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Euler's inequality in absolute geometry

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Abstract. Two results are proved synthetically in Hilbert's absolute geometry: (i) of all triangles inscribed in a circle, the equilateral one has the greatest area; (ii) of all triangles inscribed in a circle, the equilateral one has the greatest radius of the inscribed circle (which amounts, in the Euclidean case, to Euler's inequality $R \geq 2r$).

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1. Introduction

The fact that, in Euclidean geometry, in every triangle that is not equilateral, the radius of the circumscribed circle is greater than twice the radius of the inscribed circle, and in the equilateral case we have equality between the two magnitudes, i. e. that $R \geq 2r$, with equality if and only if the triangle is equilateral, can be easily deduced from the paper [2] Euler published in 1767, but cannot be found explicitly mentioned in these terms there, as pointed out in [10]. While stated earlier, by Chapple in [1]—"To inscribe and circumscribe a triangle in and about two eccentric circles, the radius of the lesser circle must be less than half the radius of the greater circle."—the proof provided there for it is faulty. That flawed proof starts with a correct proof of the fact that among all the triangles inscribed in a given circle the equilateral one has the greatest area.

If we ask for an absolute geometry version of Euler's inequality, then we cannot expect it to be $R \geq 2r$. There are two reasons why that inequality cannot hold in absolute geometry. First, there are triangles in hyperbolic geometry for which there is no circumscribed circle, so R makes no sense at all in those cases. Even if we were to state the inequality in a manner that ensures that the circumscribed circle exists—one could state that, for all triangles inscribed in a circle of radius R, we have $R \geq 2r$—the inequality is false for triangles in absolute geometries in which the sum of the angles in a triangle exceeds two right angles (to be denoted henceforth by π), as shown in [8, Lemma 3].

An inequality that both makes perfect sense in plane absolute geometry and which happens to also be true, as we shall prove in this paper, is:

Theorem 1. *Of all triangles inscribed in a given circle, the equilateral triangle and only it has the greatest radius of the inscribed circle. Put differently, if* ABC *is a non-equilateral triangle inscribed in a circle* $\mathcal{C}$*, and* EFG *is an equilateral triangle inscribed in* $\mathcal{C}$*, then* $r_{ABC} < r_{EFG}$*, where by* r_{XYZ} *we have denoted the radius of the inscribed circle of triangle* XYZ*.*

It is also true that Chapple's result,

Theorem 2. *Of all triangles inscribed in a given circle, the equilateral triangle has the greatest area.*

holds in plane absolute geometry.

The aim of this paper is to prove these two theorems inside Hilbert's plane absolute geometry $\mathcal{A}$ (whose axioms are the plane axioms of groups I, II, and III of Hilbert's *Grundlagen der Geometrie*, or equivalently the axioms A1–A9 in [11]). The core results true in $\mathcal{A}$ can be found in [4] or [5]. Their proofs in Euclidean geometry can be found in [6, pp. 79–84, pp. 100–104].

2. Comparing areas and angles of triangles inscribed in a circle

There are three kinds of models of Hilbert's plane absolute geometry, which we will refer to as *Hilbert planes*. In Hilbert planes, the sum of the angles in any triangle can be: (i) π, and then we say they are of *Euclidean type*; (ii) less than π, and then we say they are of *hyperbolic type*; (iii) greater than π, and then we say they are of *elliptic type*.

To compare the areas of two triangles, we will use Hilbert's notion of equidecomposability (*Zerlegungsgleichheit*), and will say that the area of triangle ABC is less than that of triangle $A'B'C'$ if triangle ABC can be decomposed into a finite number of triangles that can be recomposed to form a proper part of $A'B'C'$. For our purposes, this equidecomposability notion can be turned into one of first-order logic, by restricting the number of triangles used in the decomposition to three. We thus do not need to assign a measure to triangles to make sense of the phrase "the area of the equilateral triangle inscribed in $\mathcal{C}$ is greater than that of any other non-equilateral triangle inscribed in $\mathcal{C}$." Area measures can indeed be introduced in Hilbert planes, as shown in [3], but we have decided to avoid referring to them, given that we can express our theorem without area measures. Notice, however, that, if an area measure were defined, then the equidecomposability of triangles and quadrilaterals implies the equality of the areas, but the equality of the areas does not imply equidecomposability, as shown by Hilbert in his *GdG* (see also [5, Chapter 5]).

