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The decomposition of almost paracontact metric manifolds in eleven classes revisited

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Abstract. This paper is a continuation of our previous work, where eleven basic classes of almost paracontact metric manifolds with respect to the covariant derivative of the structure tensor field were obtained. First we decompose one of the eleven classes into two classes and the basic classes of the considered manifolds become twelve. Also, we determine the classes of α-para-Sasakian, α-para-Kenmotsu, normal, paracontact metric, para-Sasakian, K-paracontact and quasi-para-Sasakian manifolds. Moreover, we study 3-dimensional almost paracontact metric manifolds and show that they belong to four basic classes from the considered classification. We define an almost paracontact metric structure on any 3-dimensional Lie group and give concrete examples of Lie groups belonging to each of the four basic classes, characterized by commutators on the corresponding Lie algebras.

Keywords. Almost paracontact metric manifolds, 3-Dimensional almost paracontact manifolds, α-Para-Sasakian manifolds, α-Para-Kenmotsu manifolds.

1. Introduction

Different manifolds with additional tensor structures have been classified with respect to the structure $(0,3)$ tensors, generated by the covariant derivative of the fundamental tensor of type $(1,1)$. For example, such classifications are: the Gray–Hervella classification of almost Hermitian manifolds given in [6], the Naveira classification of Riemannian almost product manifolds—in [11], the Ganchev–Borisov classification of almost complex manifolds with Norden metric—in [4], the Alexiev–Ganchev classification of almost contact metric manifolds—in [1], the Ganchev–Mihova–Gribachev classification of almost contact B-metric manifolds—in [5] and etc.

In [10] we decomposed the vector space of the structure $(0,3)$ tensors on almost paracontact metric manifolds [called almost paracontact manifolds with

semi-Riemannian metric of $(n + 1, n)$] in eleven subspaces which are orthogonal and invariant under the action of the structure group of the considered manifolds. In this paper we show that one of the eleven subspaces could be decomposed in two orthogonal and invariant subspaces and give their characteristic conditions. Then we find the dimensions of the twelve subspaces and the projections of the structure tensor in the corresponding basic classes of almost paracontact metric manifolds. Also, we obtain the classes of the following types of almost paracontact metric manifolds: α-para-Sasakian, α-para-Kenmotsu, normal, paracontact metric, para-Sasakian, K-paracontact and quasi-para-Sasakian.

We pay special attention to almost paracontact metric manifolds of dimension 3, which is the lowest dimension for these manifolds. First, we establish that such manifolds belong only to four basic classes from the considered classification. Then we define an almost paracontact metric structure on a 3-dimensional Lie group. We determine its Lie algebra by commutators such that the Lie group is a manifold belonging to some of the four basic classes of 3-dimensional almost paracontact metric manifolds. The considered Lie groups are characterized geometrically in terms of their curvature properties. Moreover, we find explicit matrix representations of these Lie groups. Let us note that Lie groups as 3-dimensional almost contact B-metric manifolds were studied in [9] and their matrix representations were obtained in [8].

2. Preliminaries

A $(2n + 1)$-dimensional smooth manifold $M^{(2n+1)}$ has an *almost paracontact structure* (φ, ξ, η) if it admits a tensor field φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying the following conditions:

$$
\begin{aligned}
(i) \quad & \varphi^2 = id - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \\
(ii) \quad & \text{there exists a distribution } \mathbb{D} \colon p \in M \longrightarrow \mathbb{D}_p \subset T_pM \colon \\
& \mathbb{D}_p = Ker\eta = \{x \in T_pM \colon \eta(x) = 0\}, \text{ called } paracontact \\
& distribution \text{ generated by } \eta.
\end{aligned}
\tag{2.1}
$$

Then the tangent space T_pM at each $p \in M$ is the following orthogonal direct sum

$$T_pM = \mathbb{D}_p \oplus span_{\mathbb{R}}\{\xi(p)\}$$

and every vector $x \in T_pM$ can be decomposed uniquely in the manner

$$x = hx + vx,$$

where $hx = \varphi^2 x \in \mathbb{D}_p$ and $vx = \eta(x) \cdot \xi(p) \in span_{\mathbb{R}}\{\xi(p)\}$. Using the conditions (2.1) we have

$$h\xi = 0, \quad h^2 = h, \quad h \circ \varphi = \varphi \circ h = \varphi, \quad v \circ h = h \circ v = 0.$$

The tensor field φ induces an almost paracomplex structure [7] on each fibre on $\mathbb{D}$ and $(\mathbb{D}, \varphi, g_{|\mathbb{D}})$ is a $2n$-dimensional almost paracomplex manifold.

Since g is non-degenerate metric on M and ξ is non-isotropic, the paracontact distribution $\mathbb{D}$ is non-degenerate.

As immediate consequences of the definition of the almost paracontact structure we have that the endomorphism φ has rank $2n$, and $\eta \circ \varphi = 0$ (see [2,3] for the almost contact case).

From now on, we will use x, y, z for arbitrary elements of $\chi(M)$ or vectors in the tangent space T_pM at $p \in M$.

If a manifold $M^{(2n+1)}$ with (φ, ξ, η)-structure admits a pseudo-Riemannian metric g such that

$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y),$$

then we say that $M^{(2n+1)}$ has an almost paracontact metric structure and g is called *compatible* metric. Any compatible metric g with a given almost paracontact structure is necessarily of signature $(n+1, n)$.

Setting $y = \xi$, we have $\eta(x) = g(x, \xi)$.

Any almost paracontact structure admits a compatible metric.

The fundamental 2-form

$$\phi(x, y) = g(\varphi x, y)$$

is non-degenerate on the horizontal distribution $\mathbb{D}$ and $\eta \wedge \phi^n \neq 0$.

Let ϕ be the fundamental 2-form on $(M, \varphi, \xi, \eta, g)$ and F be the covariant derivative of ϕ with respect to the Levi-Civita connection ∇ of g, i.e the tensor field F of type $(0,3)$ is defined by

$$F(x, y, z) = (\nabla\phi)(x, y, z) = (\nabla_x\phi)(y, z) = g\left((\nabla_x\varphi)y, z\right).$$

Because of (1.1) and (1.3) the tensor F has the following properties:

$$F(x, y, z) = -F(x, z, y),$$
$$F(x, \varphi y, \varphi z) = F(x, y, z) + \eta(y)F(x, z, \xi) - \eta(z)F(x, y, \xi). \tag{2.2}$$

The following 1-forms are associated with F:

$$\theta(x) = g^{ij}F(e_i, e_j, x); \; \theta^*(x) = g^{ij}F(e_i, \varphi e_j, x); \; \omega(x) = F(\xi, \xi, x),$$

where $\{e_i, \xi\}$ $(i = 1, \ldots, 2n)$ is a basis of TM, and (g^{ij}) is the inverse matrix of (g_{ij}).

We express $\nabla\eta$, $d\eta$, $L_\xi g$ and $d\phi$ in terms of the structure tensor F in the following lemma

Lemma 2.1. *For arbitrary x, y, z we have:*

$$(\nabla_x\eta)y = g(\nabla_x\xi, y) = -F(x, \varphi y, \xi);$$
$$d\eta(x, y) = \frac{1}{2}\left((\nabla_x\eta)y - (\nabla_y\eta)x\right) = \frac{1}{2}(-F(x, \varphi y, \xi) + F(y, \varphi x, \xi)); \tag{2.3}$$
$$(L_\xi g)(x, y) = (\nabla_x\eta)y + (\nabla_y\eta)x = -F(x, \varphi y, \xi) - F(y, \varphi x, \xi); \tag{2.4}$$
$$d\phi(x, y, z) = \underset{(x,y,z)}{\mathfrak{S}} \; F(x, y, z),$$

where $\mathfrak{S}_{(x,y,z)}$ denotes the cyclic sum over x, y, z.

In [12] it is proved that a $(2n+1)$-dimensional almost paracontact metric manifold is normal if and only if the following condition holds:

$$\varphi(\nabla_x\varphi)y - (\nabla_{\varphi x}\varphi)y + (\nabla_x\eta)(y)\xi = 0. \tag{2.5}$$

Moreover, we have that (2.5) is equivalent to the following equality

$$F(x, y, \varphi z) + F(\varphi x, y, z) + F(x, \varphi y, \xi) = 0. \tag{2.6}$$

Definition 2.2. A $(2n+1)$-dimensional almost paracontact metric manifold is called

- *normal* if $N(x, y) - 2d\eta(x, y)\xi = 0$, where

$$N(x, y) = \varphi^2[x, y] + [\varphi x, \varphi y] - \varphi[\varphi x, y] - \varphi[x, \varphi y]$$

  is the Nijenhuis torsion tensor of φ (see [13]);
- *paracontact metric* if $\phi = d\eta$;
- α-*para-Sasakian* if $(\nabla_x\varphi)y = \alpha(g(x, y)\xi - \eta(y)x)$, where $\alpha \neq 0$ is constant;
- *para-Sasakian* if it is normal and paracontact metric;
- α-*para-Kenmotsu* if $(\nabla_x\varphi)y = -\alpha(g(x, \varphi y)\xi + \eta(y)\varphi x)$, where $\alpha \neq 0$ is constant, in particular, para-Kenmotsu if $\alpha = -1$;
- *K-paracontact* if it is paracontact and ξ is Killing vector field;
- *quasi-para-Sasakian* if it is normal and $d\phi = 0$.

Remark 2.3. In [13] it was proved that $(M, \varphi, \xi, \eta, g)$ is para-Sasakian if and only if $(\nabla_x\varphi)y = -g(x, y)\xi + \eta(y)x$. This result is obtained by $\phi(x, y) = g(x, \varphi y)$. We note that if $\phi(x, y) = g(\varphi x, y)$, then $(M, \varphi, \xi, \eta, g)$ is para-Sasakian if and only if $(\nabla_x\varphi)y = g(x, y)\xi - \eta(y)x$. Hence, if $\phi(x, y) = g(x, \varphi y)$ (resp. $\phi(x, y) = g(\varphi x, y)$), then an α-para-Sasakian manifold is para-Sasakian if $\alpha = -1$ (resp. $\alpha = 1$).

