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Real hypersurfaces of complex two-plane Grassmannians with certain parallel conditions

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Abstract. In Chen (Bull Korean Math Soc 54(3):975–992, 2017), we introduce the notion of *-Ricci tensor in a real hypersurface of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2}), m \geq 3$, and in the present paper we study the characterizations of the Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel, Reeb parallel and Lie Reeb parallel *-Ricci tensor, respectively.

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1. Introduction

A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ consists of all complex two dimensional linear subspaces of $\mathbb{C}^{m+2}$, which is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyper Kähler manifold (See Berndt and Suh [1,2]). Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$. The Kähler structure J on $G_2(\mathbb{C}^{m+2})$ induces a structure vector field ξ called *Reeb vector field* on M by $\xi := -JN$, where N is a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. For the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$, its canonical basis $\{J_1, J_2, J_3\}$ induces the almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ on M by $\xi_v := -J_vN, v = 1, 2, 3$. It is well known that for the real hypersurface M there exist two natural geometrical conditions that $[\xi] = \text{Span}\{\xi\}$ or $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator A of M. Denote by $\mathfrak{D}$ the orthogonal complement of the distribution $\mathfrak{D}^\perp$. By

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using such geometrical conditions Berndt and Suh proved that the Reeb vector field ξ either belongs to $\mathfrak{D}$ or $\mathfrak{D}^{\perp}$ and gave the following classification:

Theorem 1.1. ([1]) *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2}), m \geq 3$. If $\mathfrak{D}^{\perp}$ and $[\xi]$ are invariant under shape operator, then*

- *(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ for $\xi \in \mathfrak{D}^{\perp}$, or*
- *(B) M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$ for $\xi \in \mathfrak{D}$, where $m = 2n$.*

If the Reeb vector field ξ is invariant under by shape operator, M is said to be a *Hopf hypersurface*, that is, $A\xi = \alpha\xi$, where $\alpha = g(A\xi, \xi)$ is a smooth function. Based on the classification of Theorem 1.1 Berndt and Suh later gave a new characterization for the type (B) hypersurfaces of $G_2(\mathbb{C}^{m+2})$.

Theorem 1.2. ([6]) *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2}), m \geq 3$. Then the Reeb vector ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

For the classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, the assumption that the Ricci tensor satisfies certain conditions is key. For example, Suh and Jeong classified the real Hopf hypersurfaces of $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor and pseudo anti-commuting Ricci tensor, respectively (cf.[5,8]). Also, in the series of articles Suh studied respectively the real hypersurfaces admitting a parallel, Reeb parallel, and Reeb invariant Ricci tensor (see [9–11]).

As the corresponding of Ricci tensor, we note that Hamada in [4] defined the *-Ricci tensor of a real hypersurface in non-flat complex space forms by

$$Ric^*(X,Y) = \frac{1}{2} trace\{\phi \circ R(X, \phi Y)\}, \quad \forall X, Y \in TM. \tag{1.1}$$

In [3], we considered a real hypersurface of $G_2(\mathbb{C}^{m+2})$ with commuting *-Ricci tensor and pseudo anti-commuting *-Ricci tensor, respectively. Motivated by the present work, in this paper we study a real Hopf hypersurface whose *-Ricci tensor satisfies certain parallel conditions. We first consider the real hypersurface with parallel *-Ricci tensor, i.e. $\nabla S^* = 0$, where the *-Ricci operator S^* is defined by $Ric^*(X,Y) = g(S^*X, Y)$ for any vector fields X, Y on M. We assert the following:

Theorem 1.3. *There do not exist any Hopf hypersurfaces with parallel *-Ricci tensor in $G_2(\mathbb{C}^{m+2}), m \geq 3$.*

However, by relaxing the parallel condition to Reeb parallel, i.e. $\nabla_\xi S^* = 0$, we have the following result.

Theorem 1.4. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2}), m \geq 3$ with Reeb parallel *-Ricci tensor. If $S^*\mathfrak{D}^{\perp} \subset \mathfrak{D}^{\perp}$ then either M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$, where*

$m = 2n$, or M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Finally we assume that the *-Ricci tensor is Lie Reeb parallel, i.e. $\mathfrak{L}_\xi Ric^* = 0$, where $\mathfrak{L}_\xi$ denotes the Lie derivative along Reeb vector field ξ, and prove the following:

Theorem 1.5. *Let M be a Hopf hypersurface of complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2}), m \geq 3$. If the *-Ricci tensor is Lie Reeb parallel, then M is an open part of a tube around a totally geodesic in $G_2(\mathbb{C}^{m+2})$.*

This paper is organized as follows: In Sect. 2, some basic concepts and formulas for real hypersurfaces in complex two-plane Grassmannian are presented. In Sect. 3 we consider Hopf hypersurfaces with parallel *-Ricci tensor and give the proofs of Theorem 1.3 and Theorem 1.4. In Sect. 4 we assume that the *-Ricci tensor of Hopf hypersurface is Lie Reeb parallel and give the proof of Theorem 1.5.

2. Preliminaries

In this section we will summarize some basic notations and formulas about the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$. For more detail please refer to [1,2,7–9]. Let $G_2(\mathbb{C}^{m+2})$ be the complex Grassmannian manifold of all complex 2-dimensional linear spaces of $\mathbb{C}^{m+2}$. In fact $G_2(\mathbb{C}^{m+2})$ can be identified with a homogeneous space $SU(m+2)/(S(U(2) \times U(m))$. Up to scaling there exists the unique $S(U(2) \times U(m))$-invariant Riemannian metric $\widetilde{g}$ on $G_2(\mathbb{C}^{m+2})$. The Grassmannian manifold $G_2(\mathbb{C}^{m+2})$ equipped such a metric becomes a symmetric space of rank two, which is both Kähler and quaternionic Kähler. From now on we always assume $m \geq 3$ because it is well known that $G_2(\mathbb{C}^3)$ is isometric to $\mathbb{C}P^2$ and $G_2(\mathbb{C}^4)$ is isometric to the real Grassmannian manifold $G_2^+(\mathbb{R}^6)$ of oriented 2-dimensional linear subspace of $\mathbb{R}^6$.

Denote by J and $\mathfrak{J}$ the Kähler structure and quaternionic Kähler structure on $G_2(\mathbb{C}^{m+2})$, respectively. A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of almost Hermitian structures J_v such that $J_v J_{v+1} = J_{v+2} = -J_{v+1} J_v$, where the index is taken modulo three. As is well known the Kähler structure J and quaternionic Kähler structure $\mathfrak{J}$ satisfy the following relations:

$$JJ_v = J_v J, \quad trace(JJ_v) = 0, \quad v = 1, 2, 3.$$

We denote $\widetilde{\nabla}$ by the Levi-Civita connection with respect to $\widetilde{g}$, there exist 1-forms q_1, q_2, q_3 such that $\widetilde{\nabla}_X J_v = q_{v+2}(X) J_{v+1} - q_{v+1}(X) J_{v+2}$ for any vector field X on $G_2(\mathbb{C}^{m+2})$.