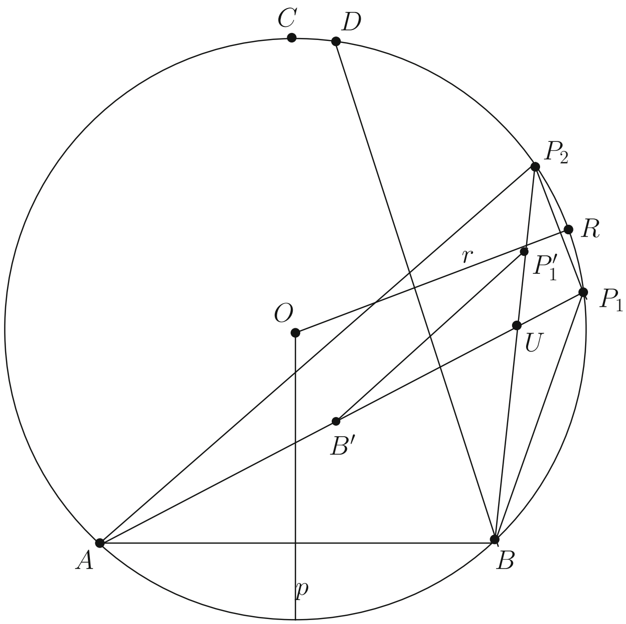

FIGURE 1 The area of triangle ABP_2 is greater than that of triangle ABP_1

To prove Theorem 2, we will first prove the following

Lemma 1. *Let A, B, and C be three points on a circle $\mathcal{C}$, such that $CA \equiv CB$, and let P_1 and P_2 be two points on the arc $\overset{\frown}{CB}$ on which A does not lie, such that P_1 lies on the arc $\overset{\frown}{P_2B}$ on which C does not lie. Then the area of triangle ABP_1 is less than that of triangle ABP_2.*

Proof. Let O be the center of the circle $\mathcal{C}$. Let U be the point of intersection of segments AP_1 and BP_2, and let b denote the internal angle bisector of $\widehat{AUB}$. Let p denote the perpendicular bisector of AB (see Fig. 1).

Since p passes through C, the points B and P_2 lie on the same side of p, and thus U and B lie on the same side of p. Since B and A lie on different sides of p, the points U and A also lie on different sides of p, i.e., p intersects the segment AU. This means that $UA > UB$. Let r denote the perpendicular bisector of P_1P_2 and let D denote the reflection of B in r. Let R denote the intersection of r with the arc $\overset{\frown}{P_2P_1}$ on which C does not lie. Since $\widehat{BOR} \equiv \widehat{ROD}$, $\widehat{BOR} < \widehat{BOC}$, we have that D lies on the arc $\overset{\frown}{AP_2}$ on which C lies. Since $\Delta P_2P_1B \equiv \Delta P_1P_2D$, we have $\widehat{BP_2P_1} \equiv \widehat{DP_1P_2}$, and, since $\widehat{DP_1P_2} < \widehat{AP_1P_2}$ (given that D lies on the arc $\overset{\frown}{AP_2}$ on which C lies), we also have

$\widehat{UP_2P_1} < \widehat{AP_1P_2}$, so $UP_1 < UP_2$ (given that opposite the greater angle lies the greater side in triangle UP_2P_1). Let B' denote the reflection of B in b and P_1' denote the reflection of P_1 in b. Since $UB < UA$ and $UP_1 < UP_2$, we have that B' lies between A and U and that P_1' lies between U and P_2.