Let $\mathbb{U}^\pi(n)$ be the paraunitary group, i.e. $\mathbb{U}^\pi(n)$ consists of paracomplex matrices $\beta = A + \epsilon B$ ($\epsilon^2 = 1$; A, B are real matrices of type $(n \times n)$) such that $\beta^{-1} = \bar{\beta}^t$. If r is the real representation of $\mathbb{U}^\pi(n)$ then

$$r(\beta) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad A^t A - B^t B = I_n, \quad A^t B - B^t A = 0,$$

where $\beta \in \mathbb{U}^\pi(n)$, I_n denotes the identity matrix of type $(n \times n)$. We consider the group $\mathbb{U}^\pi(n) \times \{1\}$ which consists of matrices α of type $((n+1) \times (n+1))$

such that $\alpha = \begin{pmatrix} & & 0 \\ \beta & & \vdots \\ & & 0 \\ 0 \ldots 0 & & 1 \end{pmatrix}, \beta \in \mathbb{U}^\pi(n)$. Then $r(\alpha) = \begin{pmatrix} A & B & 0 \\ B & A & \vdots \\ & & 0 \\ 0 & \ldots 0 & 1 \end{pmatrix}$.

For $\alpha \in \mathbb{U}^\pi(n) \times id$ we have $\alpha\xi = \xi$, $\alpha \circ \varphi = \varphi \circ \alpha$ and g is an isometry with respect to α, i.e. the matrices of $\mathbb{U}^\pi(n) \times \{1\}$ preserve the structures ξ, φ, g, η. Hence, $\mathbb{U}^\pi(n) \times \{1\}$ is the structure group of the almost paracontact metric manifolds.

Let V be a $(2n+1)$-dimensional real vector space with an almost paracontact structure (φ, ξ, η) and a compatible metric g with this structure. We denote by $\otimes_3^0 V$ the space of the tensors of type $(0,3)$ over V. Let $\mathcal{F}$ be the subspace of $\otimes_3^0 V$ defined by

$$\mathcal{F} = \{F \in \otimes_3^0 V \colon F(x,y,z) = -F(x,z,y) = F(x,\varphi y, \varphi z) - \eta(y)F(x,z,\xi) + \eta(z)F(x,y,\xi)\}.$$

The metric g on V induces an inner product $\langle , \rangle$ on $\mathcal{F}$ which is defined by

$$\langle F_1, F_2 \rangle = g^{ip} g^{jq} g^{kr} F_1(f_i, f_j, F_k) F_2(f_p, f_q, F_r),$$

where $F_1, F_2 \in \mathcal{F}$ and $\{f_1, \ldots, f_{2n}\}$ is a basis of V.

The standard representation of the structure group $\mathbb{U}^\pi(n) \times \{1\}$ in V induces a representation λ of $\mathbb{U}^\pi(n) \times \{1\}$ in $\mathcal{F}$ in the following manner:

$$(\lambda(\alpha)F)(x,y,z) = F(\alpha^{-1}x, \alpha^{-1}y, \alpha^{-1}z),$$

for $\alpha \in \mathbb{U}^\pi(n) \times \{1\}$ and $F \in \mathcal{F}$. Also, $\lambda(\alpha)$ preserves the inner product $\langle , \rangle$ in $\mathcal{F}$.

3. On the decomposition of $\mathcal{F}$

In [10] we obtained a decomposition of a vector space $\mathcal{F}$ into eleven subspaces $\mathcal{F}_i (i = 1, \ldots, 11)$, which are mutually orthogonal and invariant under the action of the structure group $\mathbb{U}^\pi(n) \times \{1\}$. First we found the following partial decomposition of $\mathcal{F}$ in a direct sum of its subspaces W_i $(i = 1, 2, 3, 4)$, i.e.

$$\mathcal{F} = W_1 \oplus W_2 \oplus W_3 \oplus W_4,$$

where W_i $(i = 1, 2, 3, 4)$ were defined by

$$\begin{aligned}
W_1 &= \{F \in \mathcal{F} \colon F(x,y,z) = F(hx, hy, hz)\}, \\
W_2 &= \{F \in \mathcal{F} \colon F(x,y,z) = -\eta(y)F(hx, hz, \xi) + \eta(z)F(hx, hy, \xi)\}, \\
W_3 &= \{F \in \mathcal{F} \colon F(x,y,z) = \eta(x)F(\xi, hy, hz), \\
W_4 &= \{F \in \mathcal{F} \colon F(x,y,z) = \eta(x)\{\eta(y)F(\xi, \xi, hz) - \eta(z)F(\xi, \xi, hy)\},
\end{aligned} \tag{3.1}$$

for arbitrary vectors $x, y, z \in V$. The subspaces W_i $(i = 1, 2, 3, 4)$ are mutually orthogonal and invariant under the action of $\mathbb{U}^\pi(n) \times \{1\}$.

3.1. The subspace W_1 of $\mathcal{F}$

In [10] we obtained that $W_1 = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$, where the subspaces $\mathcal{F}_i$ $(i = 1, 2, 3)$ of W_1 are mutually orthogonal and invariant under the action of $\mathbb{U}^\pi(n) \times \{1\}$. They were characterized by

$$\mathcal{F}_1 = \left\{ F \in \mathcal{F} \colon F(x,y,z) = \frac{1}{2(n-1)} \{g(x, \varphi y)\theta_F(\varphi z) - g(x, \varphi z)\theta_F(\varphi y) \right.$$

$$\left. - g(\varphi x, \varphi y)\theta_F(hz) + g(\varphi x, \varphi z)\theta_F(hy)\} \right\}, \tag{3.2}$$

$$\mathcal{F}_2 = \{F \in \mathcal{F} \colon F(\varphi x, \varphi y, z) = -F(x,y,z), \quad \theta_F = 0\}, \tag{3.3}$$

$$\mathcal{F}_3 = \{F \in \mathcal{F} \colon F(\varphi x, \varphi y, z) = F(x,y,z)\}. \tag{3.4}$$

Now, we will show that the subspace $\mathcal{F}_3$ could be decomposed in an orthogonal direct sum of two its subspaces. For this purpose we define the following linear map

$$k\colon \mathcal{F}_3 \longrightarrow \mathcal{F}_3 \quad \text{by} \quad k(F)(x,y,z) = \frac{1}{3}\left\{F(x,y,z) + F(y,z,x) + F(z,x,y)\right\}.$$

We note that from (3.4) it follows that $F(\xi,y,z) = F(x,\xi,z) = 0$ for an arbitrary $F \in \mathcal{F}_3$. Then one can easily verify that $k(F)(\varphi x, \varphi y, z) = k(F)(x,y,z)$, i.e. $k(F)$ belongs to $\mathcal{F}_3$. By direct computations we check that k is a projection (i.e. $k^2 = k$) and it commutes with the action of $\mathbb{U}^\pi(n) \times \{1\}$. We put

$$\mathbb{G}_3 = \text{Im}k = \left\{F \in \mathcal{F}_3\colon F(x,y,z) = \frac{1}{3}\left\{\underset{(x,y,z)}{\mathfrak{S}} F(x,y,z)\right\}\right\}, \qquad (3.5)$$

$$\mathbb{G}_4 = \text{Ker}k = \left\{F \in \mathcal{F}_3\colon \underset{(x,y,z)}{\mathfrak{S}} F(x,y,z) = 0\right\}. \qquad (3.6)$$

Proposition 3.1. *The subspace $\mathcal{F}_3$ is an orthogonal direct sum of the subspaces $\mathbb{G}_3$ and $\mathbb{G}_4$. These subspaces are invariant under the action of $\mathbb{U}^\pi(n) \times \{1\}$.*

Proof. Taking into account that k is a projection in $\mathcal{F}_3$, we have $\mathcal{F}_3 = \text{Im}k \oplus \text{Ker}k = \mathbb{G}_3 \oplus \mathbb{G}_4$. Further, we will show that $\mathbb{G}_3$ and $\mathbb{G}_4$ are orthogonal. Because for an arbitrary $F \in \mathcal{F}$ the conditions $F(x,y,z) = -F(y,x,z)$ and $F(x,y,z) = \frac{1}{3}\{F(x,y,z) + F(y,z,x) + F(z,x,y)\}$ are equivalent, (3.5) becomes

$$\mathbb{G}_3 = \text{Im}k = \{F \in \mathcal{F}_3\colon F(x,y,z) = -F(y,x,z)\}. \qquad (3.7)$$

Now, we take $F' \in \mathbb{G}_3$ and $F'' \in \mathbb{G}_4$. Using (3.6) and (3.7) we obtain

$$\begin{aligned}
\langle F', F'' \rangle &= g^{ip}g^{jq}g^{kr} F_1(f_i, f_j, F_k) F_2(f_p, f_q, F_r) \\
&= -g^{ip}g^{jq}g^{kr} F'(f_i, f_j, F_k) F''(f_q, f_r, F_p) \\
&\quad - g^{ip}g^{jq}g^{kr} F'(f_i, f_j, F_k) F''(f_r, f_p, F_q) \\
&= -g^{jq}g^{ip}g^{kr} F'(f_j, f_i, F_k) F''(f_q, f_p, F_r) \\
&\quad - g^{kr}g^{ip}g^{jq} F'(f_k, f_i, F_j) F''(f_r, f_p, F_q) = -2\langle F', F'' \rangle.
\end{aligned}$$

Thus we find $\langle F', F'' \rangle = 0$. Hence, $\mathbb{G}_3$ and $\mathbb{G}_4$ are orthogonal.