Let M be an immersed real hypersurface of $G_2(\mathbb{C}^{m+2})$ with induced metric g. There exists a local defined unit normal vector field N on M and we write $\xi := -JN$ by the structure vector field of M. An induced one-form η is defined by $\eta(\cdot) = \widetilde{g}(J\cdot, N)$, which is dual to ξ. For any vector field X on M the tangent part of JX is denoted by $\phi X = JX - \eta(X)N$. Moreover, the following identities hold:

$$\phi^2 = -Id + \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \phi \circ \xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{2.2}$$

where $X, Y \in \mathfrak{X}(M)$. By these formulas, we know that (ϕ, η, ξ, g) is an almost contact metric structure on M. Similarly, for every almost Hermitian structure J_v, it induces an almost contact structure $(\phi_v, \eta_v, \xi_v, g)$ on M by

$$\xi_v = -J_v N, \quad \eta_v(X) = g(\xi_v, X), \quad \phi_v X = J_v X - \eta_v(X)N,$$

for any vector field X. Thus the relations (2.1) and (2.2) hold for $(\phi_v, \eta_v, \xi_v, g)$. Denote by ∇, A the induced Riemannian connection and the shape operator on M, respectively. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX.$$

Also, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX. \tag{2.3}$$

Moreover, the following equations are proved (see [7]):

$$\phi_{v+1}\xi_v = -\xi_{v+2}, \quad \phi_v \xi_{v+1} = \xi_{v+2}, \tag{2.4}$$

$$\phi\xi_v = \phi_v \xi, \quad \eta(\xi_v) = \eta_v(\xi), \tag{2.5}$$

$$\phi\phi_v X = \phi_v \phi X + \eta_v(X)\xi - \eta(X)\xi_v, \tag{2.6}$$

$$\nabla_X \xi_v = q_{v+2}(X)\xi_{v+1} - q_{v+1}(X)\xi_{v+2} + \phi_v AX, \tag{2.7}$$

$$\begin{aligned}(\nabla_X \phi_v)Y &= -q_{v+1}(X)\phi_{v+2}Y + q_{v+2}(X)\phi_{v+1}Y \\ &\quad + \eta_v(Y)AX - g(AX, Y)\xi_v,\end{aligned} \tag{2.8}$$

$$\begin{aligned}\nabla_X(\phi_v \xi) &= q_{v+2}(X)\phi_{v+1}\xi - q_{v+1}(X)\phi_{v+2}\xi \\ &\quad + \phi_v \phi AX - g(AX, \xi)\xi_v + \eta(\xi_v)AX.\end{aligned} \tag{2.9}$$

The curvature tensor R and Codazzi equation of M are given respectively as follows:

$$\begin{aligned}R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y \\ &\quad + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y + 2g(X, \phi Y)\phi Z \\ &\quad + \sum_{v=1}^{3} \Big\{ g(\phi_v Y, Z)\phi_v X - g(\phi_v X, Z)\phi_v Y - 2g(\phi_v X, Y)\phi_v Z \Big\} \\ &\quad + \sum_{v=1}^{3} \Big\{ g(\phi_v \phi Y, Z)\phi_v \phi X - g(\phi_v \phi X, Z)\phi_v \phi Y \Big\} \\ &\quad - \sum_{v=1}^{3} \{ \eta(Y)\eta_v(Z)\phi_v \phi X - \eta(X)\eta_v(Z)\phi_v \phi Y \} \\ &\quad - \sum_{v=1}^{3} \Big\{ \eta(X)g(\phi_v \phi Y, Z) - \eta(Y)g(\phi_v \phi X, Z) \Big\}\xi_v \\ &\quad + g(AY, Z)AX - g(AX, Z)AY\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ &\quad + \sum_{v=1}^{3}\Big\{\eta_v(X)\phi_v Y - \eta_v(Y)\phi_v X - 2g(\phi_v X, Y)\xi_v\Big\} \\ &\quad + \sum_{v=1}^{3}\Big\{\eta_v(\phi X)\phi_v\phi Y - \eta_v(\phi Y)\phi_v\phi X\Big\} \\ &\quad + \sum_{v=1}^{3}\Big\{\eta(X)\eta_v(\phi Y) - \eta(Y)\eta_v(\phi X)\Big\}\xi_v \qquad (2.11)\end{aligned}$$

for any vector fields X, Y, Z on M.

Notice that Berndt and Suh [1] proved the following two properties for the real hypersurfaces of types (B) and (A).

Proposition 2.1. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \delta = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \frac{\pi}{4})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\delta) = 4m - 4 = m(\mu),$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathfrak{J}J\xi, \quad T_\gamma = \mathfrak{J}\xi,\ T_\delta,\ T_\mu,$$

where

$$T_\delta \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\delta = T_\delta, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\delta = T_\mu.$$

Proposition 2.2. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three(if $r = \frac{\pi}{2}$)or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \frac{\pi}{\sqrt{8}})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned}T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N, \\ T_\lambda &= \{X \mid X \perp \mathbb{H}\xi, JX = J_1X\}, \\ T_\mu &= \{X \mid X \perp \mathbb{H}\xi, JX = -J_1X\}.\end{aligned}$$

Recall that the *-Ricci operator S^* of M is defined by

$$g(S^*X,Y) = Ric^*(X,Y) = \frac{1}{2}trace\{\phi\circ R(X,\phi Y)\}, \quad \text{for all } X,Y\in TM.$$

The *-Ricci operator S^* is expressed as follows([3]):

$$S^*X = -(4m+6)\phi^2X - (\phi A)^2X + 2\sum_{v=1}^{3}\Big\{\eta_v(\phi X)\phi\xi_v - \eta_v(X)\xi_v + \eta(\xi_v)\eta_v(X)\xi + \eta_v(\xi)\phi\phi_v X\Big\} \tag{2.12}$$

for all $X\in TM$. Making use of (2.12), a straightforward computation gives the following formulas:

$$(\phi S^* - S^*\phi)X = \phi[(A\phi)^2 - (\phi A)^2]X - 4\sum_{v=1}^{3}\eta_v(\xi)\eta(X)\phi\xi_v, \quad \forall X\in TM, \tag{2.13}$$

$$S^*\xi = -(\phi A)^2\xi + 4\sum_{v=1}^{3}\Big\{-\eta_v(\xi)\xi_v + \eta(\xi_v)\eta_v(\xi)\xi\Big\}. \tag{2.14}$$

From now on we always assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. As in [1], by taking the inner product of the Codazzi equation (2.11) with ξ, we have

$$\begin{aligned} A\phi AX = {} & \frac{1}{2}\alpha(A\phi X + \phi AX) + \phi X \\ & - 2\sum_{v=1}^{3}\Big\{\eta(\xi_v)\eta(\phi_v X)\xi + \eta(\xi_v)\eta(X)\phi_v\xi\Big\} \\ & + \sum_{v=1}^{3}\Big\{\eta_v(X)\phi\xi_v + \eta(\phi_v X)\xi_v + \eta(\xi_v)\phi_v X\Big\}. \end{aligned} \tag{2.15}$$

From this we assert the following

Lemma 2.3. ([1]) *If $A\xi=\alpha\xi$ and $X\in\mathcal{D}$ with $AX=\lambda X$, then*

$$\begin{aligned} &(2\lambda-\alpha)A\phi X - (\lambda\alpha+2)\phi X \\ &\quad = -2\sum_{v=1}^{3}\Big\{2\eta(\xi_v)\eta(\phi_v X)\xi - \eta_v(X)\phi\xi_v - \eta(\phi_v X)\xi_v - \eta(\xi_v)\phi_v X\Big\}. \end{aligned}$$

Here $\mathcal{D}$ denotes the orthogonal complement of the real span $[\xi]$ of the Reeb vector ξ in TM.

