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Regularizing transformations of polygons

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Abstract. We start with a generic n-gon Q_0 with vertices $q_{j,0}$ ($j = 0, \ldots, n-1$) in the d-dimensional Euclidean space $\mathbb{E}^d$. Additionally, $m+1$ real numbers $u_0, \ldots, u_m \in \mathbb{R}$ ($m < n$) with $\sum_{\mu=0}^{m} u_\mu = 1$ are given. From these initial data we iteratively define generations of n-gons Q_k in $\mathbb{E}^d$ for $k \in \mathbb{N}$ with vertices $q_{j,k} := \sum_{\mu=0}^{m} u_\mu \, q_{j+\mu,k-1}$. We can show that this affine iteration generally regularizes in an affine sense.

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1. Introduction

Schoenberg [6], Ziv [7], Nicollier [2] and Donisi et al. [1] studied geometric iteration processes starting with a generic n-gon Q_0 in $\mathbb{E}^2$. They use homotheties to construct vertices of a *next generation polygon* Q_1. Reiterating this process creates a series of generations Q_k. This iteration, in general, has a regularizing effect on the polygon. Surprisingly, the result for n-gons in the plane $\mathbb{E}^2$ presented by Roeschel in [5] is also valid for n-gons in higher dimensions. In [5] the proof for $\mathbb{E}^2$ is based on the fact that the space of planar n-gons is spanned by the planar prototype n-gons of $\mathbb{E}^2$. As this does not hold for higher dimensions the proof for $\mathbb{E}^d$ with $d \geq 3$ demands another approach with different arguments. We prove an affine regularization theorem: these iterations in higher dimensions also deliver generations Q_k approaching the affine shape of regular planar polygons.

2. The spatial affine iteration

We use vectors in $\mathbb{R}^d$ to describe points of the d-dimensional Euclidean space $\mathbb{E}^d$ ($d > 2$) with respect to a Cartesian coordinate frame $\{O; x_1, \ldots, x_d\}$. We start with some spatial n-gon $Q_0 \subset \mathbb{E}^d$ with vertices $\{q_{0,0}, q_{1,0}, \ldots, q_{n-1,0}\}$

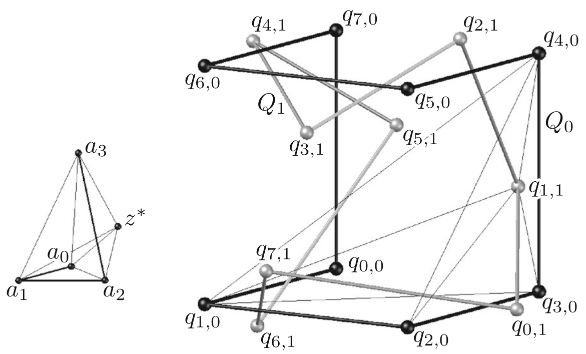

FIGURE 1 An example for $n = 8$ and $m = d = 3$: the polygon Q_0 with vertices of a cube and the first generation polygon Q_1 for $(u_0, u_1, u_2, u_3) = (0.2, -0.35, 0.75, 0.4)$

($n > 2$, $q_{j,0} \in \mathbb{R}^d$). Our starting polygon Q_0 shall be called *polygon of generation 0*.

On the other hand in an m-dimensional affine space $\mathbb{R}^m$ $(0 < m < n)$ with a simplex $S := \{a_0, \ldots, a_m\}$ we choose a *reference point* z^* with respect to S: Let $z^* := \sum_{\mu=0}^{m} u_\mu \, a_\mu$ be given by its barycentric coordinates $(u_0, \ldots, u_m) \in \mathbb{R}^{1\times(m+1)}$ with $\sum_{\mu=0}^{m} u_\mu = 1$.

Let $\alpha_{j,1}$ be the affine mappings from the ordered reference simplex vertex set S to ordered sets of m consecutive vertices $q_{j,0}, \ldots, q_{j+m,0}$ of Q_0 ($j \in \mathbb{J} := \{0, \ldots, n-1\}$; first index *mod* n). Each of these n affine mappings is applied to the reference point z^*; this way we get n image points $q_{j,1} := \alpha_{j,1}(z^*) = \sum_{\mu=1}^{m} u_\mu \, q_{j+\mu-1,0}$ which form a new n-gon Q_1 called the *generation* 1 polygon.

