \mathcal{T}'$. Theories $\mathcal{T}$ and $\mathcal{T}'$ are *mutually interpretable* if $\mathcal{T} \trianglelefteq \mathcal{T}'$ and $\mathcal{T}' \trianglelefteq \mathcal{T}$.

We call an interpretation δ of $\mathcal{T}$ in $\mathcal{T}'$ *faithful* or *conservative* if for every sentence φ in $\mathcal{T}$ the following holds:

$$\mathcal{T} \vdash \varphi \quad \text{if and only if} \quad \mathcal{T}' \vdash \varphi^{\delta}. \tag{2.2}$$

If δ is a faithful interpretation of $\mathcal{T}$ in $\mathcal{T}'$ and ε a faithful interpretation of $\mathcal{T}'$ in $\mathcal{T}$ then the following holds (for all sentences φ of L and ψ of L'):

$$\mathcal{T} \vdash \varphi \leftrightarrow \varphi^{\delta\varepsilon} \quad \text{and} \quad \mathcal{T}' \vdash \psi \leftrightarrow \psi^{\varepsilon\delta}. \tag{2.3}$$

Two theories are called *mutually faithfully interpretable* if each faithfully interprets the other.

These notions can be transferred from δ to the associated translation τ_δ. If δ is an interpretation of $\mathcal{T}$ in $\mathcal{T}'$ then τ_δ is called a translation from $\mathcal{T}$ in $\mathcal{T}'$ (which maps consequences of $\mathcal{T}$ into consequences of $\mathcal{T}'$). If δ is a faithful interpretation then τ_δ is called a faithful translation.

2.2. Relationships between theories

Intuitively, T and $\mathcal{T}'$ are representations of the 'same' theory if they are theories about the same objects (sorts) and if they have the same consequences.[13] In order to state this informal notion precisely, let $\mathcal{T}$ and $\mathcal{T}'$ be theories which are formulated in languages L resp. L'. We consider the cases $L = L'$ resp. $L \subsetneq L'$ resp. $L \nsubseteq L'$.

If $L = L'$ then $\mathcal{T}$ and $\mathcal{T}'$ represent the same theory if $\mathcal{T} = \mathcal{T}'$, i.e., if $\mathcal{T}$ and $\mathcal{T}'$ are *logically equivalent*.

If $L \neq L'$ and $L \subseteq L'$ then $\mathcal{T}$ and $\mathcal{T}'$ represent the same theory if they are *definitionally equivalent*, i.e., if every symbol of L' is explicitly definable in $\mathcal{T}'$ in terms of L and if the definitional extensions of $\mathcal{T}$ and $\mathcal{T}'$ to $L \cup L'$ are logically equivalent (see Hodges [16]).[14]

If $L \nsubseteq L'$ then $\mathcal{T}$ and $\mathcal{T}'$ are representations of the 'same' theory if there exists a faithful interpretation δ of $\mathcal{T}$ in $\mathcal{T}'$ and a faithful interpretation ε of $\mathcal{T}'$ in $\mathcal{T}$ which are both sort-preserving. We call $\mathcal{T}$ and $\mathcal{T}'$ in this case *mutually faithfully interpretable* (see Button et al. [10, §5.5]) or *sententially equivalent*.[15]

Remark 2.1. There exists a stronger (syntactical) criterion for the sameness of theories with $L \nsubseteq L'$ (see Pinter [23]): Theories $\mathcal{T}$ and $\mathcal{T}'$ are called *definitionally equivalent with respect to* δ *and* ε, in symbols $\mathcal{T} \equiv_{\delta,\varepsilon} \mathcal{T}'$, if there exists

[13] In the literature theories are considered as 'equivalent' with respect to different aspects, such as provability, definability or the set of models (see, e.g., Barrett and Halvorson [6]).
[14] Hence a definitional extension of a theory consists of adding some new symbols together with explicit definitions of them. Thus a definitional extension introduces only 'abbreviations' which can be eliminated at any time.
[15] Cp. Visser [35] who introduces the notion of sentential congruence.

an interpretation δ of $\mathcal{T}$ in $\mathcal{T}'$ and an interpretation ε of $\mathcal{T}'$ in $\mathcal{T}$ with

$$\mathcal{T} \cup \varepsilon \vdash \mathcal{T}' \cup \delta \quad \text{and} \quad \mathcal{T}' \cup \delta \vdash \mathcal{T} \cup \varepsilon \tag{2.4}$$

This, however, is a criterion with respect to definability. If one is interested in *aspects of provability*—which are of main interest in the investigations of the calculus of reflections (see footnote 3)—it seems appropriate to consider theories as 'equivalent' if they are mutually faithfully interpretable.

Next, we want to introduce the notion of an *extension of a theory*. Intuitively, a theory $\mathcal{T}'$, which is formulated in a language L', is an extension of a theory $\mathcal{T}$ in L if the objects of $\mathcal{T}$ are objects of $\mathcal{T}'$ and if the sentences of $\mathcal{T}$ are sentences of $\mathcal{T}'$.

If $L = L'$ then $\mathcal{T}'$ is an extension of $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$.

If $L \neq L'$ and $L \subseteq L'$ then $\mathcal{T}'$ is an extension of $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$.

If $L \nsubseteq L'$ then $\mathcal{T}'$ is an extension of $\mathcal{T}$, in symbols $\mathcal{T} \preceq \mathcal{T}'$, if there exists a faithful and sort-preserving interpretation δ of $\mathcal{T}$ in $\mathcal{T}'$.

For theorems, which are given by axiom systems, we provide the following criterium.

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be axiom systems which are formulated in languages L resp. L'. Then $\mathcal{T}_{\mathcal{B}}$ is a (conservative) extension of $\mathcal{T}_{\mathcal{A}}$ if and only if there exists a interpretation δ of L in L' with the following properties:*

(1) *The interpretation δ is sort-preserving.*
(2) *The relations of $\mathcal{A}$ are explicitly defined in $\mathcal{B}$.*
(3) *$\mathcal{A} \vdash \varphi$ if and only if $\mathcal{B} \vdash \varphi^\delta$ (for every sentence φ of $\mathcal{T}_{\mathcal{A}}$).*[16]

Proof. If $\mathcal{T}_{\mathcal{B}}$ is an extension of $\mathcal{T}_{\mathcal{A}}$ then there exists an interpretation δ of $\mathcal{T}_{\mathcal{A}}$ in $\mathcal{T}_{\mathcal{B}}$ which satisfies the conditions (1), (2) and (3) according to our definitions.

For a proof of the converse statement let δ be an interpretation of the language L in L' which satisfies (1), (2) and (3). According to (1) δ is sort-preserving. It remains to show that δ is an interpretation of the theory $\mathcal{T}_{\mathcal{A}}$ in $\mathcal{T}_{\mathcal{B}}$, that is, $\mathcal{T}_{\mathcal{A}}^\delta \subseteq \mathcal{T}_{\mathcal{B}}$. This holds since $\mathcal{A} \vdash \varphi$ implies $\mathcal{B} \vdash \varphi^\delta$ for every sentence φ of $\mathcal{T}_{\mathcal{A}}$ according to (3). The interpretation δ is faithful by (3). □

2.3. Semantics

A Σ-*structure* $\mathfrak{A}$ is an 'interpretation'[17] of the symbols of Σ (in the sense of Chang and Keisler [11]) and assigns to every sort σ a non-empty set U_σ (the *universe* of σ) and to every predicate ρ of arity $\sigma_1 \times \cdots \times \sigma_m$ a subset of $U_{\sigma_1} \times \cdots \times U_{\sigma_m}$.