Moreover, by (2.6) and (2.15), a straightforward computation leads to

$$(\phi A)^2X = (A\phi)^2X \tag{2.16}$$

for all vector field X on M.

3. Real hypersurfaces with parallel *-Ricci tensors

In this section we first assume that M is a Hopf hypersurface admitting parallel *-Ricci tensor, i.e. $\nabla S^* = 0$. Using (2.3) and (2.7), we compute the covariant derivative $(\nabla_Y S^*)X$ for all vector fields X, Y.

$$\begin{aligned}(\nabla_Y S^*)X = &-(4m+6)[g(\phi AY, X)\xi + \eta(X)\phi AY] - [\nabla_Y(\phi A)^2]X \\ &+ 2\sum_{v=1}^{3}\Big\{[q_{v+2}(Y)\eta_{v+1}(\phi X) - q_{v+1}(Y)\eta_{v+2}(\phi X) \\ &+ g(\phi_v AY, \phi X)]\phi\xi_v + [\eta(X)\eta_v(AY) \\ &- g(AY, X)\eta_v(\xi)]\phi\xi_v + \eta_v(\phi X)[q_{v+2}(Y)\phi_{v+1}\xi \\ &- q_{v+1}(Y)\phi_{v+2}\xi + \phi\phi_v AY - g(AY,\xi_v)\xi + \eta(\xi_v)AY] \\ &- [q_{v+2}(Y)\eta_{v+1}(X) - q_{v+1}(Y)\eta_{v+2}(X) + g(\phi_v AY, X)]\xi_v \\ &- \eta_v(X)[q_{v+2}(Y)\xi_{v+1} - q_{v+1}(Y)\xi_{v+2} + \phi_v AY] + \eta_v(\phi AY)\eta_v(X)\xi \\ &+ [q_{v+2}(Y)\eta(\xi_{v+1}) - q_{v+1}(Y)\eta(\xi_{v+2}) + \eta(\phi_v AY)]\eta_v(X)\xi \\ &+ \eta(\xi_v)[q_{v+2}(Y)\eta_{v+1}(X) - q_{v+1}(Y)\eta_{v+2}(X) + g(\phi_v AY, X)]\xi \\ &+ \eta(\xi_v)\eta_v(X)\phi AY + [q_{v+2}(Y)\eta_{v+1}(\xi) - q_{v+1}(Y)\eta_{v+2}(\xi) \\ &+ \eta(\phi_v AY)]\phi\phi_v X + \eta_v(\phi AY)\phi\phi_v X + \eta_v(\xi)\nabla_Y(\phi\phi_v)X\Big\}. \quad (3.1)\end{aligned}$$

Putting $Y = \xi$ in (3.1), by (2.8) and a straightforward computation, we derive

$$(\nabla_\xi S^*)X = -\phi(\nabla_\xi A)\phi AX - \phi A\phi(\nabla_\xi A)X - 4\sum_{v=1}^{3}\alpha\eta(\xi_v)\eta(X)\phi\xi_v. \quad (3.2)$$

Moreover, taking $X = \xi$ we obtain

$$(\nabla_\xi S^*)\xi = -4\sum_{v=1}^{3}\alpha\eta_v(\xi)\phi\xi_v. \quad (3.3)$$

Thus the parallel condition $\nabla S^* = 0$ yields

$$\sum_{v=1}^{3}\alpha\eta_v(\xi)\phi\xi_v = 0. \quad (3.4)$$

Taking an inner product of (3.4) with ϕY for $Y \in \mathfrak{D}$ gives

$$\alpha\eta(Y)\sum_{v=1}^{3}\eta_v(\xi)^2 = 0.$$

That means that $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$ for $\alpha \neq 0$. If $\alpha = 0$, as the proof of [5, Lemma 3.1], we can get the same conclusion. Namely we prove the following lemma.

Lemma 3.1. *Let M be a Hopf hypersurface of $G_2(\mathbb{C}^{m+2})$. If the *-Ricci tensor of M is parallel, then the Reeb vector field ξ either belongs to $\mathfrak{D}$ or $\mathfrak{D}^\perp$.*

Proof of Theorem 1.3. According to Lemma 3.1, the Reeb vector field ξ either belongs to $\mathfrak{D}$ or $\mathfrak{D}^{\perp}$. When $\xi \in \mathfrak{D}$, by Theorem 1.2, M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.

In the following we need to check wether a hypersurface of type (B) in $G_2(\mathbb{C}^{m+2})$ admits a parallel *-Ricci tensor or not. For $\xi \in \mathfrak{D}$, the formula (3.1) becomes

$$\begin{aligned}
&(\nabla_Y S^*)X \\
&\quad = -(4m+6)[g(\phi AY, X)\xi + \eta(X)\phi AY] - [\nabla_Y(\phi A)^2]X \\
&\qquad + 2\sum_{v=1}^{3}\Big\{[q_{v+2}(Y)\eta_{v+1}(\phi X) - q_{v+1}(Y)\eta_{v+2}(\phi X) + g(\phi_v AY, \phi X)]\phi\xi_v \\
&\qquad + \eta(X)\eta_v(AY)\phi\xi_v + \eta_v(\phi X)[q_{v+2}(Y)\phi_{v+1}\xi - q_{v+1}(Y)\phi_{v+2}\xi \\
&\qquad + \phi_v\phi AY - g(AY,\xi)\xi_v] - [q_{v+2}(Y)\eta_{v+1}(X) - q_{v+1}(Y)\eta_{v+2}(X) \\
&\qquad + g(\phi_v AY, X)]\xi_v - \eta_v(X)[q_{v+2}(Y)\xi_{v+1} - q_{v+1}(Y)\xi_{v+2} + \phi_v AY] \\
&\qquad + \eta_v(\phi AY)\eta_v(X)\xi + \eta(\phi_v AY)\eta_v(X)\xi + 2\eta_v(\phi AY)\phi\phi_v X\Big\} = 0. \qquad (3.5)
\end{aligned}$$

Letting $X = \xi$ and using (2.3), we have

$$-(4m+6)\phi AY + (\phi A)^3 Y + 2\sum_{v=1}^{3}\Big\{\eta_v(AY)\phi\xi_v - 3\eta_v(\phi AY)\xi_v\Big\} = 0. \qquad (3.6)$$

Now by Proposition 2.1 we consider the formula (3.6) with $Y = \xi_1 \in T_\beta$, then since $A\phi\xi_1 = 0$ we obtain

$$\begin{aligned}
&-(4m+6)\phi A\xi_1 + (\phi A)^3\xi_1 + 2\sum_{v=1}^{3}\Big\{\eta_v(A\xi_1)\phi\xi_v - 3\eta_v(\phi A\xi_1)\xi_v\Big\} \\
&\quad = -(4m+6)\beta\phi\xi_1 + 2\Big\{\beta\phi\xi_1 - 3\beta\sum_{v=1}^{3}\eta_v(\phi\xi_1)\xi_v\Big\} \\
&\quad = -(4m+4)\beta\phi\xi_1 = 0.
\end{aligned}$$

This means $4m+4 = 0$ since $\beta \neq 0$. It is impossible, thus M can not be a real hypersurface of type (B).

