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Groemer–Wallen measure of asymmetry for Reuleaux polygons

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Abstract. In this paper, we consider the Groemer–Wallen measure of asymmetry for Reuleaux polygons, and show that the n-th ($n \geq 5, n$ odd) regular Reuleaux polygons are the most symmetric among all n-th Reuleaux polygons.

Mathematics Subject Classification. 52A38.

Keywords. Measure of asymmetry, Reuleaux triangle, Reuleaux polygons, Constant width.

1. Introduction

Measures of (central) symmetry, or as we prefer, asymmetry for convex bodies have been extensively studied (for recent results see [5,7,10,12,18,19]). A survey on measures of asymmetry of convex bodies (up to 1963) has been published by Grünbaum [6]. A measure of asymmetry for domains of constant width was studied first by Besicovitch [1].

Groemer and Wallen [5] introduced a measure of asymmetry for convex domains of constant width, and determined the extremal bodies with respect to this measure. More specifically, they obtained that the most asymmetric domains are Reuleaux triangles.

Martini and Mustafaev [15] gave a new construction of curves of constant width, and proved the same result as in [5] by a different method.

Motivated by the work of Groemer and Wallen, replacing the area by the perimeter, Lu and Pan [14] introduced another measure of asymmetry for convex domains of constant width. They showed that Reuleaux triangles are the most asymmetric domains of constant width in this sense.

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In [10], using the Lu–Pan measure of asymmetry, Jin showed that the regular Reuleaux polygons have better symmetry than the irregular ones. For the Minkowski measure of asymmetry for Reuleaux polygons see [8].

In this paper, using the Groemer–Wallen measure of asymmetry, we prove the following theorem.

Theorem 1. *If K is a Reuleaux polygon of order n ($n \geq 5$, n odd), then*

$$\alpha_n \leq \alpha(K) < \alpha_0,$$

where $\alpha(\cdot)$ denotes the Groemer–Wallen measure of asymmetry for convex bodies,

$$\alpha_0 = \frac{4\pi - 3\sqrt{3}}{2\pi - 3\sqrt{3}}, \alpha_n = 1 + \frac{2\pi}{n^2 \tan(\pi/2n)\cos(\pi/n) - n^2 \sin(\pi/n) + (n-1)\pi}.$$

Moreover, equality holds on the left-hand side if and only if K is regular.

From Theorem 1, we obtain the following result (see also [5,15]):

Theorem 2. *Let K be a convex domain of constant width. Then*

$$1 \leq \alpha(K) \leq \alpha_0.$$

Equality holds on the left-hand side if and only if K is a circular disc. Equality holds on the right-hand side if and only if K is a Reuleaux triangle.

2. Preliminaries

Let C be a convex body, a closed bounded convex subset of $\mathbb{R}^d$. Let $\mathcal{K}^d$ be the set of all d-dimensional convex bodies. A convex body K is said to be of constant width if its width function, that is, the support function of $K+(-K)$, is constant (see [2,9,17]). Let $\mathcal{W}^d \subset \mathcal{K}^d$ be the set of all convex bodies of constant width. It is well-known that K is of constant width if and only if each boundary point of K is incident with (at least) one diameter (a chord of maximal length) of K.

From now on, we let $d = 2$. By a diameter of $K \in \mathcal{K}^2$ of direction u we mean a line segment of direction u in K of maximal length. If $K \in \mathcal{W}^2$ then for any u there is exactly one diameter $D(u)$ of K of direction u, and the two lines that pass through the endpoints of $D(u)$ and orthogonal to u are support lines of K. The diameter $D(u)$ splits K into two convex domains, say $K_+(u)$ and $K_-(u)$, where $K_+(u)$ lies in the 'positive' half-plane with respect to the line of direction u containing $D(u)$ [5].

For each $u \in S^1$, the unit circle, set $\alpha(K,u) = |K_+(u)|/|K_-(u)|$, $|\cdot|$ is the area. Groemer and Wallen [5] defined the asymmetry function $\alpha(K)$ of $K \in \mathcal{W}^2$, by

$$\alpha(K) = \max\left\{\alpha(K,u) : u \in S^1\right\}.$$

They proved that

$$1 \leq \alpha(K) \leq \alpha_0,$$

where $\alpha_0 = \frac{4\pi - 3\sqrt{3}}{2\pi - 3\sqrt{3}}$. Equality holds on the left-hand side if and only if K is a circular disc. Equality holds on the right-hand side if and only if K is a Reuleaux triangle.

3. Proof of Theorems 1–2

Let $K \in \mathcal{W}^2$ and $V \subset \mathrm{bd}(K)$. The set V is called a pinching set if each diameter of K is incident with (at least) one point of V. A convex body K of constant width is called a Reuleaux polygon if it admits a finite pinching set. In fact, each Reuleaux polygon contains a polygon with the vertices being same as the Reuleaux polygon, that is, the set of all vertices of the polygon is a pinching set of the Reuleaux polygon. In this case we say that the polygon generates the Reuleaux polygon. For example, each Reuleaux triangle can be generated by an equilateral triangle (see [13]).