[16] This implies that the axioms of $\mathcal{A}$ can be derived from the axioms of $\mathcal{B}$ if the undefined notions of $\mathcal{A}$ are defined in terms of $\mathcal{B}$.

[17] Please note that in model theory the notion of an 'interpretation' is used in different ways. In Chang and Keisler [11] an interpretation of a language L associates to every relation of L a relation of the underlying universe.

Structures $\mathfrak{A}$ and $\mathfrak{B}$ of the same signature are *isomorphic*, in symbols $\mathfrak{A} \cong \mathfrak{B}$, if there exists a family of bijections of the universes of $\mathfrak{A}$ onto the universes of $\mathfrak{B}$ which preserve the predicate relations.

$\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* if every (first-order) sentence which holds in one of these structures holds also in the other (see Tarski [33]).

By a *reduct* $\mathfrak{A}^-$ of $\mathfrak{A}$ we understand a structure which is obtained from $\mathfrak{A}$ by omitting some of the sorts or relations. $\mathfrak{A}^+$ is an *expansion* of $\mathfrak{A}$ if $\mathfrak{A}$ is a reduct of $\mathfrak{A}^+$.

If a Σ-structure $\mathfrak{A}$ *satisfies* a sentence ψ, then we write $\mathfrak{A} \models \psi$. A Σ-structure $\mathfrak{M}$ is a *model* of a Σ-theory $\mathcal{T}$ if $\mathfrak{M} \models \psi$ for all $\psi \in \mathcal{T}$. We denote the class of models of $\mathcal{T}$ by $\mathsf{Mod}(\mathcal{T})$. According to the Completeness Theorem of Gödel, a sentence ϕ of first-logic is derivable from an axiom system $\mathcal{A}$ (that is $\mathcal{A} \vdash \phi$) if and only if ϕ is satisfied by every model $\mathfrak{M}$ of $\mathcal{A}$ (that is $\mathfrak{M} \models \phi$).

The notion of an 'interpretation of a theory' is a purely syntactical one. From a semantical point of view an interpretation δ of $\mathcal{T}$ in $\mathcal{T}'$ associates to every model of $\mathcal{T}'$ a model of $\mathcal{T}$: If $\mathfrak{M}$ is a model of $\mathcal{T}'$ then there exists a unique expansion of $\mathfrak{M}$ to a model $\mathfrak{M}^*$ of $L \cup L'$ with $\mathfrak{M}^* \models \delta$ and the reduct of $\mathfrak{M}^*$ to L is a model of $\mathcal{T}$ (which is denoted by $\mathfrak{M}^\delta$). Let $\chi_\delta : \mathsf{Mod}(\mathcal{T}') \to \mathsf{Mod}(\mathcal{T}), \mathfrak{M} \mapsto \mathfrak{M}^\delta$ be the associated mapping. In the general case χ_δ is neither injective nor surjective (see Pinter [23]).

$\mathsf{Mod}(\mathcal{T})$ and $\mathsf{Mod}(\mathcal{T}')$ are called *bi-interpretable* if there exists an interpretation δ of $\mathcal{T}$ in $\mathcal{T}'$ and an interpretation ε of $\mathcal{T}'$ in $\mathcal{T}$ such that $\mathfrak{M} \cong \mathfrak{M}^{\delta\varepsilon}$ and $\mathfrak{N} \cong \mathfrak{N}^{\varepsilon\delta}$ (for all models $\mathfrak{M}$ of $\mathcal{T}$ and for all models $\mathfrak{N}$ of $\mathcal{T}'$).

Theories $\mathcal{T}$ and $\mathcal{T}'$ are called bi-interpretable if $\mathsf{Mod}(\mathcal{T})$ and $\mathsf{Mod}(\mathcal{T}')$ are bi-interpretable (see Button et al. [10, Definitions 5.4 and 5.5]).[18]

It is well-known that isomorphic structures are elementarily equivalent and that the converse does not hold (see Tarski [33]). An analogous relationship exists between bi-interpretable structures and sentential equivalence.

Theorem 2.3. *If $\mathsf{Mod}(\mathcal{T})$ and $\mathsf{Mod}(\mathcal{T}')$ are bi-interpretable by sort-preserving interpretations then $\mathcal{T}$ and $\mathcal{T}'$ are sententially equivalent.*

Proof. See Proposition 5.9 of [10]. □

According to [10, Proposition 5.10] bi-interpretability of theories is a strictly stronger notion than sentential equivalence.

[18] The mapping $\chi_{\delta\varepsilon} : \mathfrak{M} \mapsto \mathfrak{M}^{\delta\varepsilon}$ is a functor of the category of models of $\mathcal{T}$ and $\chi_{\varepsilon\delta} : \mathfrak{N} \mapsto \mathfrak{N}^{\varepsilon\delta}$ is a functor of the category of models of $\mathcal{T}'$. The relationship between $\mathcal{T}$ and $\mathcal{T}'$ can be described by properties of these functors: If $\mathfrak{M} = \mathfrak{M}^{\delta\varepsilon}$ and $\mathfrak{N} = \mathfrak{N}^{\varepsilon\delta}$ then $\mathcal{T}$ and $\mathcal{T}'$ are definitional equivalent. If $\mathfrak{M} \cong \mathfrak{M}^{\delta\varepsilon}$ and $\mathfrak{N} \cong \mathfrak{N}^{\varepsilon\delta}$ then $\mathcal{T}$ and $\mathcal{T}'$ are bi-interpretable. If $\mathfrak{M}$ and $\mathfrak{M}^{\delta\varepsilon}$ (and also $\mathfrak{N}$ and $\mathfrak{N}^{\varepsilon\delta}$) are elementarily equivalent then $\mathcal{T}$ and $\mathcal{T}'$ are sententially equivalent. For references see Pinter [23, Theorem (3.3) and (3.5)] and Button and Walsh [10, § 5.4 and § 5.5] and Visser [35].

3. Symmetric spaces and reflection groups

3.1. Symmetric spaces

We present the axiom system for symmetric spaces in a language with two sorts of individual variables (elements $a, b, c, \ldots$ of a set $\mathcal{L}$ and elements $A, B, C, \ldots$ of a set $\mathcal{P}$) and two ternary relations σ on $\mathcal{P} \times \mathcal{L} \times \mathcal{P}$ and ϱ on $\mathcal{L} \times \mathcal{P} \times \mathcal{L}$. The elements of $\mathcal{L}$ are to be interpreted as 'lines' and the elements of $\mathcal{P}$ as 'points' and $\sigma(A, b, C)$ as 'b is a line of symmetry of the points A and C' (or equivalently 'the points A and C are symmetric with respect to the line b') and, dually, $\varrho(a, B, c)$ as 'B is a center of symmetry of the lines a and c' (or equivalently 'the lines a and c are symmetric with respect to the point B').