Finally, taking into account that k commutes with the action of $\mathbb{U}^\pi(n) \times \{1\}$, for an arbitrary $F' \in \mathbb{G}_3 = \text{Im}k$ we have

$$\lambda(\alpha)(F') = \lambda(\alpha)(k(F')) = k(\lambda(\alpha)(F')). \qquad (3.8)$$

We note that $\lambda(\alpha)(F') \in \mathcal{F}_3$ because $\mathcal{F}_3$ is invariant under the action of $\mathbb{U}^\pi(n) \times \{1\}$. Then from (3.8) it follows that $\lambda(\alpha)(F') \in \mathbb{G}_3$ which means that $\mathbb{G}_3$ is invariant under the action of $\mathbb{U}^\pi(n) \times \{1\}$. Since $\lambda(\alpha)$ is an isometry with respect to the inner product $\langle,\rangle$ in $\mathcal{F}$, the orthogonal complement $\mathbb{G}_4$ of the invariant subspace $\mathbb{G}_3$ in $\mathcal{F}_3$ is also invariant. □

From now on, we will denote the subspaces $\mathcal{F}_1$ and $\mathcal{F}_2$ by $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. In conclusion, we state.

Proposition 3.2. *The decomposition $W_1 = \mathbb{G}_1 \oplus \mathbb{G}_2 \oplus \mathbb{G}_3 \oplus \mathbb{G}_4$ is orthogonal and invariant under the action of $\mathbb{U}^\pi(n) \times \{1\}$.*

The characteristic conditions of $\mathbb{G}_1$ and $\mathbb{G}_2$ are (3.2) and (3.3), respectively, which were obtained in [10]. According to (3.7) and (3.6), the characteristic conditions of $\mathbb{G}_3$ and $\mathbb{G}_4$ are as follows:

$$\mathbb{G}_3 = \{F \in \mathcal{F} : F(\xi, y, z) = F(x, \xi, z) = 0, \quad F(x, y, z) = -F(y, x, z)\},$$

$$\mathbb{G}_4 = \left\{F \in \mathcal{F} : F(\xi, y, z) = F(x, \xi, z) = 0, \quad \underset{(x,y,z)}{\mathfrak{S}} F(x, y, z) = 0\right\}.$$

3.2. The subspace W_2 of $\mathcal{F}$

In [10] we decomposed W_2 into 6 subspaces which are mutually orthogonal and invariant under the action of $\mathbb{U}^\pi(n) \times \{1\}$, i.e.

$$W_2 = \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9.$$

Taking into account the characteristic condition of W_2 in (3.1), we rewrite the conditions of $\mathcal{F}_6, \mathcal{F}_7, \mathcal{F}_8$ and $\mathcal{F}_9$ in an equivalent form to the one in [10, Theorem 2.1, p. 124]. Moreover, we denote the subspaces $\mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7, \mathcal{F}_8, \mathcal{F}_9$ by $\mathbb{G}_5, \mathbb{G}_6, \mathbb{G}_7, \mathbb{G}_8, \mathbb{G}_9, \mathbb{G}_{10}$, respectively. So, we have:

$$\mathbb{G}_5 = \mathcal{F}_4 = \left\{F \in \mathcal{F} : F(x, y, z) = \frac{\theta_F(\xi)}{2n}\{\eta(y)g(\varphi x, \varphi z) - \eta(z)g(\varphi x, \varphi y)\}\right\}.$$

This is the class of generalized α-para-Sasakian manifolds.

$$\mathbb{G}_6 = \mathcal{F}_5 = \left\{F \in \mathcal{F} : F(x, y, z) = -\frac{\theta_F^*(\xi)}{2n}\{\eta(y)g(x, \varphi z) - \eta(z)g(x, \varphi y)\}\right\}.$$

This is the class of generalized α-para-Kenmotsu manifolds.

$$\mathbb{G}_7 = \mathcal{F}_6 = \{F \in \mathcal{F} : F(x, y, z) = -\eta(y)F(x, z, \xi) + \eta(z)F(x, y, \xi),$$
$$F(x, y, \xi) = -F(y, x, \xi) = -F(\varphi x, \varphi y, \xi), \quad \theta_F^*(\xi) = 0\},$$
$$\mathbb{G}_8 = \mathcal{F}_7 = \{F \in \mathcal{F} : F(x, y, z) = -\eta(y)F(x, z, \xi) + \eta(z)F(x, y, \xi),$$
$$F(x, y, \xi) = F(y, x, \xi) = -F(\varphi x, \varphi y, \xi), \quad \theta_F(\xi) = 0\},$$
$$\mathbb{G}_9 = \mathcal{F}_8 = \{F \in \mathcal{F} : F(x, y, z) = -\eta(y)F(x, z, \xi) + \eta(z)F(x, y, \xi),$$
$$F(x, y, \xi) = -F(y, x, \xi) = F(\varphi x, \varphi y, \xi)\},$$
$$\mathbb{G}_{10} = \mathcal{F}_9 = \{F \in \mathcal{F} : F(x, y, z) = -\eta(y)F(x, z, \xi) + \eta(z)F(x, y, \xi),$$
$$F(x, y, \xi) = F(y, x, \xi) = F(\varphi x, \varphi y, \xi)\}.$$

Remark 3.3. We call the classes $\mathbb{G}_5$ and $\mathbb{G}_6$ the class of generalized α-para-Sasakian and generalized α-para-Kenmotsu manifolds, respectively, because $\theta_F(\xi)$ and $\theta_F^*(\xi)$ are functions in general.

3.3. The subspaces W_3 and W_4 of $\mathcal{F}$

As in [10] we put $\mathcal{F}_{10} = W_3$ and $\mathcal{F}_{11} = W_4$. Now, we denote $\mathcal{F}_{10}$ and $\mathcal{F}_{11}$ by $\mathbb{G}_{11}$ and $\mathbb{G}_{12}$, respectively. These subspaces were represented by

$$\mathbb{G}_{11} = \mathcal{F}_{10} = \{F \in \mathcal{F} : F(x, y, z) = \eta(x)F(\xi, \varphi y, \varphi z)\},$$
$$\mathbb{G}_{12} = \mathcal{F}_{11} = \{F \in \mathcal{F} : F(x, y, z) = \eta(x)\{\eta(y)F(\xi, \xi, z) - \eta(z)F(\xi, \xi, y)\}\}.$$

Corresponding to the decomposition in Sect. 3 of the space $\mathcal{F}$ into 12 mutually orthogonal and invariant subspaces, we give 12 classes of almost paracontact metric manifolds. An almost paracontact metric manifold M is said to be in the class $\mathbb{G}_i$ $(i = 1, \ldots, 12)$ (or $\mathbb{G}_i$-manifold) if at each $p \in M$ the tensor F of M belongs to the subspace $\mathbb{G}_i$. The special class $\mathbb{G}_0$, determined by the condition $F(x, y, z) = 0$, is the intersection of the basic twelve classes. Hence, $\mathbb{G}_0$ is the class of the almost paracontact metric manifolds with parallel structures, i.e. $\nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = 0$.

Finally, by using the characteristic symmetries of the structure tensor F of a $(2n + 1)$-dimensional almost paracontact metric manifold in the classes $\mathbb{G}_i$ $(i = 1, \ldots, 12)$, we obtain

Theorem 3.4. *The dimensions of the subspaces in the decomposition of the space $\mathcal{F}$ are as follows:*

$$\dim \mathbb{G}_1 = 2(n - 1), \qquad \dim \mathbb{G}_2 = (n - 1)(n^2 - 2), \ \dim \mathbb{G}_3 = \frac{(n - 2)(n - 1)n}{3}$$

$$\mathbb{G}_4 = \frac{2(n - 1)n(n + 1)}{3}, \ \dim \mathbb{G}_5 = 1, \qquad\qquad \dim \mathbb{G}_6 = 1,$$

$$\dim \mathbb{G}_7 = n^2 - 1, \qquad \dim \mathbb{G}_8 = n^2 - 1, \qquad \dim \mathbb{G}_9 = n(n - 1),$$

$$\dim \mathbb{G}_{10} = n(n + 1), \qquad \dim \mathbb{G}_{11} = n(n - 1), \qquad \dim \mathbb{G}_{12} = 2n.$$

4. The projections of the structure tensor F in the twelve basic classes of the classification of the almost paracontact metric manifolds

The decompositions of $\mathcal{F}$ in direct sums of the subspaces W_j $(j = 1, 2, 3, 4)$ and $\mathbb{G}_i$ $(i = 1, \ldots, 12)$ imply that every $F \in \mathcal{F}$ has a unique representation in the form $F(x, y, z) = \sum_{j=1}^{4} F^{W_j}(x, y, z)$ and $F(x, y, z) = \sum_{i=1}^{12} F^i(x, y, z)$, respectively, where $F^{W_j} \in W_j$ and $F^i \in \mathbb{G}_i$. Then it is clear that an almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ belongs to a direct sum of two or more basic classes, i.e. $M \in \mathbb{G}_i \oplus \mathbb{G}_j \oplus \cdots$, if and only if the structure tensor F on M is the sum of the corresponding projections F^i, F^j, ..., i.e. the following condition is satisfied $F = F^i + F^j + \cdots$.