Let K be a Reuleaux polygon generated by the polygon with vertices $e_1e_2\cdots e_n$, n odd. It is obvious that each diameter of K meets one element in $\{e_1, e_2, \ldots, e_n\}$. Define

$$\alpha(K, e_i) = \max\left\{|K_+(u)| \,/\, |K_-(u)| : u \in S^1, e_i \in D(u)\right\}.$$

Clearly, we have $\alpha(K) = \max\{\alpha(K, e_i), i = 1, 2, \ldots n\}$.

Proof of Theorem 1. Let the width of K be ω.

(1) For simplicity, we start with the case $n = 5$. By the definition of Reuleaux polygons, $|e_1e_3| = |e_1e_4| = |e_2e_4| = |e_2e_5| = |e_3e_5| = \omega$. We shall denote these vertices $e_i (i = 1, 2, 3, 4, 5)$ by cyclic index, that is, for any integers i, j, we have $e_i = e_j$ if $i \equiv j (\mathrm{mod}\ 5)$. For each vertex e_i, the two diameters e_ie_{i+2}, e_ie_{i+3} split K into three parts, sector $e_ie_{i+2}e_{i+3}$, and two curved edge triangles $e_ie_{i+1}e_{i+2}, e_ie_{i+3}e_{i+4}$. Denote the areas of sector $e_ie_{i+2}e_{i+3}$, curved edge triangles $e_ie_{i+1}e_{i+2}, e_ie_{i+3}e_{i+4}$ by A_i, S_i, S_i' respectively. Then, for the area of K, we have $|K| = A_i + S_i + S_i'$ and $\sum_{i=1}^5 A_i = \frac{\pi}{2}\omega^2$.

It is easy to prove $\alpha(K, e_i) = \max\{\frac{|K|}{S_i}, \frac{|K|}{S_i'}\} - 1$. Then, we have

$$\begin{aligned}
\alpha(K) + 1 &= \max\left\{\frac{|K|}{S_i}, \frac{|K|}{S_i'}, i = 1, 2, 3, 4, 5\right\} \\
&= \frac{|K|}{\min\{S_i, S_i', i = 1, 2, 3, 4, 5\}} \\
&\geq \frac{10|K|}{\sum_{i=1}^5 (S_i + S_i')} \\
&= \frac{10|K|}{5|K| - \sum_{i=1}^5 A_i}
\end{aligned}$$

$$= \frac{10|K|}{5|K| - \frac{1}{2}\pi\omega^2}$$
$$= 2 + \frac{2\pi\omega^2}{10|K| - \pi\omega^2}.$$

Let $A(5)$ be the area of a regular Reuleaux pentagon with width ω. Then $A(5) = (\frac{5}{2}\tan\frac{\pi}{10}\cos\frac{\pi}{5} + \frac{\pi}{2} - \frac{5}{2}\sin\frac{\pi}{5})\omega^2$. By Firey-Sallee Theorem (see [3,16], or Corollary 2.4 of [13]), we have $|K| \le A(5)$. Therefore, we have

$$\alpha(K) \ge 1 + \frac{2\pi\omega^2}{10A(5) - \pi\omega^2} = \alpha_5.$$

Since $\min\{S_i, S_i', i = 1,2,3,4,5\} = \frac{1}{10}\sum_{i=1}^{5}(S_i + S_i')$ if and only if $S_1 = S_2 = S_3 = S_4 = S_5 = S_1' = S_2' = S_3' = S_4' = S_5'$, by the equality condition of Firey-Sallee Theorem, equality holds in $\alpha(K) \ge \alpha_5$ if and only if K is a regular Reuleaux pentagon.

Now we prove $\alpha(K) < \alpha_0$. Let l_+, l_- be the two half planes divided by a diameter ef of K, and $K_+ := K \cap l_+, K_- := K \cap l_-$. Construct a Reuleaux triangle $R(\triangle)$ with vertices e, f, g such that $R(\triangle)_+ := R(\triangle) \cap l_+ \supset K_+, R(\triangle)_- := R(\triangle) \cap l_- \subset K_-$. Let $u_{ef} := \overrightarrow{ef}/|\overrightarrow{ef}| \in S^1$. So, we have $\alpha(K, u_{ef}) = |K_+|/|K_-| < |R(\triangle)_+|/|R(\triangle)_-| = \alpha_0$. Therefore, $\alpha(K) < \alpha_0$.