Thus triangle ABP_1 has been split into two triangles, ABU and BUP_1, and these two triangles have been reassembled to form the quadrilateral $ABP_1'B'$, consisting of the two triangles ABU and $B'UP_1'$. Given that the quadrilateral $ABP_1'B'$ is a proper part of triangle ABP_2, we are done. □

Corollary 1. *Under the conditions of Lemma 1, in Hilbert planes of hyperbolic type the sum of the angles of triangle ABP_1 is greater than the sum of the angles of triangle ABP_2, whereas in Hilbert planes of elliptic type the sum of the angles of triangle ABP_1 is smaller than the sum of the angles of triangle ABP_2.*

Proof. The sum of the angles of triangle ABP_2 is $\widehat{ABP_2} + \widehat{P_1AB} + \widehat{P_1AP_2} + \widehat{AP_2B}$, whereas that of triangle ABP_1 is $\widehat{P_1AB}+\widehat{ABP_2}+\widehat{P_2BP_1}+\widehat{AP_1B}$. Given that $\widehat{ABP_2} + \widehat{P_1AB}$ is common in the two sums, we need to compare only the sums $\widehat{P_1AP_2} + \widehat{AP_2B}$ and $\widehat{P_2BP_1} + \widehat{AP_1B}$ to find out which among the angle sums of triangle ABP_1 and triangle ABP_2 is greater. Notice that the angles of the convex quadrilateral $AB'P_1'P_2$ are $\widehat{AP_2B}$, $\pi - \widehat{AP_1B}$, $\pi - \widehat{P_2BP_1}$, $\widehat{P_1AP_2}$. Thus the sum of the angles of $AB'P_1'P_2$ is $2\pi+\widehat{AP_2B}+\widehat{P_1AP_2}-\widehat{AP_1B}-\widehat{P_2BP_1}$. Since the sum of the angles of $AB'P_1'P_2$ is the sum of the angles of the two triangles $AB'P_1'$ and $AP_1'P_2$, and thus $< 2\pi$ in the hyperbolic case and $> 2\pi$ in the elliptic case, we have $\widehat{P_1AP_2} + \widehat{AP_2B} < \widehat{AP_1B} + \widehat{P_2BP_1}$ in Hilbert planes of hyperbolic type and $\widehat{P_1AP_2} + \widehat{AP_2B} > \widehat{AP_1B} + \widehat{P_2BP_1}$ in Hilbert planes of elliptic type. □

Lemma 2. *If $\mathcal{C}, A, B, C, P_1$, and P_2 are as in Lemma 1, then $\widehat{AP_2B} < \widehat{AP_1B}$ in Hilbert planes of hyperbolic type, whereas $\widehat{AP_2B} > \widehat{AP_1B}$ in Hilbert planes of elliptic type.*

Proof. Let P be an arbitrary point of the arc $\overset{\frown}{CB}$ on which P_1 and P_2 lie. We distinguish several cases: (i) the center O of $\mathcal{C}$ and C lie one the same side of AB; (ii) O and C lie on different sides of AB; and (iii) O lies on AB. Case (i) splits into three subcases: (α) O lies inside triangle ABP; (β) O lies outside triangle ABP; and (γ) O lies on AP.

In case (α), the sum of the angles of triangle ABP is $\widehat{OAB} + \widehat{OBA} + \widehat{OAP} + \widehat{OPA} + \widehat{OPB} + \widehat{PBO}$. Given that the triangles OAB, OAP, and OPB are isosceles, the above sum can be written as $2\widehat{OAB} + 2(\widehat{OPA} + \widehat{OPB})$, i. e., $2\widehat{OAB} + 2\widehat{APB}$. In case ($\beta$), the sum of the angles of triangle ABP is $\widehat{OAB} - \widehat{OAP} + \widehat{OPB} - \widehat{OPA} + \widehat{OBP} + \widehat{OBA}$. Given that the triangles OAB, OAP, and OPB are isosceles, the above sum can be written as $2\widehat{OAB} + 2\widehat{APB}$. In case ($\gamma$) the sum of the angles of triangle ABP is $\widehat{OAB}+\widehat{OBA}+\widehat{OBP}+\widehat{OPB}$. Given that the triangles OAB and OPB are isosceles, the above sum can be written as $2\widehat{OAB}+2\widehat{APB}$.

In case (ii), the sum of the angles of triangle ABP is $\widehat{OAP}-\widehat{OAB}+\widehat{OBP}-\widehat{ABO}+\widehat{OPB}+\widehat{OPA}$. Given that the triangles OAB, OAP, and OPB are isosceles, the above sum can be written as $2(\widehat{OPA}+\widehat{OPB})-2\widehat{OAB}$, i. e., $2\widehat{APB}-2\widehat{OAB}$.