Following the operators defined in [10, Theorem 2.1, p. 124] and the operator k defined in this paper in Sect. 3.1, we find the projections F^{W_j} $(j = 1, 2, 3, 4)$ and F^i $(i = 1, \ldots, 12)$ of $F \in \mathcal{F}$ in the subspaces W_j and $\mathbb{G}_i$, respectively. These projections are given below:

$$F^{W_1} = F(\varphi^2 x, \varphi^2 y, \varphi^2 z),$$

$$F^{W_2} = -\eta(y)F(\varphi^2 x, \varphi^2 z, \xi) + \eta(z)F(\varphi^2 x, \varphi^2 y, \xi),$$

$$F^{W_3} = \eta(x)F(\xi, \varphi y, \varphi z),$$

$$F^{W_4} = \eta(x)\{\eta(y)F(\xi, \xi, z) - \eta(z)F(\xi, \xi, y)\}.$$

$$F^1(x, y, z) = \frac{1}{2(n - 1)} \{g(x, \varphi y)\theta_{F^1}(\varphi z) - g(x, \varphi z)\theta_{F^1}(\varphi y)$$

$$- g(\varphi x, \varphi y)\theta_{F^1}(\varphi^2 z) + g(\varphi x, \varphi z)\theta_{F^1}(\varphi^2 y)\};$$

$$F^2(x,y,z) = \frac{1}{2}\left\{F(\varphi^2 x, \varphi^2 y, \varphi^2 z) - F(\varphi x, \varphi^2 y, \varphi z)\right\}$$

$$- \frac{1}{2(n-1)}\left\{g(x,\varphi y)\theta_{F^1}(\varphi z) - g(x,\varphi z)\theta_{F^1}(\varphi y)\right.$$

$$\left. - g(\varphi x, \varphi y)\theta_{F^1}(\varphi^2 z) + g(\varphi x, \varphi z)\theta_{F^1}(\varphi^2 y)\right\};$$

$$F^3(x,y,z) = \frac{1}{6}\left\{F(\varphi^2 x, \varphi^2 y, \varphi^2 z) + F(\varphi x, \varphi^2 y, \varphi z)\right.$$

$$+ F(\varphi^2 y, \varphi^2 z, \varphi^2 x) + F(\varphi y, \varphi^2 z, \varphi x)$$

$$\left. + F(\varphi^2 z, \varphi^2 x, \varphi^2 y) + F(\varphi z, \varphi^2 x, \varphi y)\right\};$$

$$F^4(x,y,z) = \frac{1}{2}\left\{F(\varphi^2 x, \varphi^2 y, \varphi^2 z) + F(\varphi x, \varphi^2 y, \varphi z)\right\}$$

$$- \frac{1}{6}\left\{F(\varphi^2 x, \varphi^2 y, \varphi^2 z) + F(\varphi x, \varphi^2 y, \varphi z)\right.$$

$$+ F(\varphi^2 y, \varphi^2 z, \varphi^2 x) + F(\varphi y, \varphi^2 z, \varphi x)$$

$$\left. + F(\varphi^2 z, \varphi^2 x, \varphi^2 y) + F(\varphi z, \varphi^2 x, \varphi y)\right\};$$

$$F^5(x,y,z) = \frac{\theta_{F^5}(\xi)}{2n}\{\eta(y)g(\varphi x, \varphi z) - \eta(z)g(\varphi x, \varphi y)\};$$

$$F^6(x,y,z) = -\frac{\theta^*_{F^6}(\xi)}{2n}\{\eta(y)g(x, \varphi z) - \eta(z)g(x, \varphi y)\};$$

$$F^7(x,y,z) = -\frac{1}{4}\eta(y)\left\{F(\varphi^2 x, \varphi^2 z, \xi) - F(\varphi x, \varphi z, \xi)\right.$$

$$\left. - F(\varphi^2 z, \varphi^2 x, \xi) + F(\varphi z, \varphi x, \xi)\right\} + \frac{1}{4}\eta(z)\left\{F(\varphi^2 x, \varphi^2 y, \xi)\right.$$

$$\left. - F(\varphi x, \varphi y, \xi) - F(\varphi^2 y, \varphi^2 x, \xi) + F(\varphi y, \varphi x, \xi)\right\}$$

$$+ \frac{\theta^*_{F^6}(\xi)}{2n}\{\eta(y)g(x, \varphi z) - \eta(z)g(x, \varphi y)\};$$

$$F^8(x,y,z) = -\frac{1}{4}\eta(y)\left\{F(\varphi^2 x, \varphi^2 z, \xi) - F(\varphi x, \varphi z, \xi)\right.$$

$$\left. + F(\varphi^2 z, \varphi^2 x, \xi) - F(\varphi z, \varphi x, \xi)\right\} + \frac{1}{4}\eta(z)\left\{F(\varphi^2 x, \varphi^2 y, \xi)\right.$$

$$\left. - F(\varphi x, \varphi y, \xi) + F(\varphi^2 y, \varphi^2 x, \xi) - F(\varphi y, \varphi x, \xi)\right\}$$

$$- \frac{\theta_{F^5}(\xi)}{2n}\{\eta(y)g(\varphi x, \varphi z) - \eta(z)g(\varphi x, \varphi y)\};$$

$$F^9(x,y,z) = -\frac{1}{4}\eta(y)\left\{F(\varphi^2 x, \varphi^2 z, \xi) + F(\varphi x, \varphi z, \xi)\right.$$

$$\left. - F(\varphi^2 z, \varphi^2 x, \xi) - F(\varphi z, \varphi x, \xi)\right\} + \frac{1}{4}\eta(z)\left\{F(\varphi^2 x, \varphi^2 y, \xi)\right.$$

$$\left. + F(\varphi x, \varphi y, \xi) - F(\varphi^2 y, \varphi^2 x, \xi) - F(\varphi y, \varphi x, \xi)\right\};$$

$$F^{10}(x,y,z) = -\frac{1}{4}\eta(y)\left\{F(\varphi^2 x, \varphi^2 z, \xi) + F(\varphi x, \varphi z, \xi)\right.$$

$$\left. + F(\varphi^2 z, \varphi^2 x, \xi) + F(\varphi z, \varphi x, \xi)\right\} + \frac{1}{4}\eta(z)\left\{F(\varphi^2 x, \varphi^2 y, \xi)\right.$$

$$+ F(\varphi x, \varphi y, \xi) + F(\varphi^2 y, \varphi^2 x, \xi) + F(\varphi y, \varphi x, \xi)\};$$
$$F^{11}(x, y, z) = \eta(x)F(\xi, \varphi^2 y, \varphi^2 z);$$
$$F^{12}(x, y, z) = \eta(x)\left\{\eta(y)F(\xi, \xi, \varphi^2 z) - \eta(z)F(\xi, \xi, \varphi^2 y)\right\}.$$

Further in this section, using the characteristic conditions of the twelve classes of almost paracontact metric manifolds and the projections of the structure tensor F, we relate the obtained classes with those studied in the literature.

Theorem 4.1. *A $(2n + 1)$-dimensional almost paracontact metric manifold $M(\varphi, \xi, \eta, g)$ is normal if and only if M belongs to one of the classes $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_5$, $\mathbb{G}_6$, $\mathbb{G}_7$, $\mathbb{G}_8$ or to the classes which are their direct sums.*

Proof. Let M belongs to $\mathbb{G}_i$ $(i = 1, 2, 5, 6, 7, 8)$ or to their direct sums. By direct computations we check that for the structure tensor F of M the condition (2.6) holds. Hence, M is normal.

Now, let us assume that M is normal. Then (2.6) is fulfilled. Replacing x and y with ξ in (2.6) we have

$$F(\xi, \xi, z) = 0. \tag{4.1}$$

Replacing x with ξ in (2.6) and using (4.1) we get

$$F(\xi, y, z) = 0. \tag{4.2}$$

Another consequence from (2.6) is

$$F(\varphi x, \varphi y, \xi) = -F(x, y, \xi), \tag{4.3}$$

which we obtain substituting in (2.6) y and z with φy and ξ, respectively. The equalities (4.1) and (4.2) mean that the projections F^{W_4} and F^{W_3} of F vanish. Hence, $F(x, y, z) = F^{W_1}(x, y, z) + F^{W_2}(x, y, z)$. Taking into account (4.3) we conclude that $F^{W_2} = F^i$ or F^{W_2} is a sum of F^i $(i = 5, 6, 7, 8)$.

Next, we replace x, y and z in (2.6) with $\varphi^2 x$, $\varphi^2 y$ and φz, respectively. By using (2.2) and (4.1) we obtain

$$F(\varphi^2 x, \varphi^2 y, \varphi^2 z) + F(\varphi x, \varphi y, \varphi^2 z) + F(x, \varphi y, \xi) = 0. \tag{4.4}$$

We substitute x and y in (4.4) with φx and φy, respectively. So we get

$$F(\varphi x, \varphi y, \varphi^2 z) + F(\varphi^2 x, \varphi^2 y, \varphi^2 z) + F(\varphi x, y, \xi) = 0. \tag{4.5}$$

From (4.3) by using (4.1) we derive $F(x, \varphi y, \xi) = -F(\varphi x, y, \xi)$. Then (4.4) and (4.5) imply $F(\varphi^2 x, \varphi^2 y, \varphi^2 z) = -F(\varphi x, \varphi y, \varphi^2 z)$, which shows that $F^{W_1} = F^i$ $(i = 1, 2)$ or $F^{W_1} = F^1 + F^2$. Summarizing the obtained results we conclude that $F = F^i$ $(i = 1, 2, 5, 6, 7, 8)$ or $F = F^1 + F^2 + F^5 + F^6 + F^7 + F^8$, which completes the proof. $\qquad\square$

Now, by using (2.3) we compute $d\eta$ for a $(2n + 1)$-dimensional almost paracontact metric manifold M belonging to each of the basic classes. We obtain

Lemma 4.2. (a) *If $M \in \mathbb{G}_i$, $i = 1, 2, 3, 4, 6, 7, 10, 11$, then $d\eta = 0$;*
(b) *If $M \in \mathbb{G}_5$, then $d\eta = \frac{\theta_F(\xi)}{2n} g(\varphi x, y)$;*
(c) *If $M \in \mathbb{G}_i$, $i = 8, 9$, then $d\eta = -F^i(x, \varphi y, \xi)$;*
(d) *If $M \in \mathbb{G}_{12}$, then $d\eta = \frac{1}{2}\left(\eta(x)F(\xi, \xi, \varphi y) - \eta(y)F(\xi, \xi, \varphi x)\right)$.*

Let $\overline{\mathbb{G}}_5$ be the subclass of $\mathbb{G}_5$ which consists of all $(2n + 1)$-dimensional $\mathbb{G}_5$-manifolds such that $\theta_F(\xi) = 2n$ (resp. $\theta_F(\xi) = -2n$) by $\phi(x, y) = g(\varphi x, y)$ (resp. $\phi(x, y) = g(x, \varphi y)$).