Next let us assume $\xi \in \mathfrak{D}^{\perp} = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$, without loss general we thus may assume $\xi = \xi_1$. Applying (2.7), it is easy to get $q_v(\xi) = 0$ for $v = 2, 3$. Furthermore, from (2.4) we have

$$\phi\xi_2 = \phi_2\xi_1 = -\xi_3, \quad \phi_1\xi_2 = \xi_3, \quad \phi\xi_3 = \phi_3\xi_1 = \xi_2. \qquad (3.7)$$

In this case the Eq. (2.15) becomes

$$A\phi AX = \frac{1}{2}\alpha(A\phi X + \phi AX) + \phi X + \eta_3(X)\xi_2 - \eta_2(X)\xi_3 + \phi_1 X. \qquad (3.8)$$

Because for all $X \in TM$, by (2.7) we obtain

$$g(\phi AX, \xi_2) = g(\nabla_X \xi, \xi_2) = -g(\xi, \nabla_X \xi_2) = q_3(X) - g(\xi_2, \phi AX),$$

i.e. $q_3(X) = 2\eta_2(\phi AX) = 2\eta_3(AX)$ from (3.7). Similarly, $q_2(X) = 2\eta_2(AX)$. Making use of (3.7) and (3.8) we compute the formula (3.1) with $X = \xi$.

$$\begin{aligned}
(\nabla_Y S^*)\xi &= -(4m+6)\phi AY + (\phi A)^3 Y + 2\sum_{v=1}^{3}\Big\{\eta_v(AY)\phi\xi_v \\
&\quad - [q_{v+2}(Y)\eta_{v+1}(\xi) - q_{v+1}(Y)\eta_{v+2}(\xi)]\xi_v \\
&\quad - 3\eta_v(\phi AY)\xi_v\Big\} - 2\phi_1 AY + 2\phi AY + 2\nabla_Y(\phi\phi_1)\xi \\
&= -(4m+6)\phi AY + (\phi A)^3 Y + 2\Big\{ - [q_2(Y)\xi_3 - q_3(Y)\xi_2] \\
&\quad - \phi_1 AY + \phi AY - 2\eta_3(AY)\xi_2 + 2\eta_2(AY)\xi_3 - \phi\phi_1\phi AY\Big\} \\
&= -(4m+4)\phi AY + (\phi A)^3 Y - 2[q_2(Y)\xi_3 - q_3(Y)\xi_2] \\
&\quad - 4\eta_3(AY)\xi_2 + 4\eta_2(AY)\xi_3 \\
&= -\left(4m+5+\frac{1}{4}\alpha^2\right)\phi AY - \frac{1}{4}\alpha^2 A\phi Y \\
&\quad - \frac{1}{2}\alpha\phi Y - \alpha\{-\eta_2(Y)\xi_3 + \eta_3(Y)\xi_2\} - \frac{1}{2}\alpha\phi_1 Y \\
&\quad - \frac{1}{2}\alpha\phi A^2 Y - 2\{\eta_2(Y)\phi A\xi_2 + \eta_3(Y)\phi A\xi_3\} + \phi A\phi\phi_1 Y. \qquad (3.9)
\end{aligned}$$

Since $\nabla S^* = 0$ putting $Y = \xi_2$ and $Y = \xi_3$ respectively in the formula (3.9) yields

$$0 = \left(4m+6+\frac{1}{4}\alpha^2\right)A\xi_2 - \frac{1}{4}\alpha^2\phi A\phi\xi_2 + \alpha\xi_2 + \frac{1}{2}\alpha A^2\xi_2,$$

$$0 = \left(4m+6+\frac{1}{4}\alpha^2\right)A\xi_3 - \frac{1}{4}\alpha^2\phi A\phi\xi_3 + \alpha\xi_3 + \frac{1}{2}\alpha A^2\xi_3.$$

Write $T = (4m+6+\frac{1}{4}\alpha^2)A - \frac{1}{4}\alpha^2\phi A\phi + \frac{1}{2}\alpha A^2$, then T is a linear transformation on $\mathfrak{D}^\perp$ with $T\xi_1 = T\xi = (4m+6+\frac{1}{4}\alpha^2)\alpha\xi_1, T\xi_2 = -\alpha\xi_2$ and $T\xi_3 = -\alpha\xi_3$. We further find $AT = TA$ by (2.16). Thus there exists a basis X_1, X_2, X_3 of $\mathfrak{D}^\perp$ with $AX_i = \lambda_i X_i$ and $TX_i = \lambda_i X_i,\ i = 1, 2, 3$, which satisfies

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = SO(3)\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix},$$

where $SO(3)$ denotes the special orthogonal group. Accordingly, we prove that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.

In terms of Proposition 2.2, we let $Y \in T_\lambda, \lambda = -\sqrt{2}\tan(\sqrt{2}r)$, i.e. $\phi Y = \phi_1 Y$, $AY = \lambda Y$ and $A\phi Y = \lambda\phi Y$, then it follows from (3.9) that

$$\left(4m+4+2\tan^2(\sqrt{2}r)\right)\phi Y = 0.$$

Namely $4m + 4 + 2\tan^2(\sqrt{2}r) = 0$, which is impossible. This shows that ξ can not belong to the distribution $\mathfrak{D}^\perp$. Therefore we complete the proof of Theorem 1.3. □

Proof of Theorem 1.4. Let M be a real Hopf hypersurface of $G_2(\mathbb{C}^{m+2})$ with Reeb parallel *-Ricci tensor, i.e. $\nabla_\xi S^* = 0$. By the proof of Lemma 3.1 we also know $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.

In the following we consider these two cases respectively. When $\xi \in \mathfrak{D}$, by taking $Y = \xi$ in (3.5) and using (2.11), we obtain

$$\begin{aligned}(\nabla_\xi S^*)X &= -\phi(\nabla_\xi A)\phi AX - \phi A\phi(\nabla_\xi A)X \\ &= \alpha A\phi AX + 2(\phi A)^3X + 2\phi AX - \sum_{v=1}^{3}\{\eta_v(\phi AX)\xi_v - 3\eta_v(AX)\phi\xi_v\} \\ &\quad + \alpha\phi A^2X - \sum_{v=1}^{3}\{\eta_v(X)\phi A\xi_v + 3\eta_v(\phi X)\phi A\phi\xi_v\}.\end{aligned}$$

Namely if the *-Ricci tensor is Reeb parallel, the following equation holds:

$$\begin{aligned}&\alpha A\phi AX + 2(\phi A)^3X + 2\phi AX - \sum_{v=1}^{3}\Big\{\eta_v(\phi AX)\xi_v - 3\eta_v(AX)\phi\xi_v\Big\} \\ &\quad + \alpha\phi A^2X - \sum_{v=1}^{3}\Big\{\eta_v(X)\phi A\xi_v + 3\eta_v(\phi X)\phi A\phi\xi_v\Big\} = 0. \qquad (3.10)\end{aligned}$$

Now by Proposition 2.1, we check the formula (3.10) as follows:

Case I $X = \xi \in \mathfrak{D}$. It is obvious.