In case (iii), the sum of the angles of triangle ABP is $\widehat{OAP}+\widehat{OPA}+\widehat{OPB}+\widehat{OBP}$. Given that the triangles OAP and OPB are isosceles, the above sum can be written as $2\widehat{APB}$.

Thus, in all cases, the sum of the angles of triangle ABP is $2\widehat{APB}+2\epsilon\widehat{OAB}$, where $\epsilon \in \{-1, 0, 1\}$. The desired conclusion now follows by Corollary 1. □

For completeness' sake, let us mention that in case the metric of the Hilbert plane is Euclidean, under the conditions of Lemma 1, the sum of the angles of triangles ABP_1 and ABP_2 are the same (namely π), and $\widehat{AP_2B} \equiv \widehat{AP_1B}$.

3. Proof of Theorem 2

We denote by $\sigma(XYZ) < \sigma(UVW)$ the fact that the area of triangle XYZ is less than that of triangle UVW and by $\sigma(XYZ) = \sigma(UVW)$ the fact that the two triangles have the same area (in the sense that they can be decomposed into a collection of pairwise congruent triangles). By [7] there is an angle of $\frac{2\pi}{3}$ in $\mathcal{A}$. Let ABC be an arbitrary triangle inscribed in the circle $\mathcal{C}$ of center O. If one of $\widehat{AOB}$, $\widehat{BOC}$, or $\widehat{COA}$ is $\frac{2\pi}{3}$, say $\widehat{AOB}$, and we let C' denote the point on $\mathcal{C}$ for which $C'A \equiv C'B \equiv AB$, then, if C and C' lie on the same side of AB, by Lemma 1, $\sigma(ABC) \leq \sigma(ABC')$, with equality if and only if $C = C'$. If C and C' lie on the different sides of AB, then we let C_1 be the reflection of C in AB. Since C_1 lies inside $\mathcal{C}$, if C_2 denotes the second intersection of AC_1 with $\mathcal{C}$, then $\sigma(ABC) = \sigma(ABC_1) < \sigma(ABC_2) < \sigma(ABC')$, the last inequality holding by Lemma 1.

If none of $\widehat{AOB}$, $\widehat{BOC}$, or $\widehat{COA}$ is $\frac{2\pi}{3}$, then one of them, say $\widehat{AOB}$, must be less than $\frac{2\pi}{3}$, and one of the remaining two angles, say $\widehat{BOC}$, must be greater than $\frac{2\pi}{3}$. Let D denote the midpoint of the arc $\overset{\frown}{CA}$ on which B lies. If $\widehat{AOD} \geq \frac{2\pi}{3}$, then let B' be the point on the arc $\overset{\frown}{CA}$ on which B lies such that $\widehat{AOB'}$ is $\frac{2\pi}{3}$. By Lemma 1, $\sigma(ABC) < \sigma(AB'C)$. Let EFG denote an equilateral triangle inscribed in $\mathcal{C}$. We can argue as above to deduce that $\sigma(AB'C) \leq \sigma(EFG)$, and thus $\sigma(ABC) < \sigma(EFG)$. If $\widehat{AOD} < \frac{2\pi}{3}$, then let B' be the point on the arc $\overset{\frown}{CA}$ on which B lies such that $\widehat{COB'}$ is $\frac{2\pi}{3}$.

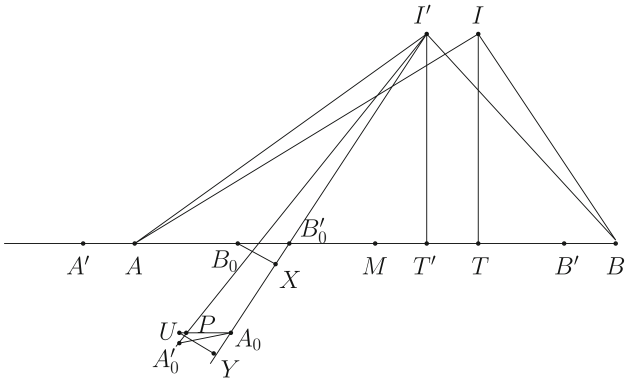

FIGURE 2 If $IT \equiv I'T'$, $IT \perp AB$, $I'T' \perp AB$, then $\widehat{AIB} < \widehat{AI'B}$