Taking into account Lemma 4.2 we establish the truth of the following proposition.

Proposition 4.3. *Let M be a $(2n + 1)$-dimensional $\mathbb{G}_5$-manifold. Then M is paracontact metric if and only if M belongs to $\overline{\mathbb{G}}_5$.*

Theorem 4.4. *A $(2n + 1)$-dimensional almost paracontact metric manifold $M(\varphi, \xi, \eta, g)$ is paracontact metric if and only if M belongs to the class $\overline{\mathbb{G}}_5$ or to the classes which are direct sums of $\overline{\mathbb{G}}_5$ with $\mathbb{G}_4$ and $\mathbb{G}_{10}$.*

Proof. Let M belongs to $\overline{\mathbb{G}}_5$. Then from Proposition 4.3 it follows that M is paracontact metric. If M belongs to a direct sum of $\overline{\mathbb{G}}_5$ with $\mathbb{G}_4$, $\mathbb{G}_{10}$, by using Lemma 4.2 we verify that M is also paracontact metric.

Now, let M be a paracontact metric manifold. Then $d\eta(x, y) = \phi(x, y) = g(\varphi x, y)$ and from (2.3) we have

$$- F(x, \varphi y, \xi) + F(y, \varphi x, \xi) = 2g(\varphi x, y). \tag{4.6}$$

Replacing x and y in (4.6) with ξ and φy, respectively, we get

$$F(\xi, \xi, y) = 0. \tag{4.7}$$

Moreover, replacing y in (4.6) with φy we derive

$$F(x, y, \xi) = F(\varphi y, \varphi x, \xi) - 2g(\varphi x, \varphi y). \tag{4.8}$$

From the condition $d\eta = \phi$ it follows that $d\phi = 0$. Now, $d\phi(x, y, \xi) = 0$ implies

$$F(x, y, \xi) - F(y, x, \xi) + F(\xi, x, y) = 0. \tag{4.9}$$

Substituting (4.8) in (4.9) we have $F(\varphi y, \varphi x, \xi) - F(\varphi x, \varphi y, \xi) + F(\xi, x, y) = 0$. In the last equality we replace x with φx, y with φy and by using (4.7) we obtain

$$F(y, x, \xi) - F(x, y, \xi) + F(\xi, x, y) = 0. \tag{4.10}$$

The equalities (4.9) and (4.10) imply $F(\xi, x, y) = 0$. Also, substituting $F(\xi, x, y) = 0$ in (4.9) we get

$$F(x, y, \xi) = F(y, x, \xi). \tag{4.11}$$

Since $F(\xi, \xi, y) = F(\xi, x, y) = 0$, we conclude that $F = F^{W_1} + F^{W_2}$. Hence

$$d\eta_F = d\eta_{F^{W_1}} + d\eta_{F^{W_2}}. \tag{4.12}$$

Using (4.11) and taking into account the characteristic conditions of the twelve classes we obtain $F^{W_2} = F^5 + F^8 + F^{10}$. From Lemma 4.2 it follows that $d\eta_{F^{W_1}} = 0$, $d\eta_{F^5}(x, y) = \frac{\theta_{F^5}(\xi)}{2n} g(\varphi x, y)$, $d\eta_{F^8}(x, y) = -F^8(x, \varphi y, \xi)$ and $d\eta_{F^{10}}(x, y) = 0$. Then (4.12) becomes

$$g(\varphi x, y) = \frac{\theta_{F^5}(\xi)}{2n} g(\varphi x, y) - F^8(x, \varphi y, \xi). \tag{4.13}$$

The equality (4.13) implies that either $F^8 = 0$ or $F^8 = F^5$. The case $F^8 = F^5$ leads a contradiction. Therefore $F^{W_2} = F^5 + F^{10}$. Now, because g is non-degenerate, as an immediate consequence from (4.13) we obtain $\theta_{F^5}(\xi) = 2n$. This means that $F^5 = \overline{F}^5$, where by $\overline{F}^5$ we denote the projection of F in $\overline{\mathbb{G}}_5$. Thus, for F^{W_2} we obtain $F^{W_2} = \overline{F}^5 + F^{10}$ or $F^{W_2} = \overline{F}^5$. The conditions $d\eta_{F^{W_1}} = d\eta_{F^{10}} = 0$ and (4.12) imply that the case $F^{W_2} = F^{10}$ is impossible. In both cases for F^{W_2} we have $\mathfrak{S}_{(x,y,z)} F^{W_2}(x,y,z) = 0$. Then from the equality $\mathfrak{S}_{(x,y,z)} F(x,y,z) = \mathfrak{S}_{(x,y,z)} F^{W_1}(x,y,z) + \mathfrak{S}_{(x,y,z)} F^{W_2}(x,y,z)$ it follows that $\mathfrak{S}_{(x,y,z)} F^{W_1}(x,y,z) = 0$. Taking into account the characteristic conditions of the classes $\mathbb{G}_i$ ($i = 1, 2, 3, 4$) we conclude that $F^{W_1} = F^4$. Finally, for F we obtain $F = \overline{F}^5$, or $F = \overline{F}^5 + F^4$, or $F = \overline{F}^5 + F^{10}$, or $F = \overline{F}^5 + F^4 + F^{10}$. Since $d\eta_{F^4} = 0$, the case $F = F^4$ is impossible. $\qquad\square$

As an immediate consequence from Theorems 4.1 and 4.4 we obtain

Corollary 4.5. *A $(2n + 1)$-dimensional almost paracontact metric manifold $M(\varphi, \xi, \eta, g)$ is para-Sasakian if and only if M belongs to the class $\overline{\mathbb{G}}_5$.*

Remark 4.6. The result in Corollary 4.5 is the same as that obtained in [13].

It is known that ξ is Killing vector field if $(L_\xi g)(x, y) = 0$. By using (2.4) we get

Proposition 4.7. *The vector field ξ is Killing only in the classes $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_3$, $\mathbb{G}_4$, $\mathbb{G}_5$, $\mathbb{G}_8$, $\mathbb{G}_9$, $\mathbb{G}_{11}$ and in the classes which are their direct sums.*

Using Theorem 4.4 and Proposition 4.7 we obtain that among the classes of paracontact metric manifolds the vector field ξ is Killing only in $\overline{\mathbb{G}}_5$ and $\overline{\mathbb{G}}_5 \oplus \mathbb{G}_4$. Then we state.

Theorem 4.8. *A $(2n + 1)$-dimensional almost paracontact metric manifold $M(\varphi, \xi, \eta, g)$ is K-paracontact metric if and only if M belongs to the classes $\overline{\mathbb{G}}_5$ and $\overline{\mathbb{G}}_5 \oplus \mathbb{G}_4$.*

Finally, we check that $d\phi$ vanishes only for normal almost paracontact metric manifolds belonging to the classes $\mathbb{G}_5$, $\mathbb{G}_8$ and $\mathbb{G}_5 \oplus \mathbb{G}_8$. Thus we establish the truth of the following theorem.

Theorem 4.9. *A $(2n + 1)$-dimensional almost paracontact metric manifold $M(\varphi, \xi, \eta, g)$ is quasi-para-Sasakian if and only if M belongs to the classes $\mathbb{G}_5$, $\mathbb{G}_8$ and $\mathbb{G}_5 \oplus \mathbb{G}_8$.*

5. The projections of the structure tensor F for dimension 3

Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost paracontact metric manifold and $\{e_i\}_{i=1}^3 = \{e_1, e_2, e_3\}$ be a φ-basis of T_pM, which satisfies the following conditions:

$$\varphi e_1 = e_2, \quad \varphi e_2 = e_1, \quad e_3 = \xi,$$

$$g(e_1, e_1) = g(e_3, e_3) = -g(e_2, e_2) = 1, \quad g(e_i, e_j) = 0, \quad i \neq j \in \{1, 2, 3\}.$$

We denote the components of the structure tensor F with respect to the φ-basis $\{e_i\}_{i=1}^3$ by $F_{ijk} = F(e_i, e_j, e_k)$. By direct computations for arbitrary x, y, z, given by $x = x^i e_i$, $y = y^i e_i$, $z = z^i e_i$ with respect to $\{e_i\}_{i=1}^3$, we obtain

$$\begin{aligned}
F(x, y, z) = {} & x^1 \left\{ F_{113}(y^1 z^3 - y^3 z^1) + F_{123}(y^2 z^3 - y^3 z^2) \right\} \\
& + x^2 \left\{ F_{213}(y^1 z^3 - y^3 z^1) + F_{223}(y^2 z^3 - y^3 z^2) \right\} \\
& + x^3 \left\{ F_{331}(y^3 z^1 - y^1 z^3) + F_{332}(y^3 z^2 - y^2 z^3) \right\}.
\end{aligned}$$

For the components $\theta_F^i = \theta_F(e_i)$, $\theta_F^{*i} = \theta_F^*(e_i)$, $\omega_F^i = \omega(e_i)$ of the Lee forms θ_F, θ_F^*, ω_F of F we have

$$\begin{aligned}
\theta_F^1 = \theta_F^2 = \theta_F^{*1} = \theta_F^{*2} = 0, \quad \theta_F^3 = F_{113} - F_{223}, \\
\theta_F^{*3} = F_{123} - F_{213}, \quad \omega_F^1 = F_{331}, \quad \omega_F^2 = F_{332}, \quad \omega_F^3 = F_{333} = 0. \quad (5.1)
\end{aligned}$$