Case II $X = \xi_\mu \in T_\beta$, then $A\phi\xi_\mu = 0$ for $\mu = 2, 3$. Making use of (2.7) and (2.8), we have

$$\begin{aligned}&\alpha A\phi A\xi_\mu + 2(\phi A)^3\xi_\mu + 2\phi A\xi_\mu - \sum_{v=1}^{3}\Big\{\eta_v(\phi A\xi_\mu)\xi_v - 3\eta_v(A\xi_\mu)\phi\xi_v\Big\} \\ &\quad + \alpha\phi A^2\xi_\mu - \sum_{v=1}^{3}\Big\{\eta_v(\xi_\mu)\phi A\xi_v + 3\eta_v(\phi\xi_\mu)\phi A\phi\xi_v\Big\} \\ &= (4\beta + \alpha\beta^2)\phi\xi_\mu.\end{aligned}$$

Since $\alpha\beta = -4$ the above equation is zero.

Case III $X = \phi\xi_\mu \in T_\gamma, \gamma = 0$, i.e. $A\phi\xi_\mu = 0$. Then

$$\begin{aligned}&\alpha A\phi A\phi\xi_\mu + 2(\phi A)^3\phi\xi_\mu + 2\phi A\phi\xi_\mu - \sum_{v=1}^{3}\Big\{\eta_v(\phi A\phi\xi_\mu)\xi_v - 3\eta_v(A\phi\xi_\mu)\phi\xi_v\Big\} \\ &\quad + \alpha\phi A^2\phi\xi_\mu - \sum_{v=1}^{3}\Big\{\eta_v(\phi\xi_\mu)\phi A\xi_v + 3\eta_v(\phi^2\xi_\mu)\phi A\phi\xi_v\Big\}\end{aligned}$$

$$= -\sum_{v=1}^{3} \eta_v(\phi\xi_\mu)\phi A\xi_v = 0.$$

The last equality is followed from (2.4) and (2.5).

Case IV $X \in T_\delta, \delta = \cot r$. Then $AX = \delta X$, $A\phi X = \mu\phi X$. The left-hand side of (3.10) becomes

$$\begin{aligned}&\alpha\delta\mu\phi X - 2\delta^2\mu\phi X + 2\delta\phi X + \alpha\delta^2\phi X\\ &\quad = (\alpha\delta\mu - 2\delta^2\mu + 2\delta + \alpha\delta^2)\phi X\\ &\quad = (-\alpha + 4\delta + \alpha\delta^2)\phi X.\end{aligned}$$

Substituting $\alpha = -2\tan(2r)$ and $\delta = \cot(r)$ into above formula, we find that it is equal to zero.

Case V $X \in T_\mu, \mu = -\tan r$. Then $AX = \mu X$ and $A\phi X = \delta\phi X$. In a same way we know the formula (3.10) holds as Case IV.

On the other hand, when $\xi \in \mathfrak{D}$ the formula (2.12) becomes

$$\begin{aligned}S^*X = -(4m+7)\phi^2 X - \frac{1}{2}\alpha(A\phi^2 X + \phi A\phi X)\\ + \sum_{v=1}^{3}\Big\{\eta_v(\phi X)\phi\xi_v - \eta_v(X)\xi_v\Big\}.\end{aligned}$$

By Proposition 2.2, putting $X = \xi_\mu \in T_\beta$ for $\mu = 1, 2, 3$, we have

$$S^*\xi_\mu = -\,(4m+7)\phi^2\xi_\mu + \frac{1}{2}\alpha\beta\xi_\mu - \xi_\mu = (4m+4)\xi_\mu.$$

It shows that the condition $S^*\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$ holds for a hypersurface of type (B).

Next we consider the case where $\xi \in \mathfrak{D}^\perp$. We first prove

Proposition 3.2. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+1}), m \geq 3$. If $\xi \in \mathfrak{D}^\perp$ then its *-Ricci tensor is Reeb parallel.*

Proof. By assumption, as before we may set $\xi = \xi_1$. First by the Codazzi equation (2.11),

$$(\nabla_\xi A)X = \alpha\phi AX - A\phi AX + \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3,$$

then we compute the formula (3.2):

$$\begin{aligned}(\nabla_\xi S^*)X = &-\alpha\phi(\phi A)^2 X + \phi(A\phi)^2 AX + \phi AX\\ &+ \phi\phi_1\phi AX - 2\eta_3(\phi AX)\phi\xi_2 + 2\eta_2(\phi AX)\phi\xi_3\\ &- \alpha\phi A\phi^2 AX + \phi(A\phi)^2 AX - \phi A\phi^2 X - \phi A\phi\phi_1 X\\ &- 2\eta_3(X)\phi A\phi\xi_2 + 2\eta_2(X)\phi A\phi\xi_3.\end{aligned}\tag{3.11}$$

Moreover, making use of (3.8) and (3.7), the Eq. (3.11) is simplified as

$$\begin{aligned}(\nabla_\xi S^*)X = &\,\phi_1 AX - 2\eta_2(AX)\xi_3 + 2\eta_3(AX)\xi_2\\ &+ \phi A\phi\phi_1 X - 2\eta_3(X)\phi A\xi_3 - 2\eta_2(X)\phi A\xi_2.\end{aligned}\tag{3.12}$$

On the other hand, by (2.8) we get

$$\begin{aligned}0 = \nabla_X(\phi_1\xi) &= q_3(X)\phi_2\xi - q_2(X)\phi_3\xi\\ &\quad + \phi_1\phi AX - g(AX,\xi)\xi_1 + \eta(\xi_1)AX\\ &= -2\eta_3(AX)\xi_3 - 2\eta_2(AX)\xi_2 + \phi_1\phi AX - \phi^2 AX,\end{aligned}$$

that is,

$$\phi_1\phi AX = 2\eta_3(AX)\xi_3 + 2\eta_2(AX)\xi_2 + \phi^2 AX. \tag{3.13}$$

Substituting (3.13) into (3.12) gives

$$(\nabla_\xi S^*)X = -\phi^2 AX + A\phi\phi_1 X - 2\eta_3(X)A\xi_3 - 2\eta_2(X)A\xi_2 \tag{3.14}$$

for all $X \in TM$. By (2.3) and (2.7), for all vector field X we have

$$\phi AX = \nabla_X\xi = \nabla_X\xi_1 = q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX.$$

It is well known that the distribution $\mathfrak{D}$ can be decomposed as $\mathfrak{D} = \mathfrak{D}_1 \oplus \mathfrak{D}_2$, where

$$\begin{aligned}\mathfrak{D}_1 &= \{X \in \mathfrak{D} | \phi X = \phi_1 X\},\\ \mathfrak{D}_2 &= \{X \in \mathfrak{D} | \phi X = -\phi_1 X\}.\end{aligned}$$