By Lemma 1, $\sigma(ABC) < \sigma(AB'C)$, and we argue as above to deduce that $\sigma(ABC) < \sigma(EFG)$. □

4. Additional Lemmas needed to prove Theorem 1

To prove Theorem 1, we first need no show that the radius of the inscribed circle of the triangle ABP increases if we keep A and B fixed, and allow P to move from B toward C on the arc $\widehat{CB}$ that does not contain A (notations being as in Lemma 1). To be precise

Lemma 3. *If $\mathcal{C}, A, B, C, P_1$, and P_2 are as in Lemma 1, then $r_{ABP_1} < r_{ABP_2}$.*

Proof. To prove Lemma 3, however, we need three additional lemmas.

Lemma 4. *If M is the midpoint of a segment AB, T is a point between M and B, T' is a point between M and T, and I and I' are two points on the same side of AB such that $IT \equiv I'T'$ and $IT \perp AB$, $I'T' \perp AB$, then $\widehat{AIB} < \widehat{AI'B}$.*

Proof. Let B' be between T' and B such that $T'B' \equiv TB$, and A' be such that A lies between A' and M and $T'A' \equiv TA$. Notice that $\Delta AIB \equiv \Delta A'I'B'$ and that $BB' \equiv AA'$. We claim that $\widehat{A'I'A} < \widehat{B'I'B}$. To see this, let B_0 and B'_0 be the reflections of B and B' in T' respectively (see Fig. 2). Then $B_0B'_0 \equiv AA'$ and the order on the segment $T'A'$ is $T'B'_0B_0AA'$.

Since $\widehat{B'I'B} \equiv \widehat{B'_0I'B_0}$, we need to show that $\widehat{A'I'A} < \widehat{B'_0I'B_0}$. To see this, notice that $\widehat{I'B'_0B_0} < \widehat{I'AA'}$, since the latter is an exterior angle of triangle $I'B'_0A$. Note also that $I'B'_0 < I'A$, given that $\widehat{I'B'_0A}$ is, as an exterior angle of the right triangle $I'T'B'_0$, obtuse. Let A_0 be a point such that B'_0 lies between I' and A_0, and such that $I'A_0 \equiv I'A$ (i. e., A_0 is the other endpoint of the segment

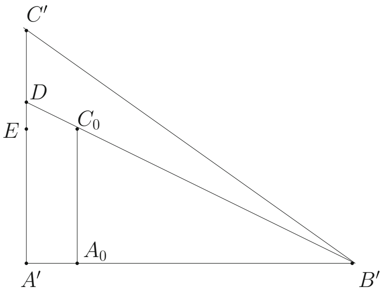

FIGURE 3 $A_0C_0 < A'D$

$I'A$ transported on the ray $\overrightarrow{I'B'_0}$ from I'). Let A'_0 be such that T' and A'_0 are on different sides of $I'A_0$, $A_0A'_0 \equiv AA'$ and $\widehat{I'A_0A'_0} \equiv \widehat{I'AA'}$. Notice that, by construction, $\Delta I'AA' \equiv \Delta I'A_0A'_0$. Let U be a point such that U and T' are on different sides of $I'A_0$ and such that $A_0U \equiv B'_0B_0$ and $\widehat{I'B'_0B_0} \equiv \widehat{I'A_0U}$. If X and Y denote the feet of the perpendiculars from B_0 and from U to $I'A_0$, then $XYUB_0$ is a Saccheri quadrilateral (since $\Delta A_0UY \equiv \Delta B'_0B_0X$ (AAS), we have $UY \equiv B_0X$), and so B_0U is parallel to B'_0A_0. This means that U lies inside the angle $\widehat{A_0I'B_0}$ (for else, the line UB_0 would have to intersect the segment $I'A_0$ by the Pasch axiom). Since $\widehat{I'A_0U} < \widehat{I'A_0A'_0}$, by the Crossbar Theorem (see [4, p. 116]), the ray $\overrightarrow{A_0U}$ intersects $I'A'_0$ in some point P. Notice that P cannot be such that U lies between A_0 and P or such that $P = U$, for then, in triangle $A_0PA'_0$, $\widehat{A_0PA'_0} > \widehat{I'A_0U} > \frac{\pi}{2}$, and $A_0P \geq A_0A'_0$, a contradiction, for opposite the greater angle lies the greater side. Thus P is between A_0 and U, and we are done, as this implies that $\widehat{A'I'A} < \widehat{B'I'B}$. This implies $\widehat{A'I'B'} < \widehat{AI'B}$ and, since $\widehat{A'I'B'} \equiv \widehat{AIB}$, we deduce that $\widehat{AIB} < \widehat{AI'B}$. □