The same process can now be applied, in turn, to the polygon Q_1 with the same reference simplex S and the same reference point z^*, creating a subsequent polygon Q_2. Iteration yields a series of polygons. $Q_k := \{q_{0,k}, \ldots, q_{n-1,k}\}$ is the kth *generation polygon* with vertices

$$q_{j,k} = \sum_{\mu=0}^{m} u_\mu \, q_{j+\mu,k-1} \ \in \mathbb{R}^d \quad (j \in \mathbb{J},\, k \in \mathbb{N}\backslash\{0\})\,. \tag{2.1}$$

The procedure is a d-dimensional generalisation of the geometric iteration presented in [5]. Figure 1 shows the first iteration step for an example with $n = 8$ and $m = d = 3$.

3. The iteration process

We describe the polygons Q_k by $d \times n$-matrices $Q_k := (q_{0,k}, \ldots, q_{n-1,k})$ in $\mathbb{R}^{d\times n}$ with $q_{j,k}$ (2.1). Formula (2.1) can be rewritten as a product of matrices $Q_k := Q_{k-1}.\, M$ with the circulant $n \times n$-matrix $M \in \mathbb{R}^{n\times n}$:

$$M = \begin{pmatrix} u_0 & 0 & \cdots & \cdots & 0 & u_m & u_{m-1} & \cdots & u_2 & u_1 \\ u_1 & \ddots & \ddots & & & \ddots & \ddots & \ddots & & u_2 \\ \vdots & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & \ddots & \ddots & u_{m-1} \\ u_{m-1} & & & \ddots & \ddots & \ddots & & & \ddots & u_m \\ u_m & \ddots & & & \ddots & \ddots & \ddots & & & 0 \\ 0 & \ddots & \ddots & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & u_m & u_{m-1} & \cdots & \cdots & u_1 & u_0 \end{pmatrix} \tag{3.1}$$

The nth complex roots of unity $\in \mathbb{C}$ shall be termed $\zeta_j := exp(i\frac{2j\pi}{n}) = \cos\frac{2j\pi}{n} + i\sin\frac{2j\pi}{n}$ $(j \in \mathbb{Z})$. We define the vectors

$$P_j := (\zeta_j^0, \ldots, \zeta_j^{n-1}) \in \mathbb{C}^{1\times n} \quad (j \in \mathbb{Z}) \tag{3.2}$$

and have $P_j \cdot M = P_j \sum_{\mu=0}^m u_\mu\, \zeta_j^\mu$ and $M \cdot P_{n-j}^t = (\sum_{\mu=0}^m u_\mu\, \zeta_j^\mu)\, P_{n-j}^t$. Thus, the vectors P_j and P_{n-j}^t $(j \in \mathbb{J})$ are left and right eigenvectors of M. The corresponding eigenvalue is

$$\lambda_j := \sum_{\mu=0}^m u_\mu\, \zeta_j^\mu \quad (j \in \mathbb{J}). \tag{3.3}$$

As $(u_0, \ldots, u_m) \in \mathbb{R}^{1\times(m+1)}$ and $\overline{\zeta_j^\mu} = \zeta_{n-j}^\mu$ we have $\overline{\lambda}_j = \lambda_{n-j}$ for all $j \in \mathbb{J}\backslash\{0\}$.

We now regard two matrices out of $\mathbb{C}^{n\times n}$

$$L := \frac{1}{\sqrt{n}} \begin{pmatrix} P_0 \\ \vdots \\ P_{n-1} \end{pmatrix} \quad \text{and} \quad R := \frac{1}{\sqrt{n}} \begin{pmatrix} P_n \\ \vdots \\ P_1 \end{pmatrix}. \tag{3.4}$$