To improve the readability of the axioms, we introduce the following abbreviations:

- $\pi(A, a) \quad :\Leftrightarrow (\forall Bb)[(\sigma(B, b, B) \wedge \sigma(B, a, C) \wedge \varrho(b, A, c)) \rightarrow \sigma(C, c, C)]$
- $\sigma(a, b, c) \quad :\Leftrightarrow (\forall A)[(\neg\pi(A, a) \wedge \sigma(A, a, A) \wedge \sigma(A, b, C)) \rightarrow \sigma(C, c, C)]$
- $\varrho(A, B, C) :\Leftrightarrow (\forall a)[(\neg\pi(A, a) \wedge \varrho(a, A, a) \wedge \varrho(a, B, c)) \rightarrow \varrho(c, C, c)]$

$\pi(A, a)$ stands for 'A and a are *polar* to each other'. The point A is called the *pole of* a, the line a is called the *polar of* A and the relation $\pi(A, a)$ the *pole–polar relation*. The intended interpretation of $\pi(A, a)$ is that points (resp. lines) are 'symmetric to A' if and only if they are 'symmetric to a' see Theorem 3.2(6)].

$\sigma(a, b, c)$ stands for 'b is a line of symmetry of a and c' and $\varrho(A, B, C)$ stands for 'B is a center of symmetry of A and C'.

We present the axioms in informal language (their formalization being straightforward) and divide them in four groups.

I. Axioms regarding the relation σ

A1. If $\sigma(A, b, A')$ then $\sigma(A', b, A)$ and if $\sigma(a, b, a')$ then $\sigma(a', b, a)$.
A2. For A, b there exists a unique element A' with $\sigma(A, b, A')$.
A3. For a, b there exists a unique element a' with $\sigma(a, b, a')$.
A4. If $\sigma(A, b, C)$ and $\sigma(A, a, A')$, $\sigma(b, a, b')$, $\sigma(C, a, C')$ then $\sigma(A', b', C')$.

II. Axioms regarding the relation ϱ

A1*. If $\varrho(a, B, a')$ then $\varrho(a', B, a)$ and if $\varrho(A, B, A')$ then $\varrho(A', B, A)$.
A2*. For a, B there exists a unique element a' with $\varrho(a, B, a')$.
A3*. For A, B there exists a unique element A' with $\varrho(A, B, A')$.
A4*. If $\varrho(a, B, c)$ and $\varrho(a, A, a')$, $\varrho(B, A, B')$, $\varrho(c, A, c')$ then $\varrho(a', B', c')$.

III. Axioms regarding the compatibility of σ and ϱ

A5. $\sigma(A, a, A)$ if and only if $\varrho(a, A, a)$.
A6. If $\sigma(A, b, C), \varrho(A, B, A'), \varrho(b, B, b'), \varrho(C, B, C')$ then $\sigma(A', b', C')$.
A6*. If $\varrho(a, B, c), \sigma(a, b, a'), \sigma(B, b, B'), \sigma(c, b, c')$ then $\varrho(a', B', c')$.

IV. Existence assumptions

A7. There exist at least a point and a line.
A8. For every line b there exist two distinct points A and A' with $\sigma(A, b, A')$.
A8*. For every point B there exist two distinct lines a and a' with $\varrho(a, B, a')$.

The axioms make the following statements: **A1** states that the symmetry of points (resp. lines) with respect to a given line is symmetric. **A2** states that for every point A and every line b there exists one and only one point A' such that b is a line of symmetry of A and A'. **A3** states that for every line a and every line b there exists one and only one line a' such that b is a line of symmetry of a and a'. **A4** states that the relation $\sigma(A, b, C)$ is preserved if A, b, C are substituted by elements A', b', C' which correspond to A, b, C by a line a of symmetry.

The axioms **A1***, **A2***, **A3***, **A4*** are the dual statements of **A1**, **A2**, **A3** and **A4**, i.e., they are obtained by interchanging the words 'point' and 'line' and the relations σ and ϱ.

A5 states that a is a line of symmetry of the pair of points A, A if and only if A is a center of symmetry of a, a. **A6** states that the relation $\sigma(A, b, C)$ is preserved if A, b, C are substituted by elements A', b', C' which correspond to A, b, C by a center B of symmetry. **A6*** is the dual statement of **A6**. According to **A7** there exist at least a point and a line. **A8** states that the symmetry of points with respect to a given line b is not the identity on $\mathcal{P}$. **A8*** is the dual statement of **A8**.

We denote the unique point F with $\sigma(E, x, F)$, for brevity, by $E\sigma_x$.

The axiom system, which we denote by $\mathcal{A}$, contains with each axiom the dual statement (obtained by interchanging 'points' and 'lines' and the relations σ and ϱ; the relation $\pi(A, a)$ is self-dual since $\sigma(A, a, A)$ and $\varrho(a, A, a)$ are, according to **A5**, equivalent). Hence the *principle of duality* holds.

Definition 3.1. An *isomorphism* of symmetric spaces $(\mathcal{L}, \mathcal{P}, \sigma, \varrho)$ and $(\mathcal{L}', \mathcal{P}', \sigma', \varrho')$ is a bijection φ from $\mathcal{L}$ to $\mathcal{L}'$ and from $\mathcal{P}$ to $\mathcal{P}'$ which preserves the relations σ and ϱ.[19]

We now study the pole–polar relation. To a line a of a symmetric space there may exist a point A with the property that two points B and C (resp. two lines b and c) are symmetric with respect to a if and only if they are symmetric with respect to A. This is the case if and only if a and A are polar to each other.

Theorem 3.2. *For a point A and a line a of a symmetric space are equivalent:*

(1) *A is a pole of a.*
(2) *$\sigma(B, a, C)$ implies $\varrho(B, A, C)$ for all points B and C.*

[19] I.e., if $\sigma(A, b, C)$ then $\sigma'(A\varphi, b\varphi, C\varphi)$ and, dually, if $\varrho(a, B, c)$ then $\varrho'(a\varphi, B\varphi, c\varphi)$ for all points A, B, C and lines a, b, c. The converse implications hold since φ is bijective and **A1** and **A1*** hold.

(3) *$\varrho(B, A, C)$ implies $\sigma(B, a, C)$ for all points B and C.*
(4) *$\varrho(b, A, c)$ implies $\sigma(b, a, c)$ for all lines b and c.*
(5) *$\sigma(b, a, c)$ implies $\varrho(b, A, c)$ for all lines b and c.*
(6) *The relation σ, restricted to $(\mathcal{P} \times \{a\} \times \mathcal{P}) \cup (\mathcal{L} \times \{a\} \times \mathcal{L})$, and the relation ϱ, restricted to $(\mathcal{L} \times \{A\} \times \mathcal{L}) \cup (\mathcal{P} \times \{A\} \times \mathcal{P})$, induce identical relations on $(\mathcal{P} \times \mathcal{P}) \cup (\mathcal{L} \times \mathcal{L})$.*

Proof. (1) $\rightarrow$ (2): Let a be a polar of A and $\sigma(B, a, C)$. We show $\varrho(B, A, C)$. Let b be a line with $\neg\pi(B, b)$ and $\varrho(b, B, b)$. If c is the line with $\varrho(b, A, c)$ then $\varrho(c, C, c)$ (since $\sigma(B, a, C)$ holds and since a is a polar of A). This shows $\varrho(B, A, C)$.