Lemma 5. *Let ABC and $A'B'C'$ be two triangles such that $AB < A'B'$, $\widehat{ABC} < \widehat{A'B'C'}$, and $\widehat{BAC}$ and $\widehat{B'A'C'}$ are right angles. Then $AC < A'C'$.*

Proof. Let A_0 be the point on the ray $\overrightarrow{B'A'}$ for which $B'A_0 \equiv BA$ (see Fig. 3). Since $AB < A'B'$, A_0 lies between B' and A'. Now transport $\widehat{ABC}$ from B' on the half-plane determined by $A'B'$ on which C' lies, such that one of its legs is $\overrightarrow{B'A'}$, and let $\vec{r}$ denote its other leg. Since $\widehat{ABC} < \widehat{A'B'C'}$, $\vec{r}$ is inside the angle $\widehat{A'B'C'}$ and thus, by the Crossbar Theorem, intersects the segment $A'C'$ in D. Let C_0 denote the point of intersection of the perpendicular p raised in A_0 on $A'B'$ with the segment $B'D$ (the point of intersection exist by the Pasch axiom applied to triangle $A'B'D$ with secant p). The triangles ABC and $A_0B'C_0$ are congruent by our construction, so $AC \equiv A_0C_0$. Now $A_0C_0 < A'D$, for if E were the point on ray $\overrightarrow{A'C'}$ such that $A_0C_0 \equiv A'E$, then, $EA'A_0C_0$ being a Saccheri quadrilateral, EC_0 is parallel to $A'A_0$, and so E can be neither D

nor can it be such that D is between A' and E (for then EC_0 would intersect the segment $A'B'$ by the Pasch axiom applied to triangle $B'A'D$ and secant EC_0). Thus $AC \equiv A_0C_0 < A'D < A'C'$. □

Lemma 6. *Let ABC and $A'B'C'$ be two triangles such that $AB \equiv A'B'$, $\widehat{CBA} < \widehat{C'B'A'}$, and $\widehat{BAC}$ and $\widehat{B'A'C'}$ are right angles. Then $\widehat{B'C'A'} < \widehat{BCA}$.*

Proof. Let D be the point on the segment $A'C'$ for which $\widehat{A'B'D} \equiv \widehat{ABC}$ (see Fig. 3). Then the triangles $A'B'D$ and ABC are congruent, and so $\widehat{A'DB'} \equiv \widehat{ACB}$. Since $\widehat{A'DB'}$ is exterior angle in triangle $C'DB'$, we have $\widehat{ACB} \equiv \widehat{A'DB'} > \widehat{A'C'B'}$. □

We now turn to the proof of Lemma 3. Let, for $i \in \{1, 2\}$, I_i denote the center of the inscribed circle of triangle ABP_i, i. e., the intersection of the internal bisectors of the angles $\widehat{P_iAB}$ and $\widehat{P_iBA}$. First, notice that, since $\widehat{P_1AB} < \widehat{P_2AB}$ and $\widehat{P_1BA} > \widehat{P_2BA}$, the same inequalities must hold for the halves of these angles, i. e., $\widehat{I_1AB} < \widehat{I_2AB}$ and $\widehat{I_1BA} > \widehat{I_2BA}$. This means that the segments AI_1 and BI_2 intersect in a point F. Let, for $i \in \{1, 2\}$, T_i, U_i, and V_i stand for the feet of the perpendiculars from I_i to AB, P_iA, and P_iB, respectively. Let M be the midpoint of AB. The order of the points T_1, T_2, M on AB thus is AMT_2T_1B.

We will consider each of three types of Hilbert planes separately.