Proposition 5.1. *The structure tensor* F^i $(i = 1, \ldots, 12)$ *of a 3-dimensional almost paracontact metric manifold* $(M, \varphi, \xi, \eta, g)$ *has the following form in the corresponding basic classes* $\mathbb{G}_i$:

$$F^1(x, y, z) = F^2(x, y, z) = F^3(x, y, z) = F^4(x, y, z) = 0;$$

$$F^5(x, y, z) = \frac{\theta_F^3}{2} \left\{ x^1(y^1 z^3 - y^3 z^1) - x^2(y^2 z^3 - y^3 z^2) \right\},$$

$$\frac{\theta_F^3}{2} = F_{113} = -F_{131} = -F_{223} = F_{232};$$

$$F^6(x, y, z) = \frac{\theta_F^{*3}}{2} \left\{ x^1(y^2 z^3 - y^3 z^2) - x^2(y^1 z^3 - y^3 z^1) \right\},$$

$$\frac{\theta_F^{*3}}{2} = F_{123} = -F_{132} = -F_{213} = F_{231};$$

$$F^7(x, y, z) = F^8(x, y, z) = F^9(x, y, z) = 0; \qquad (5.2)$$

$$\begin{aligned}
F^{10}(x, y, z) = {} & F_{113}^{10} \left\{ x^1(y^1 z^3 - y^3 z^1) + x^2(y^2 z^3 - y^3 z^2) \right\} \\
& + F_{123}^{10} \left\{ x^1(y^2 z^3 - y^3 z^2) + x^2(y^1 z^3 - y^3 z^1) \right\},
\end{aligned}$$

$$F_{113}^{10} = -F_{131}^{10} = F_{223}^{10} = -F_{232}^{10}, \quad F_{123}^{10} = -F_{132}^{10} = F_{213}^{10} = -F_{231}^{10};$$

$$F^{11}(x, y, z) = 0;$$

$$F^{12}(x, y, z) = \omega_F^1 x^3(y^3 z^1 - y^1 z^3) + \omega_F^2 x^3(y^3 z^2 - y^2 z^3),$$

$$\omega_F^1 = F_{331} = -F_{313}, \quad \omega_F^2 = F_{332} = -F_{323}.$$

By using (5.2) we have

Proposition 5.2. *The 3-dimensional almost paracontact metric manifolds belong to the classes* $\mathbb{G}_5$, $\mathbb{G}_6$, $\mathbb{G}_{10}$, $\mathbb{G}_{12}$ *and to the classes which are their direct sums.*

We note that the assertion in Proposition 5.2 follows also from Theorem 3.4. Taking into account Proposition 5.2, Theorems 4.1, 4.4, Corollary 4.5, Proposition 4.7, Theorems 4.8 and 4.9 we state:

Theorem 5.3. (a) *The classes of the 3-dimensional normal almost paracontact metric manifolds are* $\mathbb{G}_5$, $\mathbb{G}_6$ *and* $\mathbb{G}_5 \oplus \mathbb{G}_6$;

(b) *The classes of the 3-dimensional paracontact metric manifolds are* $\overline{\mathbb{G}}_5$ *and* $\overline{\mathbb{G}}_5 \oplus \mathbb{G}_{10}$;

(c) *The class of the 3-dimensional para-Sasakian manifolds is* $\overline{\mathbb{G}}_5$;

(d) *The class of the 3-dimensional K-paracontact metric manifolds is* $\overline{\mathbb{G}}_5$;

(e) *The class of the 3-dimensional quasi-para-Sasakian manifolds is* $\mathbb{G}_5$.

Remark 5.4. Well known result in the literature is that every $(2n + 1)$-dimensional para-Sasakian manifold M is a K-paracontact metric manifold but the converse is true only if M is 3-dimensional. The assertions in Corollary 4.5, Theorem 4.8 and (c), (d) from Theorem 5.3 agree with this result.

6. 3-Dimensional Lie algebras corresponding to Lie groups with almost paracontact metric structure

Let L be a 3-dimensional real connected Lie group and $\mathfrak{g}$ be its Lie algebra with a basis $\{E_1, E_2, E_3\}$ of left invariant vector fields. We define an almost paracontact structure (φ, ξ, η) and a semi-Riemannian metric g in the following way:

$$\varphi E_1 = E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0$$
$$\xi = E_3, \quad \eta(E_3) = 1, \quad \eta(E_1) = \eta(E_2) = 0,$$
$$g(E_1, E_1) = g(E_3, E_3) = -g(E_2, E_2) = 1,$$
$$g(E_i, E_j) = 0, \quad i \neq j \in \{1, 2, 3\}.$$

Then $(L, \varphi, \xi, \eta, g)$ is a 3-dimensional almost paracontact metric manifold. Since the metric g is left invariant the Koszul equality becomes

$$2g(\nabla_x y, z) = g([x, y], z) + g([z, x], y) + g([z, y], x), \tag{6.1}$$

where ∇ is the Levi-Civita connection of g. By using (6.1) we find the components $F_{ijk} = F(E_i, E_j, E_k)$, $(i, j, k \in \{1, 2, 3\})$ of the tensor F:

$$2F_{ijk} = g\left([E_i, \varphi E_j] - \varphi[E_i, E_j], E_k\right) + g\left([E_k, \varphi E_j] + [\varphi E_k, E_j], E_i\right)$$
$$+ g\left([\varphi E_k, E_i] - \varphi[E_k, E_i], E_j\right). \tag{6.2}$$

Let the commutators of $\mathfrak{g}$ be defined by $[E_i, E_j] = C_{ij}^k E_k$, where the structure constants C_{ij}^k are real numbers and $C_{ij}^k = -C_{ji}^k$. Then from (6.2) for the non-zero components F_{ijk} we obtain

$$F_{113} = -F_{131} = \frac{1}{2}(C_{12}^3 + C_{13}^2 - C_{23}^1),$$
$$F_{223} = -F_{232} = \frac{1}{2}(C_{13}^2 - C_{12}^3 - C_{23}^1),$$
$$F_{123} = -F_{132} = -C_{13}^1, \quad F_{213} = -F_{231} = C_{23}^2,$$
$$F_{331} = -F_{313} = C_{23}^3, \quad F_{332} = -F_{323} = C_{13}^3. \tag{6.3}$$

Taking into account (5.1) and (6.3), for the non-zero components of θ_F, θ_F^*, ω_F we have

$$\theta_F^3 = C_{12}^3, \quad \theta_F^{*3} = -C_{13}^1 - C_{23}^2, \quad \omega_F^1 = C_{23}^3, \quad \omega_F^2 = C_{13}^3. \tag{6.4}$$

Using (5.2), (6.3), (6.4) and applying the Jacobi identity

$$\mathop{\mathfrak{S}}_{E_i, E_j, E_k} \left[[E_i, E_j], E_k \right] = 0$$

we deduce the following

Theorem 6.1. *The manifold* $(L, \varphi, \xi, \eta, g)$ *belongs to the class* $\mathbb{G}_i (i \in \{5, 6, 10, 12\})$ *if and only if the corresponding Lie algebra* $\mathfrak{g}$ *is determined by the following commutators:*

$$\mathbb{G}_5 : [E_1, E_2] = C_{12}^1 E_1 + C_{12}^2 E_2 + C_{12}^3 E_3, \quad [E_1, E_3] = C_{13}^2 E_2,$$
$$[E_2, E_3] = C_{13}^2 E_1 : \quad \theta_{F_5}^3 = C_{12}^3 \neq 0, \quad C_{12}^1 C_{13}^2 = 0, \quad C_{12}^2 C_{13}^2 = 0; \tag{6.5}$$
$$\mathbb{G}_6 : [E_1, E_2] = C_{12}^1 E_1 + C_{12}^2 E_2, \quad [E_1, E_3] = C_{13}^1 E_1 + C_{13}^2 E_2,$$
$$[E_2, E_3] = C_{13}^2 E_1 + C_{13}^1 E_2 : \theta_{F_6}^{*3} = -2C_{13}^1 \neq 0,$$
$$C_{12}^2 C_{13}^2 - C_{13}^1 C_{12}^1 = 0, \quad C_{12}^1 C_{13}^2 - C_{13}^1 C_{12}^2 = 0; \tag{6.6}$$
$$\mathbb{G}_{10} : [E_1, E_2] = C_{12}^1 E_1 + C_{12}^2 E_2, \quad [E_1, E_3] = C_{13}^1 E_1 + C_{13}^2 E_2,$$
$$[E_2, E_3] = C_{23}^1 E_1 - C_{13}^1 E_2 : C_{13}^2 \neq C_{23}^1 \quad \text{or} \quad C_{13}^1 \neq 0,$$
$$C_{12}^1 C_{13}^1 + C_{12}^2 C_{23}^1 = 0, \quad C_{12}^1 C_{13}^2 - C_{12}^2 C_{13}^1 = 0; \tag{6.7}$$
$$\mathbb{G}_{12} : [E_1, E_2] = C_{12}^1 E_1 + C_{12}^2 E_2, \quad [E_1, E_3] = C_{13}^2 E_2 + C_{13}^3 E_3,$$
$$[E_2, E_3] = C_{13}^2 E_1 + C_{23}^3 E_3 : C_{13}^3 \neq 0 \quad \text{or} \quad C_{23}^3 \neq 0,$$
$$(C_{12}^1 - C_{23}^3) C_{13}^2 = 0, \quad (C_{12}^2 + C_{13}^3) C_{13}^2 = 0,$$
$$(C_{12}^1 - C_{23}^3) C_{13}^3 + (C_{12}^2 + C_{13}^3) C_{23}^3 = 0. \tag{6.8}$$

7. Matrix Lie groups as 3-dimensional almost paracontact metric manifolds

Let $(L, \varphi, \xi, \eta, g)$ be a 3-dimensional almost paracontact metric manifold from Sect. 6 belonging to some of the classes $\mathbb{G}_i (i \in \{5, 6, 10, 12\})$. By G we denote the simply connected Lie group isomorphic to L, both with one and the same Lie algebra $\mathfrak{g}$. Further, we find the adjoint representation Ad of G, which is the following Lie group homomorphism

$$\text{Ad}: G \longrightarrow \text{Aut}(\mathfrak{g}).$$

For $X \in \mathfrak{g}$, the map $\text{ad}_X : \mathfrak{g} \longrightarrow \mathfrak{g}$ is defined by $\text{ad}_X(Y) = [X, Y]$, where by ad_X is denoted as $\text{ad}(X)$. Due to the Jacobi identity, the map

$$\text{ad}: \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g}): X \longrightarrow \text{ad}_X$$

is Lie algebra homomorphism, which is called adjoint representation of $\mathfrak{g}$. Since the set $\text{End}(\mathfrak{g})$ of all $\mathbb{K}$-linear maps from $\mathfrak{g}$ to $\mathfrak{g}$ is isomorphic to the set of all $(n \times n)$ matrices $\text{M}(n, \mathbb{K})$ with entries in $\mathbb{K}$, ad is a matrix representation of $\mathfrak{g}$. We denote by M_i the matrices of ad_{E_i} (i = 1, 2, 3) with respect to the basis

$\{E_1, E_2, E_3\}$ of $\mathfrak{g}$. Then for an arbitrary $X = aE_1 + bE_2 + cE_3$ $(a, b, c \in \mathbb{R})$ in $\mathfrak{g}$ the matrix A of ad_X is $A = aM_1 + bM_2 + cM_3$. By using the well known identity $e^A = \mathrm{Ad}\left(e^X\right)$ we find the matrix representation of the Lie group G.