(2) Now we consider the general case $n \ge 5$ with n odd. Set $n = 2m+1, m \ge 2$. Then by the definition of Reuleaux polygon, we have $|e_i e_{m+i}| = \omega, i = 1, 2, \ldots, 2m+1$, where $e_{k+2m+1} = e_k, k = 1, 2, \ldots, 2m+1$. For each vertex e_i, the two diameters $e_i e_{i+m}, e_i e_{i+m+1}$ split K into three parts, sector $e_i e_{i+m} e_{i+m+1}$, and the two curved edge polygons $e_i e_{i+1} \ldots e_{i+m}, e_i e_{i+m+1}$ e_{i+2m}. Denote the areas of sector $e_i e_{i+m} e_{i+m+1}$, curved edge polygons $e_i e_{i+1} e_{i+m}, e_i e_{i+m+1} e_{i+2m}$ by A_i, S_i, S_i' respectively. Then the area of K, $|K| = A_i + S_i + S_i'$ and $\sum_{i=1}^{n} A_i = \frac{\pi}{2}\omega^2$.

It is easy to prove that $\alpha(K, e_i) = \max\{\frac{|K|}{S_i}, \frac{|K|}{S_i'}\} - 1$. Then, we have

$$\alpha(K) + 1 = \max\left\{\frac{|K|}{S_i}, \frac{|K|}{S_i'}, i = 1, 2, \ldots, n\right\}$$
$$= \frac{|K|}{\min\{S_i, S_i', i = 1, 2, \ldots, n\}}$$
$$\ge \frac{2n|K|}{\sum_{i=1}^{n}(S_i + S_i')}$$
$$= \frac{2n|K|}{n|K| - \sum_{i=1}^{n} A_i}$$
$$= \frac{2n|K|}{n|K| - \frac{1}{2}\pi\omega^2}$$
$$= 2 + \frac{2\pi\omega^2}{2n|K| - \pi\omega^2}.$$

Let $A(n)$ be the area of a regular Reuleaux pentagon with width ω. Then $A(n) = (\frac{n}{2}\tan\frac{\pi}{2n}\cos\frac{\pi}{n} + \frac{\pi}{2} - \frac{n}{2}\sin\frac{\pi}{n})\omega^2$. By Firey-Sallee Theorem, we have $|K| \leq A(n)$. Therefore, we have

$$\alpha(K) \geq 1 + \frac{2\pi\omega^2}{2nA(n) - \pi\omega^2} = \alpha_n.$$

Since $\min\{S_i, S_i', i = 1, 2, \ldots, n\} = \frac{1}{2n}\sum_{i=1}^{n}(S_i + S_i')$ if and only if $S_i, S_i', i = 1, 2, \ldots, n$ are all equal, by the equality condition of Firey-Sallee Theorem, equality holds in $\alpha(K) \geq \alpha_n$ if and only if K is regular.

The proof of $\alpha(K) < \alpha_0$ is same as in (1). □

Remark 1. In the proof for $\alpha(K) < \alpha_0$, we constructed a Reuleaux triangle. This method was introduced by Martini and Mustafaev [15] firstly.

Proof of Theorem 2. For each convex domain K of constant width ω, there exists Reuleaux polygons $K_i, i = 1, 2, \ldots,$ such that $K_i \to K$, as $i \to \infty$ with respect to the Hausdorff metric. Since $|\cdot|$ is continuous, we have $1 \leq \alpha(K) \leq \alpha_0$.

If K is a circular disc, then $\alpha(K) = 1$. Conversely, if $\alpha(K) = 1$, then for every $u \in S^1$ we have $|K_-(u)| = |K_+(u)|$. This implies that K is centrally symmetric (see [4] Theorem 4.5.9 , or [5] p. 519). But since K is of constant width it must be a circular disc.

If K is a Reuleaux triangle, then $\alpha(K) = \alpha_0$. Conversely, if $\alpha(K) = \alpha_0$, then there exists a direction u and a diameter $|e_1e_2| = D(u)$ of K such that $\alpha(K) = \alpha(K, u)$. Let l_+, l_- be the two half planes divided by a diameter e_1e_2 of K and $K_+ := K \cap l_+, K_- := K \cap l_-$. Construct a Reuleaux triangle $R(\triangle)$ with vertices e_1, e_2, e_3 such that $R(\triangle)_+ := R(\triangle) \cap l_+ \supset K_+, R(\triangle)_- := R(\triangle) \cap l_- \subset K_-$. Let $u_{e_1e_2} := \overrightarrow{e_1e_2}/|\overrightarrow{e_1e_2}| \in S^1$. So, we have $\alpha(K, u_{e_1e_2}) = |K_+|/|K_-| \leq |R(\triangle)_+|/|R(\triangle)_-| = \alpha_0$. Therefore, $\alpha(K) \leq \alpha_0$. Since $\alpha(K) = \alpha_0$, we have $K = R(\triangle)$. □

Compliance with ethical standards
Conflict of interest The authors declare that we have no conflict of interest.

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