In the Euclidean case (see Fig. 4), since $\widehat{AP_1B} \equiv \widehat{AP_2B}$, we have $\widehat{AI_iB} = \frac{\pi + \widehat{AP_iB}}{2}$, for $i \in \{1, 2\}$, so $\widehat{AI_1B} = \widehat{AI_2B}$. Thus I_1 and I_2 lie on an arc $\mathcal{U}$ of a circle that goes through A and B, both on the arc $\widehat{NB}$ of that circle, where by N we have denoted the intersection of MC with $\mathcal{U}$. We are thus in the situation described by Lemma 1, so the area of triangle ABI_2 is greater than that of triangle ABI_1. Since in Hilbert planes of Euclidean type, one can associate an area measure of a triangle to be half of the product of the altitude to a side with that side, and the side AB is common to both triangles, we conclude that $I_1T_1 < I_2T_2$, i. e., $r_{ABP_1} < r_{ABP_2}$.

In the hyperbolic case (see Fig. 5), let I_2' be the point on ray $\overrightarrow{T_2I_2}$ for which $T_2I_2' \equiv T_1I_1$. By Lemma 4, $\widehat{AI_1B} < \widehat{AI_2'B}$. Let U_2' and V_2' be the images of T_2 under the reflections in AI_2' and BI_2' respectively. Since $2\widehat{AI_1B} > \pi$ (else $\overrightarrow{AU_1}$ and $\overrightarrow{BV_1}$ would not intersect), we have, *a fortiori*, $2\widehat{AI_2'B} > \pi$. Let b denote the internal angle bisector of $\widehat{U_2'I_2'V_2'}$. Since $\widehat{U_1I_1P_1} = \pi - \widehat{AI_1B}$, the angle between b and $\overrightarrow{I_2'U_2'}$, whose measure is $\pi - \widehat{AI_2'B}$, is less than $\widehat{U_1I_1P_1}$, which means that b must intersect AU_2' in a point P_2'. By Lemma 6, we have $\widehat{U_1P_1I_1} < \widehat{U_2'P_2'I_2'}$, and thus the same can be said about twice the corresponding angles, i. e., $\widehat{AP_1B} < \widehat{AP_2'B}$. By Lemma 2, $\widehat{AP_2B} < \widehat{AP_1B}$, so $\widehat{AP_2B} < \widehat{AP_2'B}$. If $I_2' = I_2$,