7.1. Matrix Lie groups as manifolds from the class $\mathbb{G}_5$

Let $\mathfrak{g}_5$ be the Lie algebra obtained from (6.5) by $C_{12}^1 = C_{12}^2 = C_{13}^2 = 0$, i.e.

$$[E_1, E_2] = \alpha E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0, \tag{7.1}$$

where $\alpha = \theta_{F_5}^3 = C_{12}^3$. Then from Theorem 6.1 it follows that $(L, \varphi, \xi, \eta, g)$, where L is a Lie group with a Lie algebra $\mathfrak{g}_5$, belongs to the class $\mathbb{G}_5$. In particular, the manifold is paracontact metric if and only if $\alpha = 2$. The Levi-Civita connection ∇ is given by

$$\begin{array}{lll} \nabla_{E_1} E_1 = 0, & \nabla_{E_1} E_2 = \frac{\alpha}{2} E_3, & \nabla_{E_1} E_3 = \frac{\alpha}{2} E_2, \\ \nabla_{E_2} E_1 = -\frac{\alpha}{2} E_3, & \nabla_{E_2} E_2 = 0, & \nabla_{E_2} E_3 = \frac{\alpha}{2} E_1, \\ \nabla_{E_3} E_1 = \frac{\alpha}{2} E_2, & \nabla_{E_3} E_2 = \frac{\alpha}{2} E_1, & \nabla_{E_3} E_3 = 0. \end{array}$$

It is not hard to see that the Ricci tensor Ric is equal to

$$Ric(x, y) = scal\, g(x, y) - 2scal\, \eta(x)\eta(y),$$

where $scal = \frac{\alpha^2}{2}$ is the scalar curvature. Consequently, L is an η-Einstein manifold. For the matrices M_i $(i = 1, 2, 3)$ and A we have:

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -b\alpha & a\alpha & 0 \end{pmatrix}. \tag{7.2}$$

Theorem 7.1. *The matrix representation of the Lie group G_5 corresponding to the Lie algebra $\mathfrak{g}_5$, determined by (7.1) and having the matrix representation (7.2), is*

$$G_5 = \left\{ e^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b\alpha & a\alpha & 1 \end{pmatrix} \right\}. \tag{7.3}$$

Proof. The matrix A is nilpotent of degree $q = 2$. Therefore we compute e^A directly from

$$e^A = E + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^{q-1}}{(q-1)!}, \tag{7.4}$$

i.e. $e^A = E + A$. So we obtain (7.3). $\qquad\square$

7.2. Matrix Lie groups as manifolds from the class $\mathbb{G}_6$

We consider the Lie algebra $\mathfrak{g}_6$ obtained from (6.6) by $C_{12}^1 = C_{12}^2 = 0$, i.e.

$$[E_1, E_2] = 0, \quad [E_1, E_3] = \alpha E_1 + \beta E_2, \quad [E_2, E_3] = \beta E_1 + \alpha E_2, \qquad (7.5)$$

where $\alpha = -\frac{\theta_{F_6}^{*3}}{2} = C_{13}^1$, $\beta = C_{13}^2 \neq 0$. Then from Theorem 6.1 it follows that $(L, \varphi, \xi, \eta, g)$, where L is a Lie group with a Lie algebra $\mathfrak{g}_6$, belongs to the class $\mathbb{G}_6$. The Levi-Civita connection ∇ is given by

$$\begin{array}{lll} \nabla_{E_1} E_1 = -\alpha E_3, & \nabla_{E_1} E_2 = 0, & \nabla_{E_1} E_3 = \alpha E_1, \\ \nabla_{E_2} E_1 = 0, & \nabla_{E_2} E_2 = \alpha E_3, & \nabla_{E_2} E_3 = \alpha E_2, \\ \nabla_{E_3} E_1 = -\beta E_2, & \nabla_{E_3} E_2 = -\beta E_1, & \nabla_{E_3} E_3 = 0. \end{array}$$

It not hard to see that the Ricci tensor Ric is equal to

$$Ric(x, y) = \frac{scal}{3} g(x, y),$$

where $scal = -6\alpha^2$ is the scalar curvature. Consequently, L is an Einstein manifold.

For the matrices M_i $(i = 1, 2, 3)$ and A we get:

$$M_1 = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} -\alpha & -\beta & 0 \\ -\beta & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} -c\alpha & -c\beta & a\alpha + b\beta \\ -c\beta & -c\alpha & b\alpha + a\beta \\ 0 & 0 & 0 \end{pmatrix}. \qquad (7.6)$$

Theorem 7.2. *The matrix representation of the Lie group G_6 corresponding to the Lie algebra $\mathfrak{g}_6$, determined by (7.5) and having the matrix representation (7.6), is as follows:*

- *If $c \neq 0$, $\beta = \alpha$ and $b = -a$, then*

$$G_6 = \left\{ e^A = \begin{pmatrix} \frac{1 + e^{-2c\alpha}}{2} & \frac{-1 + e^{-2c\alpha}}{2} & 0 \\ \frac{-1 + e^{-2c\alpha}}{2} & \frac{1 + e^{-2c\alpha}}{2} & 0 \\ \frac{-1 + e^{-2c\alpha}}{2} & \frac{-1 + e^{-2c\alpha}}{2} & 1 \end{pmatrix} \right\}. \qquad (7.7)$$

- *If $c \neq 0$, $\beta = \alpha$ and $b \neq -a$, then*

$$G_6 = \left\{ e^A = \begin{pmatrix} \frac{1 + e^{-2c\alpha}}{2} & \frac{-1 + e^{-2c\alpha}}{2} & \frac{(a+b)(1 - e^{-2c\alpha})}{2c} \\ \frac{-1 + e^{-2c\alpha}}{2} & \frac{1 + e^{-2c\alpha}}{2} & \frac{(a+b)(1 - e^{-2c\alpha})}{2c} \\ 0 & 0 & 1 \end{pmatrix} \right\}. \qquad (7.8)$$

- If $c \neq 0$, $\beta = -\alpha$ and $b = a$, then

$$
G_6 = \left\{ e^A = \begin{pmatrix} \frac{1+e^{-2c\alpha}}{2} & \frac{1-e^{-2c\alpha}}{2} & 0 \\ \frac{1-e^{-2c\alpha}}{2} & \frac{1+e^{-2c\alpha}}{2} & 0 \\ \frac{-1+e^{-2c\alpha}}{2} & \frac{1-e^{-2c\alpha}}{2} & 1 \end{pmatrix} \right\}. \tag{7.9}
$$

- If $c \neq 0$, $\beta = -\alpha$ and $b \neq a$, then

$$
G_6 = \left\{ e^A = \begin{pmatrix} \frac{1+e^{-2c\alpha}}{2} & \frac{1-e^{-2c\alpha}}{2} & \frac{(a-b)(1-e^{-2c\alpha})}{2c} \\ \frac{1-e^{-2c\alpha}}{2} & \frac{1+e^{-2c\alpha}}{2} & \frac{(a-b)(-1+e^{-2c\alpha})}{2c} \\ 0 & 0 & 1 \end{pmatrix} \right\}. \tag{7.10}
$$

- If $c \neq 0$, $\beta \neq \pm\alpha$, then

$$
G_6 = \begin{pmatrix} e^{-c\alpha}\cosh c\beta & -e^{-c\alpha}\sinh c\beta & \frac{a(1-e^{-c\alpha}\cosh c\beta)+be^{-c\alpha}\sinh c\beta}{c} \\ -e^{-c\alpha}\sinh c\beta & e^{-c\alpha}\cosh c\beta & \frac{b(1-e^{-c\alpha}\cosh c\beta)+ae^{-c\alpha}\sinh c\beta}{c} \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.11}
$$

- If $c = 0$, then

$$
G_6 = \left\{ e^A = \begin{pmatrix} 1 & 0 & a\alpha + b\beta \\ 0 & 1 & b\alpha + a\beta \\ 0 & 0 & 1 \end{pmatrix} \right\}. \tag{7.12}
$$

Proof. The characteristic polynomial of A is

$$
P_A(\lambda) = \lambda(-c\alpha + c\beta - \lambda)(c\alpha + c\beta + \lambda) = 0.
$$

Hence for the eigenvalues λ_i $(i = 1, 2, 3)$ of A we have

$$
\lambda_1 = 0, \quad \lambda_2 = c(\beta - \alpha), \quad \lambda_3 = -c(\alpha + \beta).
$$

First, we assume that $c \neq 0$. If $\beta = \alpha$ (resp. $\beta = -\alpha$) we obtain

$$
\lambda_1 = \lambda_2 = 0, \lambda_3 = -2c\alpha \quad (\text{resp. } \lambda_1 = \lambda_3 = 0, \lambda_2 = -2c\alpha).
$$

Let us consider the case $\beta = \alpha$. Then the eigenvectors

$$
p_1 = (-1, 1, 0), \quad p_2 = (a + b, 0, c),
$$

corresponding to $\lambda_1 = \lambda_2 = 0$, are linearly independent for arbitrary a and b. The coordinates (x_1, x_2, x_3) of the eigenvector p_3, corresponding to $\lambda_3 = -2c\alpha$, satisfy the following system:

$$
\begin{vmatrix} x_1 - x_2 + \frac{a+b}{c}x_3 = 0 \\ \\ -x_1 + x_2 + \frac{a+b}{c}x_3 = 0. \end{vmatrix}
$$

If we suppose that $b = -a$, then $p_1, p_2 = (0, 0, c)$ and $p_3 = (1, 1, 1)$ are linearly independent and for the change of basis matrix P we get

$$P = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & c & 1 \end{pmatrix}.$$

By using that $e^A = Pe^J P^{-1}$, where J is the diagonal matrix with elements $J_{ii} = \lambda_i$ and P^{-1} is the inverse matrix of P, we obtain the matrix representation (7.7) of G_6. When $b \neq -a$ the vectors $p_1, p_2 = (a+b, 0, c)$ and $p_3 = (1, 1, 0)$ are linearly independent and the matrix representation of G_6 is in the form (7.8).