L and R are symmetric and regular for $n > 1$ (see [3,5,6] and [7]). We have: $L = \overline{R}$ and $L \cdot R = I_{n,n}$ with the $n \times n$- unit matrix $I_{n,n}$; the matrices L and R are unitary $n \times n$-matrices in $\mathbb{C}^{n\times n}$. We have $L \cdot M \cdot R = D(\lambda_0, \ldots, \lambda_{n-1})$ with the diagonal matrix $D(\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{C}^{n\times n}$ containing the eigenvalues λ_j of M as its elements in the main diagonal. This yields $M = R \cdot D(\lambda_0, \ldots, \lambda_{n-1}) \cdot L$ and

$$\begin{aligned} Q_k \cdot R &= Q_{k-1} \cdot R \cdot D(\lambda_0, \ldots, \lambda_{n-1}) \quad \text{and} \\ Q_k \cdot R &= Q_0 \cdot R \cdot D(\lambda_0, \ldots, \lambda_{n-1})^k \quad \text{for } k \in \mathbb{N}\backslash\{0\}. \end{aligned} \tag{3.5}$$

We get $Q_k \cdot R = \frac{1}{\sqrt{n}} \left(\sum_{\nu=0}^{n-1} q_{\nu,k}\zeta_n^\nu,\ \sum_{\nu=0}^{n-1} q_{\nu,k}\zeta_{n-1}^\nu, \ldots, \sum_{\nu=0}^{n-1} q_{\nu,k}\zeta_1^\nu \right)$.

Then (3.5) yields

$$\sum_{\nu=0}^{n-1} q_{\nu,k}\zeta_{n-j}^\nu = \lambda_j^k \sum_{\nu=0}^{n-1} q_{\nu,0}\zeta_{n-j}^\nu \quad \forall\, j \in \mathbb{J}. \tag{3.6}$$

Due to $\lambda_0 = \sum_{\mu=0}^{m} u_\mu \zeta_0^\mu = 1$ and $\zeta_n^\nu = 1$, the index $j = 0$ in (3.6) delivers $\sum_{\nu=0}^{n-1} q_{\nu,k} = \sum_{\nu=0}^{n-1} q_{\nu,0}$ for all $k \in \mathbb{N}\backslash\{0\}$: All polygons Q_k have the same center of gravity.

From now on let the initial polygon Q_0 have its center of gravity in the origin $O := (0,\ldots,0)^t$. So we can be sure that for all $k \in \mathbb{N}$

$$\frac{1}{n}\sum_{\nu=0}^{n-1} q_{\nu,k} = \mathbf{o}_d := (0,\ldots,0)^t. \tag{3.7}$$

As the matrix R is regular the initial polygon Q_0 can explicitly be retrieved from the $d \times n$- matrix

$$Q_0 \cdot R =: B = (b_0,\ldots,b_{n-1}) \in \mathbb{C}^{d\times n} \quad \text{with} \quad b_j = \frac{1}{\sqrt{n}}\sum_{\nu=0}^{n-1} q_{\nu,0}\,\zeta_{n-j}^\nu \in \mathbb{C}^d. \tag{3.8}$$

From $q_{\nu,0} \in \mathbb{R}^d$ and $\zeta_{n-j}^\nu = \overline{\zeta_j^\nu}$ we get $\overline{b_j} = b_{n-j}$ for all $j \in \mathbb{J}^* := \{1,\ldots,n-1\}$. Because of (3.7) the first column vector is zero: $b_0 = \mathbf{o}_d$. Equation (3.5) yields

$$Q_k \cdot R \;=\; B \cdot D(\lambda_0,\ldots,\lambda_{n-1})^k \;=\; (\mathbf{o}_d, \lambda_1^k b_1,\ldots,\lambda_{n-1}^k b_{n-1}). \tag{3.9}$$

Thus, we do not alter the recursion in any way if we replace the diagonal matrix $D(\lambda_0,\ldots,\lambda_{n-1})$ in (3.5) by the diagonal matrix $D^* := D(0,\lambda_1,\ldots,\lambda_{n-1})$. With this in mind, the iteration process can be described by

$$Q_k = B\cdot D^{*\,k}\cdot L = \frac{1}{\sqrt{n}}\sum_{\nu=1}^{n} \lambda_\nu^k\, b_\nu\, P_\nu \;\Leftrightarrow\; q_{j,k} = \frac{1}{\sqrt{n}}\sum_{\nu=1}^{n}\lambda_\nu^k\, b_\nu\,\zeta_j^\nu \tag{3.10}$$

for $j \in \mathbb{J}$. Note that $b_\nu\, P_\nu \in \mathbb{C}^{d\times n}$ for $\nu \in \mathbb{J}^*$.