(2) $\rightarrow$ (3): Let $\varrho(B, A, C)$. If $\sigma(B, a, B')$ then by (2) it is $\varrho(B, A, B')$. By **A3*** it is $B' = C$ and $\sigma(B, a, C)$.

(3) $\rightarrow$ (1): Let (A, a) be a pair which satisfies (3) and let (B, b) be a pair with $\sigma(B, b, B)$. By **A2** and **A2*** there exist a unique point C with $\sigma(B, a, C)$ and a unique line c with $\varrho(b, A, c)$. For a proof of (1) we have to show $\sigma(C, c, C)$.

If $\varrho(B, A, B')$ then by (3) it is $\sigma(B, a, B')$. By **A2** it is $B' = C$ and $\varrho(B, A, C)$. Hence $\sigma(B, b, B)$ and $\varrho(b, A, c)$ and $\varrho(B, A, C)$ and this implies $\sigma(C, c, C)$ by **A6**.

(4) and (5) are the dual statements of (2) and (3), which are equivalent with (1). Since (1) is self-dual, the statements (4) and (5) are equivalent with (1).

Statement (6) is the conjunction of the equivalent statements (2), (3), (4) and (5) and hence equivalent with each of them. □

Theorem 3.3. *For a point A and a line a the following holds:*

(1) *$\sigma(a, a, a)$ and $\varrho(A, A, A)$*
(2) *The line a has at most one pole.*
(3) *The point A has at most one polar.*
(4) *If $\pi(A, a)$ then $\sigma(A, a, A)$ and $\varrho(a, A, a)$.*

Proof. (1) $\sigma(a, a, a)$ holds (according to our definitions), if $\sigma(A, a, A)$ and $\sigma(A, a, C)$ imply $\sigma(C, a, C)$. If $\sigma(A, a, A)$ and $\sigma(A, a, C)$ then $A = C$ (by **A2**) and hence $\sigma(C, a, C)$. The second statement of (1) is dual to the first one.

(2) Let a be a polar of A and A'. Then $\varrho(A, A, A')$ (according to our definitions) since $\varrho(b, A, b)$ implies $\sigma(b, a, b)$, according to Theorem 3.2(4), and this implies $\varrho(b, A', b)$ by Theorem 3.2, (5). Hence $\varrho(A, A, A)$ [by (1)] and $\varrho(A, A, A')$ and by **A3*** it is $A = A'$.

(3) is the dual statement of (2).

(4) Let a be a polar of A. By (1) it is $\sigma(a, a, a)$ and by Theorem 3.2(5) it is $\varrho(a, A, a)$. The second statement of (4) is dual to the first one. □

The next theorem states that the relation $\pi(A, a)$ is preserved if A, a are substituted by elements C, c which correspond to A, a by a line of symmetry or by a center of symmetry.

Theorem 3.4. *In a symmetric space the following holds:*

(1) *If* $\pi(A,a)$ *and* $\sigma(A,b,C)$ *and* $\sigma(a,b,c)$ *then* $\pi(C,c)$.
(2) *If* $\pi(A,a)$ *and* $\varrho(A,B,C)$ *and* $\varrho(a,B,c)$ *then* $\pi(C,c)$.

Proof. (1) and (2) are immediate consequences of Theorem 3.2(6) and **A4**, **A4***, **A6** and **A6***. □

Next we show that a symmetric space can be regarded as an incidence structure with orthogonality. An *incidence relation* can be introduced in the following way.

$$A\,|\,b \Leftrightarrow \neg\pi(A,b) \wedge \sigma(A,b,A) \wedge \varrho(b,A,b) \qquad (*)$$

$A \mid b$ stands for 'A and b are *incident*'. The definition is self-dual. The conditions $\sigma(A,b,A)$ and $\varrho(b,A,b)$ are equivalent (according to **A5**).

Theorem 3.5. *For a point* A *and a line* b *the following holds:*

(1) $\sigma(A,b,A)$ *is equivalent with:* $\pi(A,b)$ *or* $A\,|\,b$.
(2) $\varrho(A,b,A)$ *is equivalent with:* $\pi(A,b)$ *or* $A\,|\,b$.

Proof. (1) and (2) are immediate consequences of Theorem 3.3(4) and the definition of the incidence relation. □

Theorem 3.6. *For all lines* a,b,c *and points* A,B,C *the following holds:*

(1) $\sigma(a,b,c)$ *is equivalent with:* $A\,|\,a$ *and* $\sigma(A,b,C)$ *imply* $C\,|\,c$.
(2) $\varrho(A,B,C)$ *is equivalent with:* $a\,|\,A$ *and* $\varrho(a,B,c)$ *imply* $c\,|\,C$.

Proof. (1) Let $\sigma(a,b,c)$ and $A\,|\,a$ and $\sigma(A,b,C)$. Then $\neg\pi(A,a)$ and hence $\neg\pi(C,c)$ [by Theorem 3.4(1)]. According to our definitions $\sigma(a,b,c)$ implies $\sigma(C,c,C)$ and hence $C\,|\,c$.

For a proof of the converse let a,b,c lines such that $A\,|\,a$ and $\sigma(A,b,C)$ imply $C \mid c$. We show $\sigma(a,b,c)$, i.e., the following implication: If $\neg\pi(A,a)$ and $\sigma(A,a,A)$ and $\sigma(A,b,C)$ then $\sigma(C,c,C)$. If $\neg\pi(A,a)$ and $\sigma(A,a,A)$ then $A\,|\,a$ and hence $\sigma(A,b,C)$ implies according to our assumptions $C\,|\,c$. This proves $\sigma(a,b,c)$.

(2) is the dual statement of (1). □

Basic properties of the incidence structure of a symmetric space are summarized in the next theorem.

Theorem 3.7. *In a symmetric space the following statements hold:*

(1) *Each line is incident with at least one point and each point is incident with at least one line.*
(2) *The set of points is non-collinear and the set of lines is non-concurrent.*
(3) *If* a *and* b *are incident with the same points then* $a=b$ *and, dually, if* A *and* B *are incident with the same lines then* $A=B$.

Proof. For a proof of (1) suppose that there exists a line a which is not incident with any point A. By **A7** there exists a point and by **A8*** a line b with $b \neq a$. According to Theorem 3.6(1) it is $\sigma(a, b, a)$ and $\sigma(a, b, b)$, which is a contradiction to **A3**. The second statement of (2) is the dual statement of the first one.

(2) is a consequence of **A7** (there exist a point and a line) and of **A8** (for every line b there exists a point A which is not incident with b) and of **A8*** (for every point B there exists a line a which is not incident with B).