Thus we can decompose the tangent bundle TM as $TM = \mathfrak{D}^\perp \oplus \mathfrak{D}_1 \oplus \mathfrak{D}_2$. Taking the inner product of the above equation with $Y \in \mathfrak{D}_2$, we conclude that $A\phi Y = 0$, i.e. $AY = 0$ since ϕ leaves $\mathfrak{D}_2$ invariant for all $Y \in \mathfrak{D}_2$. Therefore $(\nabla_\xi S^*)X = 0$ for all $X \in \mathfrak{D}_2$. As for $X \in \mathfrak{D}^\perp \oplus \mathfrak{D}_1$ it is easy to check that the right side of formula (3.14) vanishes. We complete the proof. □

When $\xi \in \mathfrak{D}^\perp$, by (3.7) and (3.8), we have

$$\begin{aligned}S^*X = &-(4m+7)\phi^2 X - \frac{1}{2}\alpha(A\phi^2 X + \phi A\phi X)\\ &- 2\Big\{\eta_2(X)\xi_2 + \eta_3(X)\xi_3\Big\} + \phi\phi_1 X.\end{aligned} \tag{3.15}$$

Since $S^*\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$, for $\mu = 1, 2, 3$ we can set $S^*\xi_\mu = \lambda_\mu\xi_\mu$. The formula (3.15) with $X = \xi_\mu$ implies

$$\lambda_\mu\xi_\mu = (4m+6)\xi_\mu + \frac{1}{2}\alpha(A\xi_\mu - \phi A\phi\xi_\mu), \quad \mu = 2, 3,$$

i.e. $T\xi_\mu = -(4m+6-\lambda_\mu)\xi_\mu$, where $T := \frac{1}{2}\alpha(A - \phi A\phi)$. Using (2.16), we have $AT = TA$. Furthermore, $T\xi_1 = T\xi = \frac{1}{2}\alpha^2\xi$. As the proof of Theorem 1.3 we thus prove that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. By Theorem 1.1, M is a hypersurface of type (A).

Finally we remain to check whether the condition $S^*\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$ holds or not for a hypersurface of type (A). For $\mu = 2, 3$ we put $X = \xi_\mu$ in the formula (3.15), then by Proposition 2.2 we have

$$\begin{aligned}S^*\xi_\mu &= (4m+6)\xi_\mu + \frac{1}{2}\alpha(A\xi_\mu + \phi A\xi_\mu)\\ &= (4m+6+\alpha\beta)\xi_\mu = (4m+4+2\cot^2(\sqrt{2}r))\xi_\mu.\end{aligned}$$

On the other hand, $S^*\xi_1 = S^*\xi = 0$. Hence the condition holds for hypersurfaces of type (A).

Summarizing the above discussion we complete the proof of Theorem 1.4. □

4. Real hypersurfaces with Reeb Lie parallel *-Ricci tensors

In this section we suppose that the *-Ricci tensors of Hopf hypersurface M is Lie Reeb parallel, i.e. $\mathfrak{L}_\xi Ric^* = 0$. For all $X, Y \in TM$ we have

$$\begin{aligned}(\mathfrak{L}_\xi Ric^*)(X,Y) &= \mathfrak{L}_\xi(Ric^*(X,Y)) - Ric^*(\mathfrak{L}_\xi X, Y) - Ric^*(X, \mathfrak{L}_\xi Y)\\ &= g((\nabla_\xi S^*)X, Y) + g(S^*\phi AX, Y) - g(A\phi S^*X, Y) = 0.\end{aligned}$$

This implies $(\nabla_\xi S^*)X = A\phi S^*X - S^*\phi AX$ for any X tangent to M. Thus by (3.3) taking $X = \xi$ gives

$$0 = (\nabla_\xi S^*)\xi - A\phi S^*\xi = -4\sum_{v=1}^{3}\Big\{\alpha\eta_v(\xi)\phi\xi_v - \eta_v(\xi)A\phi\xi_v\Big\}.$$

That is,

$$A\sum_{v=1}^{3}\eta_v(\xi)\phi\xi_v = \alpha\sum_{v=1}^{3}\eta_v(\xi)\phi_v\xi.$$

We write $Y = \sum_{i=1}^{3}\eta_i(\xi)\xi_i$, then $\phi Y \in \mathfrak{D}$ and $A\phi Y = \alpha\phi Y$. Replacing X in Lemma 2.3 by ϕY, we have

$$\begin{aligned}\alpha A\phi^2 Y &= (\alpha^2+2)\phi^2 Y - 2\sum_{v=1}^{3}\{2\eta(\xi_v)\eta(\phi_v\phi Y)\xi - \eta_v(\phi Y)\phi\xi_v\\ &\quad - \eta(\phi_v\phi Y)\xi_v - \eta(\xi_v)\phi_v\phi Y\}\\ &= (2+\alpha^2)\phi^2 Y - 4\sum_{v=1}^{3}\Big\{\eta(\xi_v)\eta(Y)\eta(\xi_v)\xi - \eta(Y)\eta(\xi_v)\xi_v\Big\}\\ &\quad + 2\sum_{v=1}^{3}\Big\{\eta_v(\phi Y)\phi\xi_v + \eta(\xi_v)\phi\phi_v Y\Big\}\\ &\quad + 2\sum_{v=1}^{3}\Big\{-\eta_v(Y)\xi_v + \eta(\xi_v)\eta_v(Y)\xi\Big\}.\end{aligned}$$

By a straightforward computation, the above formula becomes

$$\begin{aligned}\alpha A\phi^2 Y &= \Big(4+\alpha^2 - 4\sum_{v=1}^{3}\eta_v(\xi)^2\Big)\sum_{v=1}^{3}\eta_v(\xi)\phi^2\xi_v\\ &= \Big(4+\alpha^2 - 4\sum_{v=1}^{3}\eta_v(\xi)^2\Big)\phi^2 Y.\end{aligned}$$

If $\alpha = 0$ we know that ξ belongs to $\mathfrak{D}$ or $\mathfrak{D}^\perp$ (see [5, Lemma 3.1]). Next we assume $\alpha \neq 0$, then $A\phi^2 Y = \lambda\phi^2 Y$, where

$$\lambda = \frac{4 + \alpha^2 - 4\sum_{v=1}^{3}\eta_v(\xi)^2}{\alpha}. \tag{4.1}$$

Since $A\phi^2 Y = \phi^2 AY$ we further derive that $AY = \lambda Y$ or $AY - \lambda Y \in \mathbb{R}\xi$. The former shows $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$. For the latter we set $AY - \lambda Y = f\xi$, where f is a smooth function. So $AY = \lambda Y + f\xi$. In the following we compute the formula (3.2) with $X = Y$ as follows:

$$\begin{aligned}(\nabla_\xi S^*)Y &= -\phi(\nabla_\xi A)\phi AY - \phi A\phi(\nabla_\xi A)Y - 4\sum_{v=1}^{3}\alpha\eta(\xi_v)\eta(Y)\phi\xi_v \\ &= -\lambda\phi[\xi\alpha\phi Y + \alpha\phi\nabla_\xi Y - A\phi\nabla_\xi Y] \\ &\quad - \phi A\phi[(\xi\lambda)Y + \lambda\nabla_\xi Y - A\nabla_\xi Y] - 4\alpha\eta(Y)\phi Y \\ &= -\xi(\lambda\alpha)\phi^2 Y - [\alpha\lambda\phi^2\nabla_\xi Y - \phi A\phi A\nabla_\xi Y] - 4\alpha\eta(Y)\phi Y.\end{aligned}$$