4. Prototype polygons

The Gaussian plane of complex numbers $\mathbb{C}$ can be interpreted as a Euclidean plane $\mathbb{E}^2$ with a Cartesian coordinate frame $\{O;1,i\}$. We embed $\mathbb{E}^2$ into $\mathbb{E}^d$ by identifying 1 and i with the d-dimensional unit vectors $e_1 := (1,0,0,\ldots,0)^t$ and $e_2 := (0,1,0,\ldots,0)^t$, respectively. The elements of P_j (3.2) can be viewed as a collection of n points ζ_j^ν ($\nu \in \mathbb{J}$) equally distributed on the unit circle of $\mathbb{E}^2 \subset \mathbb{E}^d$ centered in O with $j \in \mathbb{J}^* := \{1,\ldots,n-1\}$. Its points can be written as

$$T_j = e_1\,\frac{P_j + \overline{P}_j}{2} + e_2\,\frac{P_j - \overline{P}_j}{2\,i} = e_1\,\frac{P_j + P_{n-j}}{2} + e_2\,\frac{P_j - P_{n-j}}{2\,i}. \tag{4.1}$$

T_j is represented by a matrix $\in \mathbb{R}^{d\times n}$ with columns

$$t_{\nu,j} := (\cos\frac{2\pi\nu j}{n}, \sin\frac{2\pi\nu j}{n}, 0,\ldots,0)^t \in \mathbb{R}^d \;(\nu \in \mathbb{J}). \tag{4.2}$$

T_j forms the so-called *'regular prototype n-gon of jth kind'*. The regular n-gon T_{n-j} is symmetric to T_j w.r.t. the axis e_1 and thus affinely equivalent to T_j. If j and n are relatively prime the polygon T_j is either a regular n-gon or an

n-sided regular star. If j is a divisor of n with $n = j\,p$ the polygon T_j is either a regular p-gon or an ordinary regular star with p vertices, each of the vertices being multiply counted (j times).

5. The concept of affine regularization

An affine mapping of $\mathbb{E}^d$ keeping the origin O in its place is described by

$$\beta : \mathbb{E}^d \longrightarrow \mathbb{E}^d,\ x \mapsto \beta(x) = Cx \quad \text{with} \quad C \in \mathbb{R}^{d\times d}. \tag{5.1}$$

The affine image of the polygon $Q_k = (q_{0,k}, \ldots, q_{n-1,k})$ is $\beta(Q_k) := C \cdot Q_k$. Our iteration (2.1) seems to regularize for certain $(u_0, \ldots, u_m) \in \mathbb{R}^{1\times(m+1)}$ irrespective of the choice of the starting polygon Q_0. In order to examine this interesting peculiarity we compare the n-gons Q_k with a regular prototype n-gon T_j (4.1) of jth kind[1]:

Definition 5.1. *We call the iteration* (2.1) affinely regularizing of kind j *with* $1 \le j \le n/2$ *if, for any generic initial polygon* Q_0*, there exist affine mappings* $\beta_k : \mathbb{E}^d \longrightarrow \mathbb{E}^d$ *transforming* $Q_k = (q_{0,k}, \ldots, q_{n-1,k})^t$ *into polygons* $\beta_k(Q_k)$ *with the property that the series* Δ_k *of sums of the squared distances*

$$\Delta_k := \sum_{\nu=0}^{n-1} \|\beta_k(q_{\nu,k}) - t_{\nu,j}\|^2 = \operatorname{tr}\Big((T_j - \beta_k(Q_k))^t \cdot \overline{(T_j - \beta_k(Q_k))}\Big) \tag{5.2}$$

of respective vertices of $\beta(Q_k)$ *and of the regular prototype polygon* T_j *of jth kind is a null series:* $\lim_{k\to\infty} \Delta_k = 0$.