For a proof of (3) let a and b be lines which are incident with the same points. Then $\sigma(A, a, A)$ and $\sigma(A, b, A)$ for all A with $A \,|\, a$ and all A with $A \,|\, b$. According to Theorem 3.6(1) it is $\sigma(a, b, a)$ and $\sigma(a, b, b)$ and hence $a = b$ (according to **A3**). The second statement of (3) is dual to the first one. □

According to Theorem 3.7(3) every line a is uniquely determined by the set of points which are incident with a and every point A is uniquely determined by the set of lines which are incident with A.

We now introduce an *orthogonality relation* for lines and the dual relation for points.

$$\begin{aligned} a \perp b \quad &\Leftrightarrow \quad a \neq b \wedge \sigma(a, b, a) \\ A \top B \quad &\Leftrightarrow \quad A \neq B \wedge \varrho(A, B, A) \end{aligned} \tag{†}$$

$a \perp b$ stands for 'a and b are *orthogonal lines*' and $A \top B$ stands for 'A and B are *polar points*'. According to this definition there are no self-orthogonal lines (i.e., no lines a with $a \perp a$) and no self-polar points. The next theorem shows that the orthogonality relation is symmetric.

Theorem 3.8. *If $a \perp b$ then $b \perp a$.*

Proof. Let $a \perp b$. Then $a \neq b$ and $\sigma(a, b, a)$. It is sufficient to show $\sigma(b, a, b)$, i.e., if $\neg\pi(B, b)$ and $\sigma(B, b, B)$ and $\sigma(B, a, C)$ then $\sigma(C, b, C)$.

Let D denote the point with $\sigma(C, b, D)$. Then $\sigma(B, a, C)$ implies $\sigma(B, a, D)$ (since $\sigma(B, b, B)$ and $\sigma(a, b, a)$ and $\sigma(C, b, D)$; by **A4**) and hence $C = D$ (by **A2**) and $\sigma(C, b, C)$. □

Theorem 3.9. *If $\pi(A, a)$ then $A|b$ if and only if $a \perp b$.*

Proof. Let $\pi(A, a)$. If $A \,|\, b$ then b is not a polar of A (according to the definition of the incidence relation) and hence $a \neq b$. Since $A \,|\, b$ it is $\varrho(b, A, b)$ and by Theorem 3.2(4) it is $\sigma(b, A, b)$ and $a \perp b$.

If $a \perp b$ then $b \neq a$ and $\sigma(b, a, b)$. By Theorem 3.2(5) it is $\varrho(b, A, b)$ and hence $A \,|\, b$ [since b is not the polar a of A; see Theorem 3.3(3)]. □

Remark 3.10. The notion of a symmetric space is of a very general kind without axioms about the existence and uniqueness of joining lines, the existence of perpendiculars, the mobility of point and lines or assumptions about the order structure and dimension of the underlying space (for examples of symmetric spaces we refer to the second part [31] of the paper).

3.2. Reflection groups

The idea of a *geometry of involutory group elements* can be formulated in various ways. A general starting point is the *Basic Assumption* that a group G is given which is generated by two invariant subsets S and P of involutory elements of G (see Thomsen [34] and Bachmann [4, §20,2]). In addition it is often assumed that G can be faithfully represented as a group of permutations of $P \cup S$ (i.e., that the identity of G is the only element of G which leaves invariant every element of P and every element of S).

To capture this idea in a first-order logic, we introduce the notion of a *reflection group* axiomatically. The axiom system, denoted by $\mathcal{B}$, is formulated in a language with one sort of individual variables (elements $\alpha, \beta, \ldots$ of a set G), a binary operation "$\cdot$" on G, two unary predicates Π and Σ and a constant symbol 1. The elements of G are to be interpreted as 'collineations', $\Pi(\alpha)$ as 'α is point-reflection', $\Sigma(\alpha)$ as 'α is line-reflection', the constant 1 as 'the identity' and the operation $\alpha \cdot \beta$ as 'the composition of α with β'.

To improve the readability of the axioms, we define two subsets P and S of G and introduce the stroke relation as an abbreviation.

- $\alpha \in P \Leftrightarrow \Pi(\alpha)$
- $\beta \in S \Leftrightarrow \Sigma(\beta)$
- $\alpha \,|\, \beta \Leftrightarrow (\alpha \in P \cup S) \wedge (\beta \in P \cup S) \wedge \alpha \neq \beta \wedge \alpha \cdot \beta = \beta \cdot \alpha.$

Elements of S are denoted by lower case Latin letters $a, b, \ldots$ and elements of P by upper case letters $A, B, \ldots$. We present the axioms in informal language (their formalization being straightforward). The axioms are:

B1. If $\alpha, \beta, \gamma \in G$ then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$
B2. $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$
B3. For α there exist β with $\alpha \cdot \beta = \beta \cdot \alpha = 1$.
B4. $a^2 = 1$ and $a \neq 1$ and $A^2 = 1$ and $A \neq 1$
B5. $a^\alpha \in S$ and $A^\alpha \in P$
B6. If $A^\alpha = A$ for all $A \in P$ and $b^\alpha = b$ for all $b \in S$ then $\alpha = 1$.
B7. There exist $a, b \in S$ and $A, B \in P$ with $a \neq b$ and $A \neq B$.
B8. For a, b with $a \neq b$ there exists C with $C \,|\, a$ and $C \nmid b$.
B9. For A, B with $A \neq B$ there exists c with $A \,|\, c$ and $B \nmid c$.

The axioms **B1**–**B3** state that G is a group with identity 1. The axioms **B4** and **B5** state that S and P are invariant subsets of involutory elements of G. Axiom **B6** states that if $\alpha \in G$ leaves invariant every element of P and every element of S then $\alpha = 1$. According to **B7** there are at least two elements in S and in P. Axiom **B8** states that $a \in S$ is uniquely determined by the set of elements $A \in P$ with $A \,|\, a$ (in other words: If $A \,|\, a$ implies $A \,|\, b$ then $a = b$). The Axiom **B9** is the dual statement of **B8**.

We call the axioms **B1**–**B5** the *Basic Assumption of a reflection group*. Since we are confined to first-order logic we cannot express that the group G is generated by $S \cup P$. Axiom **B6** is a *representation axiom* which allows a proof of the following theorem.

Theorem 3.11. *For a reflection group (G, S, P) the following holds:*

- *The center $Z(G)$ of G is trivial.*
- *G is isomorphic to the group of inner automorphisms of G.*
- *G can be faithfully represented as a group of permutations*[20] *of $P \cup S$.*

Proof. Let $\alpha \in Z(G)$. Then $A^\alpha = A$ for all $A \in P$ and $b^\alpha = b$ for all $b \in S$ and $\alpha = 1$ (according to **B6**). Hence $Z(G) = \{1\}$ and G is isomorphic to the group of inner automorphisms of G. Moreover axiom **B6** implies that G can be faithfully represented as a group of permutations of P (resp. of S). □

The axioms **B7**–**B9** are basic existence assumptions. **B8** allows to proof a stronger version of **B6** (namely: *If $A^\alpha = A$ for all $A \in P$ then $\alpha = 1$*) and **B9** implies the dual statement: *If $a^\alpha = a$ for all $a \in S$ then $\alpha = 1$.*

If $A = b$ then A, b are *polar* to each other and A is called the *pole* of b and b the *polar* of A.