Since

$$\eta(\nabla_\xi Y) = \nabla_\xi\eta(Y) = \sum_{v=1}^{3}\xi(\eta_v(\xi)^2) = 2\sum_{v=1}^{3}\xi(\eta_v(\xi))\eta_v(\xi) = 0,$$

the above formula becomes

$$(\nabla_\xi S^*)Y = -\xi(\lambda\alpha)\phi^2 Y + [\alpha\lambda\nabla_\xi Y + \phi A\phi A\nabla_\xi Y] - 4\alpha\eta(Y)\phi Y. \tag{4.2}$$

On the other hand, by (2.12) we compute

$$\begin{aligned}A\phi S^*Y - S^*\phi AY &= A\phi S^*Y - \lambda S^*\phi Y \\ &= \alpha(4m + 4 + \lambda\alpha)\phi Y - \lambda\Big[(4m+6)\phi Y + \lambda\alpha\phi Y \\ &\quad + 2\sum_{v=1}^{3}\Big\{ - \eta_v(Y)\phi\xi_v + 2\eta(Y)\eta_v(\xi)\phi\xi_v - \eta_v(\phi Y)\xi_v \\ &\quad + 2\eta(\xi_v)\eta_v(\phi Y)\xi - \eta_v(\xi)\phi_v Y\Big\}\Big] \\ &= \alpha(4m + 4 + \lambda\alpha)\phi Y - \lambda[(4m+6)\phi Y + \lambda\alpha\phi Y \\ &\quad - 2\phi Y + 4\eta(Y)\phi Y] \\ &= \Big[(\alpha - \lambda)(4m + 4 + \lambda\alpha) - 4\lambda\eta(Y)\Big]\phi Y.\end{aligned} \tag{4.3}$$

Moreover, since

$$g(\phi A\phi A\nabla_\xi Y, \phi Y) = g(\nabla_\xi Y, A\phi A\phi^2 Y) = -\alpha\lambda g(\nabla_\xi Y, \phi Y),$$

from (4.2) and (4.3), taking the inner product of the relation $(\nabla_\xi S^*)Y = A\phi S^*Y - S^*\phi AY$ with ϕY gives

$$\Big\{(\alpha - \lambda)\big[4m + 4 + \lambda\alpha + 4\eta(Y)\big]\Big\}g(\phi Y, \phi Y) = 0,$$

i.e. $\Big\{(\alpha - \lambda)(4m + 8 + \alpha^2)\Big\}g(\phi Y, \phi Y) = 0$ by (4.1). That shows $\alpha = \lambda$ or $\phi Y = 0$.

When $\alpha = \lambda$, using (4.1) again we have $\sum_{v=1}^{3}\eta_v(\xi)^2 = 1$, which implies $\xi \in \mathfrak{D}^{\perp}$. If $\phi Y = 0$ it is easy to see that $\xi \in \mathfrak{D}$.

Therefore we assert

Lemma 4.1. *Let M be a Hopf hypersurface of $G_2(\mathbb{C}^{m+2})$ with Reeb Lie parallel *-Ricci tensor, then the Reeb vector field ξ belongs to $\mathfrak{D}$ or $\mathfrak{D}^{\perp}$.*

Proof of Theorem 1.5. For all vector field X on M we first compute

$$\begin{aligned} &A\phi S^*X - S^*\phi AX \\ &\quad = (4m+6)A\phi X + A^2\phi AX + 2\sum_{v=1}^{3}\Big\{\eta_v(\phi X)[-A\xi_v + 2\alpha\eta_v(\xi)\xi] \\ &\qquad - \eta_v(X)A\phi\xi_v - \eta_v(\xi)A\phi_v X\Big\} - \Big[(4m+6)\phi AX - (\phi A)^3X \\ &\qquad + 2\sum_{v=1}^{3}\Big\{[-\eta_v(AX) + 2\alpha\eta_v(\xi)\eta(X)]\phi\xi_v - \eta_v(\phi AX)\xi_v \\ &\qquad + 2\eta(\xi_v)\eta_v(\phi AX)\xi - \eta_v(\xi)\phi_v AX\Big\}\Big]. \end{aligned} \tag{4.4}$$

In view of Lemma 4.1, we know $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^{\perp}$. In the following we consider these two cases respectively.

When $\xi \in \mathfrak{D}$, by (4.4) the condition $\mathfrak{L}_\xi Ric^* = 0$ is equivalent to that the following formula holds:

$$\begin{aligned} (\nabla_\xi S^*)X &= (4m+6)A\phi X + A^2\phi AX - 2\sum_{v=1}^{3}\Big\{\eta_v(\phi X)A\xi_v + \eta_v(X)A\phi\xi_v\Big\} \\ &\quad - \Big[(4m+6)\phi AX - (\phi A)^3X - 2\sum_{v=1}^{3}\Big\{\eta_v(AX)\phi\xi_v + \eta_v(\phi AX)\xi_v\Big\}\Big]. \end{aligned}$$

By virtue of Proposition 3.2, the left side of the above formula vanishes, thus we need to check whether the right side also vanishes or not.

By Proposition 2.1, we put $X = \xi_\mu \in T_\beta$, then $A\phi\xi_\mu = 0$ for $\mu = 2, 3$. Making use of (2.7) and (2.8), we have

$$\begin{aligned} (\nabla_\xi S^*)\xi_\mu &= (4m+6)A\phi\xi_\mu + A^2\phi A\xi_\mu - 2\sum_{v=1}^{3}\Big\{\eta_v(\phi\xi_\mu)A\xi_v + \eta_v(\xi_\mu)A\phi\xi_v\Big\} \\ &\quad - \Big[(4m+6)\phi A\xi_\mu - (\phi A)^3\xi_\mu \\ &\quad - 2\sum_{v=1}^{3}\Big\{\eta_v(A\xi_\mu)\phi\xi_v + \eta_v(\phi A\xi_\mu)\xi_v\Big\}\Big] \\ &= -\beta(4m+4)\phi\xi_\mu. \end{aligned}$$

It is clear that $(\nabla_\xi S^*)\xi_\mu \neq 0$ since $\beta \neq 0$. This shows that there do not exist any real hypersurfaces of type (B) with Lie Reeb parallel *-Ricci tensor.