6. The affine regularization theorem

The shape of the polygons Q_k depends on the input data set Q_0 and on the barycentric coordinates $(u_0, \ldots, u_m)$ of the reference point z^* with $\sum_{\mu=0}^m u_\mu = 1$. The latter determine the matrix M (3.1), the eigenvalues λ_j and the diagonal matrix $D^* = D(0, \lambda_1, \ldots, \lambda_{n-1})$. The norms $n_j := |\lambda_j|$ of λ_j for $j \in \mathbb{J}^*$ are given by

$$n_j^2 = \lambda_j \overline{\lambda_j} = \sum_{\mu,\nu=0}^{m} u_\mu u_\nu\, \zeta_j^{\mu-\nu}. \tag{6.1}$$

We put $N := \max\{n_1, \ldots, n_{n-1}\}$. Let the barycentrics $(u_0, \ldots, u_m)$ be chosen *generally* such that not all $\lambda_1, \ldots, \lambda_{n-1}$ vanish. $N = 0$ is equivalent with $\lambda_1 = \cdots = \lambda_{n-1} = 0$ and can only occur if $m = n-1$ and, additionally, $(u_0, \ldots, u_{n-1}) = (1/n, \ldots, 1/n)$. This case of iterated series of 'degenerate n-gons' Q_k, all collapsing into the center of gravity O shall be excluded further on. For $0 < N < 1$ the series Q_k gradually contracts for increasing k and tends towards the center of gravity O. For $N = 1$ the series Q_k remains finite,

[1] As the prototypes T_j and T_{n-j} are affinely equivalent, an iteration regularizing of jth kind will also be regularizing of kind $n-j$ and we can confine ourselves to $1 \le j \le n/2$.

but in general still may change its shape and its position from generation to generation. For $N > 1$ the series Q_k gradually expands for increasing k.

We will prove that for any $N > 0$, the algorithm is—in general—affinely regularizing. We divide the set of indices into two distinct subsets:

$$\mathbb{J}_1 := \{j \in \mathbb{J}^* / \, |\lambda_j| = N\} \neq \emptyset \quad \text{and} \quad \mathbb{J}_2 := \mathbb{J}^* \backslash \mathbb{J}_1. \tag{6.2}$$

According to (3.3), for any $j^* \in \mathbb{J}_1$ the index $n - j^*$ is also contained in $\mathbb{J}_1$; for even n and $j^* = n/2$ these two indices coincide. We have

$$\frac{|\lambda_j|}{N} = 1 \; \forall j \in \mathbb{J}_1 \quad \text{and} \quad 0 \leq \frac{|\lambda_j|}{N} < 1 \; \forall j \in \mathbb{J}_2. \tag{6.3}$$

Equations (3.10) yield

$$\begin{aligned} Q_k &= \tfrac{N^k}{\sqrt{n}} \left(\textstyle\sum_{\nu \in \mathbb{J}_1} \left(\tfrac{\lambda_\nu}{N}\right)^k b_\nu P_\nu + \sum_{\nu \in \mathbb{J}_2} \left(\tfrac{\lambda_\nu}{N}\right)^k b_\nu P_\nu \right) \\ \Leftrightarrow q_{j,k} &= \tfrac{N^k}{\sqrt{n}} \left(\textstyle\sum_{\nu \in \mathbb{J}_1} \left(\tfrac{\lambda_\nu}{N}\right)^k b_\nu \zeta_\nu^j + \sum_{\nu \in \mathbb{J}_2} \left(\tfrac{\lambda_\nu}{N}\right)^k b_\nu \zeta_\nu^j \right). \end{aligned} \tag{6.4}$$

Regardless of the input data b_j (3.8) the coefficients $(\frac{\lambda_\nu}{N})^k$ form null series for all $\nu \in \mathbb{J}_2$ and $k \to \infty$; the coefficients $(\frac{\lambda_\nu}{N})^k$ for all $\nu \in \mathbb{J}_1$ are complex numbers of norm 1 for all $k \in \mathbb{N}$.