Axiom system $\mathcal{B}$ contains with each axiom the dual statement, which is obtained by interchanging the elements of S and the elements of P. Hence the *principle of duality* holds (in this sense).

Every model of $\mathcal{B}$ is uniquely determined by the group G (with the binary operation "$\cdot$" and identity 1) and the invariant subsets S and P of involutory elements of G. For brevity we denote a model of $\mathcal{B}$ by (G, S, P).

Definition 3.12. An *isomorphism* of reflection groups (G, S, P) and (G', S', P') is a group-isomorphism φ from G to G' satisfying $S\varphi = S'$ and $P\varphi = P'$.

Remark 3.13. To illustrate the wide generality of the notion of a reflection group we notice:

(a) The group, which is generated by $S \cup P$, may be a proper subgroup of G. So, for example, G may be the group of similarities of the Euclidean plane over a field of characteristic $\neq 2$ and S the set of line reflections and P the set of point reflections (for this approach and a characterization of groups of similarities see Praźmowski [24] and H. Struve [28]).

(b) Following Hjelmslev, Bachmann et al. the elements of S are called 'reflections in lines'. However, G may also be generated by reflections in hyperplanes of a higher-dimensional geometry (see Bachmann [4, §20,9] or by reflections in circles or cycles of a Euclidean or non-Euclidean geometry (see Benz [9] or Yaglom [37]).

(c) A slight modification of axiom system $\mathcal{B}$ would allow G to be a semigroup (a set with an associative binary operation) and hence an arbitrary set of transformations of a set X into itself, which is closed under composition.

[20] In a geometrical language G is a group of 'collineations' which permutes the set P of 'points' and the set S of 'lines', and which preserves the stroke-relation (see Remark 3.17).

This may be considered as the most general starting point of a *geometry of involutory group elements* in the sense of Bachmann [4, § 20,2].

3.3. The correspondence between the theory of symmetric spaces and the theory of reflection groups

We now study the relationship between the theory of symmetric spaces and the theory of reflection groups.

Let $\mathcal{A}$ be the axiom system for symmetric spaces of Sect. 3.1, but now represented in a one-sorted language $L_{\mathcal{A}}$ (by the introduction of domain predicate symbols $\sigma_{\mathcal{L}}$ and $\sigma_{\mathcal{P}}$ for the sorts $\mathcal{L}$ and $\mathcal{P}$ and a corresponding modification of quantified formulas) and let $\mathcal{B}$ be the axiom system for reflection groups of Sect. 3.2, represented in a one-sorted language $L_{\mathcal{B}}$ (with unary predicate symbols σ_G, σ_S and σ_P for the sorts G and S and P).

Theorem 3.14. *The theory of reflection groups is a conservative extension of the theory of symmetric spaces.*

Proof. We show that the theory $\mathcal{T}_{\mathcal{B}}$ of reflection groups is a conservative extension of the theory $\mathcal{T}_{\mathcal{A}}$ of symmetric spaces, i.e., that there exists an interpretation δ of $L_{\mathcal{A}}$ in $L_{\mathcal{B}}$ which satisfies the conditions (1), (2) and (3) of Theorem 2.2.

Let δ be the interpretation of $L_{\mathcal{A}}$ in $L_{\mathcal{B}}$ which is defined by the following sentences:

$$\begin{aligned}
&(\forall x)[\sigma_{\mathcal{L}}(x) \leftrightarrow \sigma_S(x)]\\
&(\forall x)[\sigma_{\mathcal{P}}(x) \leftrightarrow \sigma_P(x)]\\
&(\forall x,y,z)[\sigma(x,y,z) \leftrightarrow \sigma_P(x) \wedge \sigma_S(y) \wedge \sigma_P(z) \wedge x^y = z]\\
&(\forall x,y,z)[\varrho(x,y,z) \leftrightarrow \sigma_S(x) \wedge \sigma_P(y) \wedge \sigma_S(z) \wedge x^y = z]
\end{aligned} \tag{3.1}$$

In other words, the elements of S are called 'lines' and the elements of P are called 'points' and a line b is a 'line of symmetry' with respect to points A and C if $A^b = C$ and a point B is a 'center of symmetry' of lines a and c if $a^B = c$. Hence the conditions (1) and (2) of Theorem 2.2 are satisfied.

We note that the axioms **B8** and **B9** imply that $\sigma(a, b, c)$ is equivalent with $a^b = c$ and that $\varrho(A, B, C)$ is equivalent with $A^B = C$. With these definitions in mind, simple group-theoretic calculations show that the axioms **A1**–**A6** and their dual counterparts hold. **A7** holds according to **B8**. Axiom **A8** holds since for $b \in S$ there exists $a \in S$ with $a \neq b$ (by **B8**) and by **B9** there exists $A \in P$ with $A \,|\, a$ and $A \nmid b$ and $A^b \neq A$. Axiom **A8*** holds since the principle of duality holds and since **A8*** is the dual statement of **A8**. Hence the axioms of $\mathcal{A}$ can be proven from the axioms of $\mathcal{B}$ when the undefined notions of $\mathcal{A}$ are defined by (3.1) in terms of $\mathcal{B}$. In other words, δ is an interpretation of $\mathcal{T}_{\mathcal{A}}$ in $\mathcal{T}_{\mathcal{B}}$.

It remains to show that the condition (3) of Theorem 2.2 is satisfied. Let ϕ be a sentence of $\mathcal{T}_{\mathcal{A}}$ and τ the translation from $\mathcal{L}_{\mathcal{A}}$ to $\mathcal{L}_{\mathcal{B}}$ which associates to ϕ

the formula $\tau(\phi)$ of $\mathcal{L}_{\mathcal{B}}$ which is obtained from ϕ by substituting the predicates of $\mathcal{L}_{\mathcal{A}}$ with defining formulas of $\mathcal{L}_{\mathcal{B}}$. Hence $\tau(\phi)$ contains only variables of the sorts S or P and neither the constant $1 \in G$ nor a variable $\alpha \in G$. We have to show: $\mathcal{A} \vdash \phi$ if and only if $\mathcal{B} \vdash \tau(\phi)$.