In the case $\beta = -\alpha$, by analogical computations, we obtain (7.9) and (7.10).

Now, if we take $\beta \neq \alpha$, then the eigenvalues λ_i $(i = 1, 2, 3)$ of A are different and hence the corresponding eigenvectors

$$p_1 = \left(\frac{a}{c}, \frac{b}{c}, 1 \right), \quad p_2 = (1, -1, 0) \quad \text{and} \quad p_3 = (1, 1, 0)$$

are linearly independent. Then the matrix representation of G_6 is (7.11).

Finally, the assumption $c = 0$ implies that A is nilpotent matrix of degree $q = 2$ and by using (7.4) we obtain (7.12). $\square$

7.3. Matrix Lie groups as manifolds from the class $\mathbb{G}_{10}$

We consider the Lie algebra $\mathfrak{g}_{10}$ obtained from (6.7) by $C_{12}^1 = C_{12}^2 = C_{13}^2 = C_{23}^1 = 0$, i.e.

$$[E_1, E_2] = 0, \quad [E_1, E_3] = \alpha E_1, \quad [E_2, E_3] = -\alpha E_2, \tag{7.13}$$

where $\alpha = C_{13}^1 \neq 0$. Then from Theorem 6.1 it follows that $(L, \varphi, \xi, \eta, g)$, where L is a Lie group with a Lie algebra $\mathfrak{g}_{10}$, belongs to the class $\mathbb{G}_{10}$. The Levi-Civita connection ∇ is given by

$$\begin{aligned} \nabla_{E_1} E_1 = -\alpha E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = \alpha E_1, \\ \nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = -\alpha E_3, \quad \nabla_{E_2} E_3 = -\alpha E_2, \\ \nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0. \end{aligned}$$

It is easy to see that the Ricci tensor Ric is equal to

$$Ric(x, y) = scal\,\eta(x)\eta(y),$$

where $scal = -6\alpha^2$ is the scalar curvature.

For the matrices M_i $(i = 1, 2, 3)$ and A we have:

$$M_1 = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} -\alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} -c\alpha & 0 & a\alpha \\ 0 & c\alpha & -b\alpha \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.14}$$

Theorem 7.3. *The matrix representation of the Lie group G_{10} corresponding to the Lie algebra $\mathfrak{g}_{10}$, determined by* (7.13) *and having the matrix representation* (7.14), *is as follows*:

- *If $c \neq 0$, then*

$$G_{10} = \left\{ e^A = \begin{pmatrix} e^{-c\alpha} & 0 & \frac{a\left(1-e^{-c\alpha}\right)}{c} \\ 0 & e^{c\alpha} & \frac{b(1-e^{c\alpha})}{c} \\ 0 & 0 & 1 \end{pmatrix} \right\}. \tag{7.15}$$

- *If $c = 0$, then*

$$G_{10} = \left\{ e^A = \begin{pmatrix} 1 & 0 & a\alpha \\ 0 & 1 & -b\alpha \\ 0 & 0 & 1 \end{pmatrix} \right\}. \tag{7.16}$$

Proof. From the characteristic polynomial of A

$$P_A(\lambda) = (-c\alpha - \lambda)(c\alpha - \lambda)\lambda = 0$$

we find

$$\lambda_1 = -c\alpha, \quad \lambda_2 = c\alpha, \quad \lambda_3 = 0.$$

If $c \neq 0$, then the eigenvalues λ_i $(i = 1, 2, 3)$ of A are different and hence the corresponding to λ_i $(i = 1, 2, 3)$ eigenvectors

$$p_1 = (1, 0, 0), \quad p_2 = (0, 1, 0), \quad p_3 = (a, b, c)$$

are linearly independent. For the change of basis matrix P we have

$$P = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix}.$$

Then for the matrix representation of G_{10} we obtain (7.15).

In the case when $c = 0$ the matrix A is nilpotent of degree $q = 2$. Using (7.4) we establish (7.16). $\qquad\square$

7.4. Matrix Lie groups as manifolds from the class $\mathbb{G}_{12}$

Let $\mathfrak{g}_{12}$ be the Lie algebra obtained from (6.8) by $C_{13}^2 = 0$, i.e.

$$[E_1, E_2] = \alpha E_1 + \beta E_2, \quad [E_1, E_3] = -\beta E_3, \quad [E_2, E_3] = \alpha E_3, \tag{7.17}$$

where $\alpha = \omega_F^1 = C_{23}^3 = C_{12}^1 \neq 0$, $\beta = -\omega_F^2 = -C_{13}^3 = C_{12}^2 \neq 0$. Then from Theorem 6.1 it follows that $(L, \varphi, \xi, \eta, g)$, where L is a Lie group with a Lie algebra $\mathfrak{g}_{12}$, belongs to the class $\mathbb{G}_{12}$. The Levi-Civita connection ∇ is given by

$$\begin{aligned} \nabla_{E_1} E_1 &= \alpha E_2, & \nabla_{E_1} E_2 &= \alpha E_1, & \nabla_{E_1} E_3 &= 0, \\ \nabla_{E_2} E_1 &= -\beta E_2, & \nabla_{E_2} E_2 &= -\beta E_1, & \nabla_{E_2} E_3 &= 0, \\ \nabla_{E_3} E_1 &= \beta E_3, & \nabla_{E_3} E_2 &= -\alpha E_3, & \nabla_{E_3} E_3 &= -\beta E_1 - \alpha E_2. \end{aligned}$$

Hence the matrices M_i (i=1,2,3) and A are:

$$M_1 = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\beta \end{pmatrix}, \quad M_2 = \begin{pmatrix} -\alpha & 0 & 0 \\ -\beta & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta & -\alpha & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} -b\alpha & a\alpha & 0 \\ -b\beta & a\beta & 0 \\ c\beta & -c\alpha & b\alpha - a\beta \end{pmatrix}. \tag{7.18}$$

Theorem 7.4. *The matrix representation of the Lie group G_{12} corresponding to the Lie algebra $\mathfrak{g}_{12}$, determined by (7.17) and having the matrix representation (7.18), is as follows:*

- *If $b\alpha - a\beta \neq 0$, then*

$$G_{12} = \left\{ e^A = \begin{pmatrix} \frac{a\beta - b\alpha e^{a\beta - b\alpha}}{a\beta - b\alpha} & \frac{a\alpha(e^{a\beta - b\alpha} - 1)}{a\beta - b\alpha} & 0 \\ \frac{b\beta(1 - e^{a\beta - b\alpha})}{a\beta - b\alpha} & \frac{-b\alpha + a\beta e^{a\beta - b\alpha}}{a\beta - b\alpha} & 0 \\ \frac{c\beta(1 - e^{b\alpha - a\beta})}{a\beta - b\alpha} & \frac{c\alpha(e^{b\alpha - a\beta} - 1)}{a\beta - b\alpha} & e^{b\alpha - a\beta} \end{pmatrix} \right\}, \tag{7.19}$$

  where $(a, b) \neq (0, 0)$.
- *If $b\alpha - a\beta = 0$, then*

$$G_{12} = \left\{ e^A = \begin{pmatrix} 1 - b\alpha & a\alpha & 0 \\ -b\beta & 1 + a\beta & 0 \\ c\beta & -c\alpha & 1 \end{pmatrix} \right\}, \tag{7.20}$$

  where both a and b are zero or non-zero.

Proof. From the characteristic polynomial of A

$$P_A(\lambda) = \lambda(\lambda + b\alpha - a\beta)(b\alpha - a\beta - \lambda) = 0$$

we find

$$\lambda_1 = 0, \quad \lambda_2 = -b\alpha + a\beta, \quad \lambda_3 = b\alpha - a\beta.$$

First, we assume that $b\alpha - a\beta \neq 0$. From this condition it follows that $(a, b) \neq (0, 0)$ and the eigenvalues $\lambda_i \, (i = 1, 2, 3)$ of A are different. Then the corresponding to $\lambda_i \, (i = 1, 2, 3)$ eigenvectors

$$p_1 = (a, b, c), \quad p_2 = (\alpha, \beta, 0), \quad p_3 = (0, 0, 1)$$

are linearly independent and the change of basis matrix P is

$$P = \begin{pmatrix} a & \alpha & 0 \\ b & \beta & 0 \\ c & 0 & 1 \end{pmatrix}.$$

By straightforward computations we obtain that in this case (7.19) is the matrix representation of G_{12}.

If $b\alpha - a\beta = 0$, then both a and b are zero or non-zero. In this case the matrix A is nilpotent of degree $q = 2$ and the matrix representation of G_{12} is in the form (7.20). $\qquad\square$

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