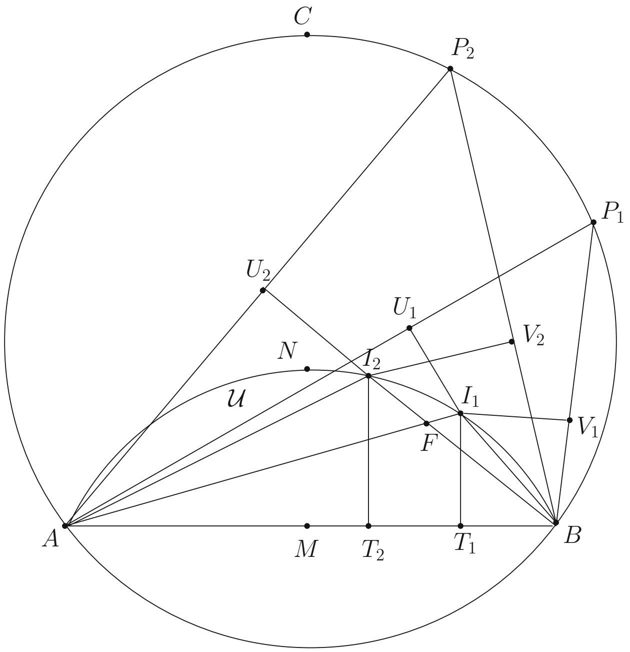

FIGURE 4 $I_1T_1 < I_2T_2$ in the Euclidean case

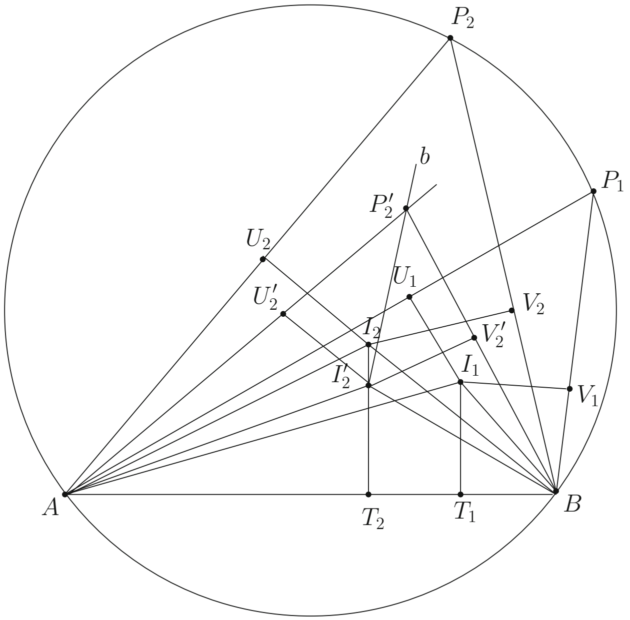

FIGURE 5 $I_1T_1 < I_2T_2$ in the hyperbolic case then $P_2' = P_2$, and so $\widehat{AP_2B} \equiv \widehat{AP_2'B}$. If I_2' were such that I_2 lies between T_2 and I_2', then P_2 would lie inside triangle ABP_2', and thus $\widehat{AP_2B} > \widehat{AP_2'B}$. We must thus have that I_2' lies between T_2 and I_2 (and thus P_2' lies inside the triangle AP_2B), which means that $r_{ABP_2} > r_{ABP_1}$.

In the elliptic case, since, for $i \in \{1, 2\}$, $AU_i \equiv AT_i$ and $BV_i \equiv BT_i$, we have that $AU_i + BV_i = AB$, so $P_iA + P_iB = P_iU_i + AU_i + P_IV_i + BV_i = 2P_iU_i + AB$. By [9, Lemma 1], we have $P_1A + P_1B < P_2A + P_2B$, so $2P_1U_1 + AB < 2P_2U_2 + AB$, i. e., $P_1U_1 < P_2U_2$. By Lemma 2, $\widehat{AP_2B} > \widehat{AP_1B}$, so the same inequality holds for the halves, i. e., $\widehat{U_2P_2I_2} > \widehat{U_1P_1I_1}$. We can now apply Lemma 5 to triangles $U_1P_1I_1$ and $U_2P_2I_2$ to conclude that $I_1U_1 < I_2U_2$, i. e., $r_{ABP_1} < r_{ABP_2}$. □

5. Proof of Theorem 1

By [7] there is an angle of $\frac{2\pi}{3}$ in $\mathcal{A}$. Let ABC be an arbitrary triangle inscribed in the circle $\mathcal{C}$ of center O. Here we distinguish several cases. If one of $\widehat{AOB}$, $\widehat{BOC}$, or $\widehat{COA}$ is $\frac{2\pi}{3}$ (case 1), say $\widehat{AOB}$, and we let C' denote the point on $\mathcal{C}$ for which $C'A \equiv C'B \equiv AB$, then, by Lemma 3, $r_{ABC} \leq r_{ABC'}$, with equality if and only if $C = C'$. We denote by r_Δ the radius of the inscribed circle in the equilateral triangle inscribed in $\mathcal{C}$.

If none of $\widehat{AOB}$, $\widehat{BOC}$, or $\widehat{COA}$ is $\frac{2\pi}{3}$ (case 2), then one of them, say $\widehat{AOB}$ must be less than $\frac{2\pi}{3}$, and one of the remaining two angles, say $\widehat{BOC}$, must be greater than $\frac{2\pi}{3}$. Let D denote the midpoint of the arc $\overset{\frown}{CA}$ on which B lies. If $\widehat{AOD} \geq \frac{2\pi}{3}$ (case 2a), then let B' be the point on the arc $\overset{\frown}{CA}$ on which B lies such that $\widehat{AOB'}$ is $\frac{2\pi}{3}$. By Lemma 3, $r_{ABC} < r_{AB'C}$, and we can argue as in case 1 to deduce that $r_{AB'C} \leq r_\Delta$, and thus $r_{ABC} < r_\Delta$. If $\widehat{AOD} < \frac{2\pi}{3}$ (case 2b), then let B' be the point on the arc $\overset{\frown}{CA}$ on which B lies such that $\widehat{COB'}$ is $\frac{2\pi}{3}$. By Lemma 3, $r_{ABC} < r_{AB'C}$, and we argue as in case 1 to deduce that $r_{ABC} < r_\Delta$.

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