If $\mathcal{A} \vdash \phi$ then $\mathcal{B} \vdash \tau(\phi)$ since δ is an interpretation of $\mathcal{T}_{\mathcal{A}}$ in $\mathcal{T}_{\mathcal{B}}$. Now suppose $\mathcal{B} \vdash \tau(\phi)$. Then there exists a proof of $\tau(\phi)$ which consists of a (finite) sequence of applications of axioms of $\mathcal{B}$. Since $\tau(\phi)$ contains only variables of the sorts S or P, we can assume that the axioms are applied on finite products of elements of S and P (and not on arbitrary elements of G).[21]

We show that under these assumptions the axioms of $\mathcal{B}$ correspond to formulas of $\mathcal{A}$. To this end we introduce the following abbreviations in $\mathcal{L}_{\mathcal{A}}$:

$$\begin{aligned} &(a) \quad a_1 \cdots a_m = 1 \ :\Leftrightarrow \ (\forall E)\,[(\sigma(E, a_1, E_1) \wedge \cdots \wedge \sigma(E_{m-1}, a_m, E)] \\ &(b) \quad a_1 \cdots a_m = b_1 \cdots b_n \ :\Leftrightarrow \ a_1 \cdots a_m \cdot b_n \cdots b_1 = 1 \\ &(c) \quad a^{b_1 \cdots b_n} = c \ :\Leftrightarrow \ b_n \cdots b_1 \cdot a \cdot b_1 \cdots b_n = c \end{aligned} \tag{3.2}$$

Obviously $a_1 \cdots a_m = 1$ if and only if $(\forall e)\,[(\sigma(e, a_1, e_1) \wedge \cdots \wedge \sigma(e_{m-1}, a_m, e)]$.

The definitions (3.2) can be dualized (by interchanging the elements of $\mathcal{P}$ and $\mathcal{L}$ and the relations σ and ϱ) and extended to 'products'[22] of arbitrary elements of $\mathcal{L}$ and $\mathcal{P}$ (such that, e.g., $aBc = 1$ is defined by $(\forall X)\,[(\sigma(X, a, Y) \wedge \varrho(Y, B, Z)] \wedge \sigma(Z, c, X)]$). We denote the extended definitions of (3.2), which are obtained in this way, by (3.2)*.

With these definitions the versions of the axioms of $\mathcal{B}$, which are used in the proof of $\tau(\phi)$, can be translated into formulas of $L_{\mathcal{A}}$, which can be derived from the axioms of $\mathcal{A}$: Axiom **B1**, restricted to $\alpha, \beta, \gamma \in (\mathcal{L} \cup \mathcal{P})^n$, is an immediate consequence of (3.2)*, b). Since in $\tau(\phi)$ the constant 1 does not occur, we can assume that there exists a proof of $\tau(\phi)$ without using **B2**. Axiom **B4** is a consequence of (3.2)*, a) and of **A1** and **A8**. Axiom **B3** holds for $\alpha = \alpha_1 \cdots \alpha_n$ with $\alpha_1, \ldots, \alpha_n \in \mathcal{L} \cup \mathcal{P}$ according to **B3** (choose $\beta := \alpha_n \cdots \alpha_1$). Axiom **B5**, restricted to $\alpha, \beta, \gamma \in (\mathcal{L} \cup \mathcal{P})^n$, is a consequence of **A4**, **A4***, **A6** and **A6*** (i.e., of the invariance of the relations σ and ϱ). For a proof of **B6** let $\alpha = \alpha_1 \cdots \alpha_n$ with $\alpha_1, \ldots, \alpha_n \in \mathcal{L} \cup \mathcal{P}$ and $A^\alpha = A$ for all $A \in \mathcal{P}$. Let A' be the point with $\tau(A, \alpha_1, A_1) \wedge \tau(A_1, \alpha_2, A_2) \wedge \cdots \wedge \tau(A_{n-1}, \alpha_n, A')$ where τ denotes the relation σ if $\alpha_i \in \mathcal{L}$ resp. the relation ϱ if $\alpha_i \in \mathcal{P}$. Then $A^\alpha = A$ implies $\sigma(E, A', F) \leftrightarrow \sigma(E, A, F)$ for all $E \in \mathcal{P}$ (according to **A4** and **A4***). Since this holds for all $A \in \mathcal{P}$ it is $\alpha = 1$, according to (3.2)*(b). This proofs that axiom **B6** is satisfied. **B7** is a consequence of **A7**, **A8** and **A8***. The axioms **B8** and **B9** hold according to Theorem 3.7(3).

[21] Since $\tau(\phi)$ contains only variables of the sorts S and P, there exists a proof where this assumption holds for the last line of the proof, for the proceeding line etc. and hence for all lines of the proof.

[22] Thus a product of elements of $\mathcal{L}$ and $\mathcal{P}$ corresponds to a product of the associated relations σ and ϱ (cp. Oberschelp et al. [20, § 10]).

Hence the derivation of $\tau(\phi)$ from the axioms of $\mathcal{B}$ can be transferred to a derivation of ϕ from the axioms of $\mathcal{A}$. This shows that the theory of reflection groups is a conservative extension of the theory of symmetric spaces. □

3.4. Formulation of geometry in the group of motions

Let δ be the interpretation of $L_{\mathcal{A}}$ in $L_{\mathcal{B}}$ which is defined by (3.1). We call the translation τ, which is associated to δ, the *natural translation* from $L_{\mathcal{A}}$ to $L_{\mathcal{B}}$. According to Theorem 3.14 the translation τ is faithful, i.e., the translation of a theorem of $\mathcal{T}_{\mathcal{A}}$ is a theorem of $\mathcal{T}_{\mathcal{B}}$, and the translation of a non-theorem of $\mathcal{T}_{\mathcal{A}}$ is a non-theorem of $\mathcal{T}_{\mathcal{B}}$. We state this as a proposition.

Theorem 3.15. *The natural translation τ from $L_{\mathcal{A}}$ to $L_{\mathcal{B}}$ is a faithful translation from $\mathcal{T}_{\mathcal{A}}$ to $\mathcal{T}_{\mathcal{B}}$.*

Proof. The theorem is an immediate consequence of Theorem 3.14. □

If $L^{+}_{\mathcal{A}}$ is a definitional extension of $L_{\mathcal{A}}$ (obtained from $L_{\mathcal{A}}$ by adding some new relation symbols together with a set ε of explicit definitions) then, obviously, the natural translation τ from $L_{\mathcal{A}}$ to $L_{\mathcal{B}}$ can be extended to a translation from $L^{+}_{\mathcal{A}}$ to $L_{\mathcal{B}}$.

So, e.g., the *orthogonality relation* $a \perp b$ on $\mathcal{L} \times \mathcal{L}$, which is defined by the formula $[a \neq b \wedge \sigma(a, b, a)]$, corresponds in $L_{\mathcal{B}}$ to the formula $[a \neq b \wedge a^b = a]$ and hence to the stroke-relation $a\,|\,b$ on $S \times S$.

Similarly, the relation $A \top B$ on $\mathcal{P} \times \mathcal{P}$, which is defined by $[A \neq B \wedge \varrho(A, B, A)]$ corresponds in $L_{\mathcal{B}}$ to the formula $[A \neq B \wedge A^B = A]$ and hence to the stroke-relation $A\,|\,B$ on $P \times P$.

In this way one can develop a *dictionary* from the language of symmetric spaces to the language of reflection groups. Since τ is a faithful translation from $\mathcal{T}_{\mathcal{A}}$ to $\mathcal{T}_{\mathcal{B}}$, the translations can be based on the assumptions that the axiom systems $\mathcal{A}$ resp. $\mathcal{B}$ are satisfied.