Q_k and any homothetic image $\rho_k(Q_k)$ have the same affine shape. Following Definition 5.1 we can apply homotheties $\rho_k : \mathbb{E}^d \longrightarrow \mathbb{E}^d$ with $x \mapsto x \, \frac{\sqrt{n}}{N^k}$. These homotheties ρ_k turn (6.4) into

$$\begin{aligned} \rho_k(Q_k) &= \textstyle\sum_{\nu \in \mathbb{J}_1} \left(\tfrac{\lambda_\nu}{N}\right)^k b_\nu P_\nu + \sum_{\nu \in \mathbb{J}_2} \left(\tfrac{\lambda_\nu}{N}\right)^k b_\nu P_\nu \\ \Leftrightarrow \rho_k(q_{j,k}) &= \textstyle\sum_{\nu \in \mathbb{J}_1} \left(\tfrac{\lambda_\nu}{N}\right)^k b_\nu \zeta_\nu^j + \sum_{\nu \in \mathbb{J}_2} \left(\tfrac{\lambda_\nu}{N}\right)^k b_\nu \zeta_\nu^j. \end{aligned} \tag{6.5}$$

With reference to the cardinal number of the index set $\mathbb{J}_1$ we have three cases:

Case A: The index set $\mathbb{J}_1$ contains just one element. This can only happen if n is an even integer and the barycentrics $(u_0, \ldots, u_m)$ lead to $\mathbb{J}_1 = \{n/2\}$. We have $\zeta_{n/2} = -1$, and $\lambda_{n/2} = \sum_{\mu=0}^m u_\mu (-1)^\mu \in \mathbb{R}$. As $N = \left|\lambda_{n/2}\right| > 0$ and therefore $\lambda_{n/2} = \pm N \neq 0$ formula (6.5) reads as

$$\rho_k(Q_k) = (\pm 1)^k b_{n/2} P_{n/2} + \sum_{\nu \in \mathbb{J}_2} (\frac{\lambda_\nu}{N})^k b_\nu P_\nu. \tag{6.6}$$

For every k we apply a further homothety $\sigma_k : \mathbb{E}^d \longrightarrow \mathbb{E}^d$ with

$$\sigma_k(x) = (\pm 1)^k x \;\Rightarrow\; \sigma_k(\rho_k(Q_k)) = b_{n/2} P_{n/2} + \sum_{\nu \in \mathbb{J}_2} (\frac{\pm \lambda_\nu}{N})^k b_\nu P_\nu. \tag{6.7}$$

We have $b_{n/2} = \sum_{\nu=0}^{n-1} q_{\nu,0} \, \zeta_{n/2}^\nu = \sum_{\nu=0}^{n-1} (-1)^\nu \, q_{\nu,0} \in \mathbb{R}^d$. For a generic input polygon Q_0 we can assume $b_{n/2} \neq \mathbf{o}_d$. In this case we choose an affine mapping τ with fixed point O and $b_{n/2} \mapsto e_1 \in \mathbb{R}^d$. The mapping τ induces an affine mapping $\mathbb{C}^d \longrightarrow \mathbb{C}^d$ transforming b_ν $(\nu \in \mathbb{J}_2)$ into $b_\nu^* := \tau(b_\nu) \in \mathbb{C}^d$; the vectors b^*_ν do not depend on k. The affine mapping $\beta_k := \tau \circ \sigma_k \circ \rho_k$ places the kth generation polygon Q_k into

$$\beta_k(Q_k) = e_1 P_{n/2} + \sum_{\nu\in\mathbb{J}_2} (\frac{\pm\lambda_\nu}{N})^k \, b^*_\nu \, P_\nu. \tag{6.8}$$

The distance vectors $d_{j,k}$ of the vertices of $\beta_k(Q_k)$ to the respective vertices of the prototype polygon $T_{n/2} = e_1 \, P_{n/2}$ (4.1) are the columns of $D_k = (d_{0,k}, \ldots, d_{n-1,k})$ with

$$D_k = \textstyle\sum_{\nu\in\mathbb{J}_2} \left(\frac{\pm\lambda_\nu}{N}\right)^k \, b^*_\nu \, P_\nu \;\Leftrightarrow\; d_{j,k} = \sum_{\nu\in\mathbb{J}_2} \left(\frac{\pm\lambda_\nu}{N}\right)^k \, b^*_\nu \, \zeta^j_\nu \quad (j \in \mathbb{J}). \tag{6.9}$$