When $\xi \in \mathfrak{D}^\perp$, as before we may assume $\xi = \xi_1$, then the formula (4.4) becomes

$$\begin{aligned}
&A\phi S^* X - S^*\phi AX \\
&\quad = (4m+6)A\phi X + A^2\phi AX - 2\sum_{v=1}^{3}\Big\{\eta_v(\phi X)A\xi_v + \eta_v(X)A\phi\xi_v\Big\} - 2A\phi_1 X \\
&\qquad - \Big[(4m+6)\phi AX - (\phi A)^3 X \\
&\qquad - 2\sum_{v=1}^{3}\Big\{\eta_v(AX)\phi\xi_v + \eta_v(\phi AX)\xi_v\Big\} - 2\phi_1 AX\Big] \\
&\quad = (4m+6)A\phi X + A^2\phi AX - 4\Big\{\eta_3(X)A\xi_2 - \eta_2(X)A\xi_3\Big\} - 2A\phi_1 X \\
&\qquad - \Big[(4m+6)\phi AX - (\phi A)^3 X - 4\Big\{-\eta_2(AX)\xi_3 + \eta_3(AX)\xi_2\Big\} - 2\phi_1 AX\Big] \\
&\quad = (4m+7)A\phi X + \frac{1}{2}\alpha A^2\phi X - 2\{-\eta_2(X)A\xi_3 + \eta_3(X)A\xi_2\} - A\phi_1 X \\
&\qquad - \frac{1}{2}\alpha\phi A^2 X - 2\{\eta_2(X)\phi A\xi_2 + \eta_3(X)\phi A\xi_3\} + \phi A\phi\phi_1 X \\
&\qquad - \Big[(4m+7)\phi AX - 4\Big\{-\eta_2(AX)\xi_3 + \eta_3(AX)\xi_2\Big\} - 2\phi_1 AX\Big].
\end{aligned}$$

From this we see that the condition $\mathfrak{L}_\xi Ric^* = 0$ yields from (3.14)

$$\begin{aligned}
&(4m+7)(A\phi X - \phi AX) + \frac{1}{2}\alpha(A^2\phi X - \phi A^2 X) + \phi_1 AX - A\phi_1 X \\
&\quad + 2\Big\{-\eta_2(AX)\xi_3 + \eta_3(AX)\xi_2\Big\} - 2\Big\{-\eta_2(X)A\xi_3 + \eta_3(X)A\xi_2\Big\} = 0.
\end{aligned} \tag{4.5}$$

By (3.13), we get $\phi_1 AX = -2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 + \phi AX$. Thus substituting this into (4.5) gives

$$\begin{aligned}
&(4m+7)(A\phi X - \phi AX) + \frac{1}{2}\alpha(A^2\phi X - \phi A^2 X) \\
&\quad + \phi AX - A\phi_1 X - 2\Big\{-\eta_2(X)A\xi_3 + \eta_3(X)A\xi_2\Big\} = 0.
\end{aligned} \tag{4.6}$$

Now we decompose the tangent bundle TM as follows:

$$TM = \mathcal{D}_1 \oplus \mathcal{D}_2,$$

where $\mathcal{D}_1 = \mathfrak{D}_1 \oplus \mathrm{Span}\{\xi_2, \xi_3\}$ and $\mathcal{D}_2 = \mathfrak{D}_2 \oplus \mathbb{R}\xi$. We know $A\mathfrak{D}_2 = \{0\}$ then $A\mathcal{D}_2 \subset \mathcal{D}_2$, which yields $A\mathcal{D}_1 \subset \mathcal{D}_1$.

Next we assume $X \in \mathcal{D}_1$, since $\xi = \xi_1 \in \mathfrak{D}^\perp$, we may write X as

$$X = \eta_2(X)\xi_2 + \eta_3(X)\xi_3 + \mathfrak{D}_1 X,$$

where $\mathfrak{D}_1 X$ denotes the orthogonal projection of X onto $\mathfrak{D}_1$. Hence using (3.7) we find

$$-A\phi_1 X - 2\Big\{-\eta_2(X)A\xi_3 + \eta_3(X)A\xi_2\Big\} = -A\phi X.$$

So the relation (4.6) becomes

$$(4m+6)(A\phi X - \phi AX) + \frac{1}{2}\alpha(A^2\phi X - \phi A^2 X) = 0. \tag{4.7}$$

Let $Y \in \mathcal{D}_1$ with $AY = \rho Y$, then we obtain from Lemma 2.3

$$A\phi Y = \delta\phi Y \text{ with } \delta = \frac{\rho\alpha + 4}{2\rho - \alpha}.$$

It yields from the formula (4.7) that

$$\Big(4m + 6 + \frac{1}{2}\alpha(\rho + \delta)\Big)(\rho - \delta) = 0.$$

From this we get $\rho^2 - \rho\alpha - 2 = 0$ or $\alpha\rho^2 + 2(4m+6)\rho - (4m+4)\alpha = 0$. We choose some real number r with $0 < r < \frac{\pi}{\sqrt{8}}$ such that $\alpha = \sqrt{8}\cot(\sqrt{8}r)$, then $\beta = \sqrt{2}\cot(\sqrt{2}r)$ and $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ are the solutions of equation $x^2 - x\alpha - 2 = 0$. Moreover, we know $\rho \neq \alpha$. Hence we prove

Proposition 4.2. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+1}), m \geq 3$, with Lie Reeb parallel *-Ricci tensor. Suppose that $A\xi = \alpha\xi$ and $\xi \in \mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has five(if $r = \frac{\pi}{2}$)or six (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r),$$
$$\mu = 0, \quad \rho_1, \quad \rho_2,$$

where

$$\rho_{1,2} = \frac{-(4m+6) \pm \sqrt{(4m+6)^2 + (4m+4)\alpha^2)}}{\alpha}.$$

We denote $T_\rho = \{X \in TM | AX = \rho X\}$ then

$$\mathcal{D} = T_\beta \oplus T_\lambda \oplus T_\mu \oplus T_{\rho_1} \oplus T_{\rho_2}.$$

As in [2, Section 6] we denote c_p by the geodesic in $G_2(\mathbb{C}^{m+2})$ for $p \in M$ with $c_p(0) = p$ and $\dot{c}_p(0) = N_p$, and by F the smooth map

$$F : M \to G_2(\mathbb{C}^{m+2}) \quad p \mapsto c_p(r).$$

Its differential d_pF can be computed using Jacobi vector fields by means of

$$d_pF(X) = Z_X(r).$$

Here, $Z_X(r)$ is the Jacobi vector field along $c_p(r)$ with $Z_X(0) = X$ and $Z'_X(0) = -AX$. In the present situation we get

$$Z_X(r) = \begin{cases} \Big(\cos(\sqrt{8}r) - \frac{\alpha}{\sqrt{8}}\sin(\sqrt{8}r)\Big)E_X(r), & X \in T_\alpha \\ \Big(\cos(\sqrt{2}r) - \frac{\rho}{\sqrt{2}}\sin(\sqrt{2}r)\Big)E_X(r), & X \in T_\rho \text{ and } \rho \in \{\beta, \lambda, \rho_1, \rho_2\} \\ E_X(r), & X \in T_\mu, \end{cases}$$

where $E_X(r)$ denotes the parallel vector field along c_p with $E_X(0) = X$. This shows the kernel of dF is $T_\alpha \oplus T_\beta$ and F is of constant rank $\dim(T_\lambda \oplus T_\mu \oplus T_{\rho_1} \oplus T_{\rho_2})$. So, locally, F is a submersion into a submanifold $\mathbf{P}$ of $G_2(\mathbb{C}^{m+2})$. As the proof of theorem in [2] we can prove that $\mathbf{P}$ is a totally geodesic in

$G_2(\mathbb{C}^{m+2})$. Rigidity of totally geodesic submanifold implies that M is an open part of totally geodesic submanifold $\mathbf{P}$ of $G_2(\mathbb{C}^{m+2})$. We complete the proof of Theorem 1.5. □

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