Thus, for example, the relation $\pi(A, a)$ holds in $\mathcal{A}$ if and only if $\sigma(B, a, C)$ implies $\varrho(B, A, C)$ for all points B, C [according to Theorem 3.2(2) and (3)]. According to τ this corresponds in $L_{\mathcal{B}}$ to: $B^a = C$ implies $B^A = C$, and hence $B^{aA} = B$ (for all points B) and $aA = 1$ (by **B6**) and $A = a$. Hence the *pole–polar relation* $\pi(A, a)$ of $\mathcal{T}_{\mathcal{A}}$ corresponds in $\mathcal{T}_{\mathcal{B}}$ to the relation $A = a$ on $P \times S$.

As a consequence, the *incidence relation* $A\,|\,b$ of $L_{\mathcal{A}}$, which is defined by the formula $[\neg\pi(A, b) \wedge \sigma(A, b, A) \wedge \varrho(b, A, b)]$ corresponds in $L_{\mathcal{B}}$ to the formula $[A \neq b \wedge A^b = A \wedge b^A = b]$ and hence to the stroke-relation $A\,|\,b$ on $P \times S$.

In addition to the basic concepts *point* and *line* we introduce (in both languages $L_{\mathcal{A}}$ and $L_{\mathcal{B}}$) the notion of an *oriented angle* $\measuredangle(a, b)$ as an ordered pair of lines which have a common point (the vertex of the angle). Angles $\measuredangle(a, b)$ and $\measuredangle(c, d)$ with the same vertex are of equal magnitude, in symbols $\measuredangle(a, b) \equiv \measuredangle(c, d)$, if the following formula holds: $(\forall E)[E\sigma_a\sigma_b = E\sigma_c\sigma_d]$. This corresponds in $L_{\mathcal{B}}$ to the equation $ab = cd$ (since $E\sigma_a\sigma_b = E\sigma_c\sigma_d$ corresponds to $(E^a)^b = (E^c)^d$ and to $E^{abdc} = E$ (for all points E) and (by **B6**) to $abdc = 1$ and to $ab = cd$).

Dually, an *oriented segment* $\overline{AB}$ is an ordered pair of points which have a joining line. Segments $\overline{AB}$ and $\overline{CD}$ are of equal magnitude, in symbols $\overline{AB} \equiv \overline{CD}$, if the following formula holds: $(\forall e)[e\varrho_A\varrho_B = e\varrho_C\varrho_D]$ (where $e\varrho_A$ denotes the unique line with $\varrho(e, A, e\varrho_A)$). This corresponds in $L_{\mathcal{B}}$ to the equation $AB = CD$.

If a, b, c, d are lines with a common perpendicular then a, b and c, d are of equal *directed distance*, in symbols $\overline{ab} \equiv \overline{cd}$, if the following formula holds: $(\forall E)[E\sigma_a\sigma_b = E\sigma_c\sigma_d]$. This corresponds in $L_{\mathcal{B}}$ to the equation $ab = cd$.

A point M is a *midpoint* of A and B if $\varrho(A, M, B)$ which corresponds in $L_{\mathcal{B}}$ to $A^M = B$. A line m is a *midline* of a and b if $\sigma(a, m, b)$ which corresponds in $L_{\mathcal{B}}$ to $a^m = b$. A *bisector* of an angle $\measuredangle(a, b)$ is a midline of a and b which is incident with a common point of a and b.

A pair (A, b) is a *flag* if A, b are incident. Flags (A, a) and (B, b) are called *parallel*, in symbols $(A, a) \parallel (B, b)$, if $(\forall E)[E\sigma_a\sigma_b = E\varrho_A\varrho_B]$, which holds, particularly, if A and B have a joining line which is orthogonal to a and b. This corresponds in $L_{\mathcal{B}}$ to $ab = AB$ and to $Aa = Bb$.

A *quadrangle* is a set of four points A, B, C, D and four lines a, b, c, d with $a \mid A, B$ and $b \mid B, C$ and $c \mid C, D$ and $d \mid D, A$.

For dictionaries, which can be found in the literature, we refer to Thomsen [34, §4] and Bachmann [4, §1,4; §20,2] and [5, §1.1] and [8, §5.1].

Now we can formulate a wide generalization of Theorem 3.14.

Theorem 3.16. *Let* $\mathbf{A}_1, \ldots, \mathbf{A}_n$ *be sentences of a definitional extension of* $L_{\mathcal{A}}$ *and* $\mathbf{B}_1, \ldots, \mathbf{B}_n$ *the translations of* $\mathbf{A}_1, \ldots, \mathbf{A}_n$ *with respect to the natural translation* τ*. Then the following holds:*

- *The theory of* $\mathcal{B} \cup \{\mathbf{B}_1, \ldots, \mathbf{B}_n\}$ *is a conservative extension of the theory of* $\mathcal{A} \cup \{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$.
- *The translation* τ *can be extended to a faithful translation from the theory of* $\mathcal{A} \cup \{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$ *to the theory of* $\mathcal{B} \cup \{\mathbf{B}_1, \ldots, \mathbf{B}_n\}$.

Proof. This theorem is a consequence of Theorems 3.14 and 3.15. □

The fundamental idea of the *Thomsen–Bachmann correspondence* which is formulated in [8, p. 129 and p.134] by the statements[23]

(1) "*The group of motions contains an image of the properties of the plane.*"
(2) "*Geometry can be formulated in the group of motions.*"

is given a precise meaning by the Theorems 3.14, 3.15 and 3.16:

[23] See also the preface to the first edition of [4] and [4, §2,4].

(1*) *The theory of the group of motions is a conservative extension of the geometric theory.*

(2*) *Geometry can be formulated in the group of motions by a faithful translation of the geometric theory.*

Remark 3.17. So far we have studied the translation from $L_{\mathcal{A}}$ to $L_{\mathcal{B}}$. Conversely one can ask whether there is an 'inverse translation' from $L_{\mathcal{B}}$ to $L_{\mathcal{A}}$. Since $L_{\mathcal{A}}$ is a purely relational language (without function symbols or constant symbols) the interpretation (3.1) induces a definitional extension[24] of $L_{\mathcal{B}}$, which we denote by $L_{\mathcal{B}}^{+}$, such that $L_{\mathcal{A}} \subseteq L_{\mathcal{B}}^{+}$.[25]

Hence the geometrical language of symmetric spaces can be used in the theory of reflection groups, i.e., a statement of $L_{\mathcal{A}}$ can be interpreted both as a statement of symmetric spaces and as a statement of reflection groups.

Since this corresponds to a common practice in the literature on reflection geometry (see, e.g., [4,5,15,22,30] and the bibliographies given there), and since we have to refer to this literature in the second part [31] of this paper, we will use the extension $L_{\mathcal{B}}^{+}$ of $L_{\mathcal{B}}$ with $L_{\mathcal{A}} \subseteq L_{\mathcal{B}}^{+}$ without further ado. This should not cause any confusion since the context will always be clear.

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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[24] A *definitional extension* L^{+} of a language L consists of adding some new relation symbols together with explicit definitions of them. Thus the formulas of L^{+} can be considered as abbreviations of formulas of L and do not increase the expressive power of L.

[25] We notice that $L_{\mathcal{A}} \subseteq L_{\mathcal{B}}^{+}$ implies—from a semantical point of view—that every reflection group has an expansion which has a reduct which is a symmetric space.

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