The vectors $b^*_\nu \zeta^j_\nu$ are independent from k. As the norms of $(\frac{\lambda_\nu}{N})^k$ form null series for all $\nu \in \mathbb{J}_2$ we can be sure that $\lim_{k\to\infty} d_{j,k} = \mathbf{o}_d$ for all $j \in \mathbb{J}$. The sum of the squared distances $\Delta_k := \sum_{j=0}^{n-1} \|d_{j,k}\|^2$ is a null series: $\lim_{k\to\infty} \Delta_k = 0$. Thus, according to our Definition 5.1 the iteration process in case A is affinely regularizing of kind $n/2$. For generic input Q_0 the polygons Q_k approach the shape of the n-gon $T_{n/2}$. The straight lines approximating the polygons Q_k tend towards the straight line through O with direction vector $b_{n/2}$.

Case B: The index set $\mathbb{J}_1$ contains exactly two different elements: $\mathbb{J}_1 = \{j^*, n-j^*\}$ with $1 \le j^* < n/2$. In a way, this could be considered the general case. We put $\lambda_{j^*} = Ne^{i\,\phi}$ and $\lambda_{n-j^*} = Ne^{-i\,\phi}$ with some real angle $\phi \in [0, 2\pi)$ and define $W := \sum_{\nu\in\mathbb{J}_2} (\frac{\lambda_\nu}{N})^k \, b_\nu \, P_\nu$. Then (6.5) yields

$$\rho_k(Q_k) \;=\; e^{i\,k\phi} \, b_{j^*} \, P_{j^*} + e^{-i\,k\phi} \, b_{n-j^*} \, \overline{P}_{j^*} + W. \tag{6.10}$$

Let $b_{j^*} := x + i\,y$ with $x,\, y \in \mathbb{R}^d$. We then have $b_{n-j^*} = \overline{b}_{j^*} = x - i\,y$ and

$$\rho_k(Q_k) = x(e^{ik\phi} P_{j^*} + e^{-ik\phi}\overline{P}_{j^*}) + i\,y(e^{ik\phi} P_{j^*} - \, e^{-ik\phi}\overline{P}_{j^*}) + W. \tag{6.11}$$

For a generic input n-gon Q_0 the two vectors $x, y \in \mathbb{R}^d$ are linearly independent. Let $\sigma : \mathbb{E}^d \longrightarrow \mathbb{E}^d$ be any affine mapping that maps the two vectors x, y into $\sigma(x) := e_1/2$ and $\sigma(y) := -e_2/2$. σ induces an affine mapping $\mathbb{C}^d \longrightarrow \mathbb{C}^d$ transforming $b_\nu (\nu \in \mathbb{J}_2)$ into $\sigma(b_\nu)$. We have

$$\begin{aligned}\sigma(\rho_k(Q_k)) = {} & (e_1 \cos k\phi + e_2 \sin k\phi) \frac{P_{j^*} + \overline{P}_{j^*}}{2} \\ & + (-e_1 \sin k\phi + e_2 \cos k\phi) \frac{P_{j^*} - \overline{P}_{j^*}}{2\,i} \\ & + \sum_{\nu\in\mathbb{J}_2} \left(\frac{\lambda_\nu}{N}\right)^k \sigma(b_\nu)\, P_\nu.\end{aligned} \tag{6.12}$$

We define the complex numbers $\theta_{\mu,\,\nu} := \sigma(b_\mu)^t \, \overline{\sigma(b_\nu)}$ for $\mu, \nu \in \mathbb{J}_2$. The matrices

$$R_k := \begin{pmatrix} \cos k\phi & \sin k\phi & 0 & \ldots & 0 \\ -\sin k\phi & \cos k\phi & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \tag{6.13}$$

describe rotations τ_k in $\mathbb{E}^d$. The induced mappings τ_k in $\mathbb{C}^d$ transform the vectors $\sigma(b_\nu)$ into vectors $\tau_k(\sigma(b_\nu)) \in \mathbb{C}^d(\nu \in \mathbb{J}_2)$. The mappings $\beta_k := \tau_k \circ \sigma \circ \rho_k$ are affine mappings from $\mathbb{E}^d$ into $\mathbb{E}^d$ and deliver

$$\beta_k(Q_k) \;=\; e_1 \frac{P_{j^*} + \overline{P}_{j^*}}{2} + e_2 \frac{P_{j^*} - \overline{P}_{j^*}}{2\,i} + \sum_{\nu \in \mathbb{J}_2} \left(\frac{\lambda_\nu}{N}\right)^k \tau_k(\sigma(b_\nu))\, P_\nu. \quad (6.14)$$

As every τ_k preserves scalar products we have $\tau_k(\sigma(b_\mu))^t\, \overline{\tau_k(\sigma(b_\nu))} = \theta_{\mu,\,\nu}$ for all $\mu, \nu \in \mathbb{J}_2$. According to (5.2) we compute the sum of squared distances of the vertices of $\beta_k(Q_k)$ to the respective vertices of the prototype polygon T_{j^*} and arrive at

$$\Delta_k = \mathrm{tr}\left((T_{j^*} - \beta_k(Q_k))^t \cdot (\overline{T_{j^*} - \beta_k(Q_k)})\right) = n \sum_{\mu \in \mathbb{J}_2} \left(\frac{\lambda_\mu \overline{\lambda}_\mu}{N^2}\right)^k \theta_{\mu, n-\mu}. \quad (6.15)$$

As the values $\theta_{\mu,\,n-\mu}$ are independent from k and $0 \le \frac{\lambda_\mu \overline{\lambda}_\mu}{N^2} < 1$ for all $\mu \in \mathbb{J}_2$ the values Δ_k $(k \in \mathbb{N})$ form a null series. Accordingly, the corresponding iteration process in case B is regularizing of kind j^* with $1 \le j^* < n/2$. For generic input n-gons Q_0 the two vectors x and y determine a plane ε^* through O. The planes ε_k approximating the polygon Q_k tend towards ε^*.

Case C: The index set $\mathbb{J}_1$ contains more than two different elements. We have $j^*, j^{**}, n - j^* \in \mathbb{J}_1$ with $1 \le j^* < j^{**} \le n/2$. According to (6.1) this is characterized by

$$\sum_{\mu,\nu=0}^{m} u_\mu u_\nu\, \zeta_{j^{**}}^{\mu-\nu} = \sum_{\mu,\nu=0}^{m} u_\mu u_\nu\, \zeta_{j^*}^{\mu-\nu}. \quad (6.16)$$

The coefficients of $u_\mu u_\nu$ in (6.16) are $\zeta_{j^{**}}^{\mu-\nu} + \zeta_{j^{**}}^{\nu-\mu} - \zeta_{j^*}^{\mu-\nu} - \zeta_{j^*}^{\nu-\mu} \in \mathbb{R}$. The corresponding barycentrics $(u_0, \ldots, u_m)$ denote points $z^* \in \mathbb{R}^m$ which, in general, are positioned on an $(m-1)$-dimensional quadric of $\mathbb{R}^m$ containing the vertices of the simplex S. In this case we cannot prove any regularizing effect of the affine iteration.

We call the barycentrics $(u_0, \ldots, u_m)$ 'generic' if they do not lead to Case C or, for $m = n-1$, they are different from $(\frac{1}{n}, \ldots, \frac{1}{n})$. Overall, we have

Theorem 6.1. *Affine Regularization Theorem. For generic barycentrics the iteration process* (2.1) *is affinely regularizing according to Definition* 5.1*. The barycentrics* $(u_0, \ldots, u_m)$ *(*$m < n$*) determine the eigenvalues* $\lambda_1, \ldots, \lambda_{n-1}$ *given by* (3.3) *and their maximal norm* $N > 0$*. If there is exactly one index* j^* *(with* $1 \le j^* \le n/2$*) with eigenvalue* λ_{j^*} *of norm* N *the iteration is regularizing of kind* j^**.*

If the iteration is affinely regularizing of kind j^* then, for a generic input n-gon Q_0, the shape of Q_k gradually approaches the shape of an affinely transformed prototype n-gon T_{j^*}.

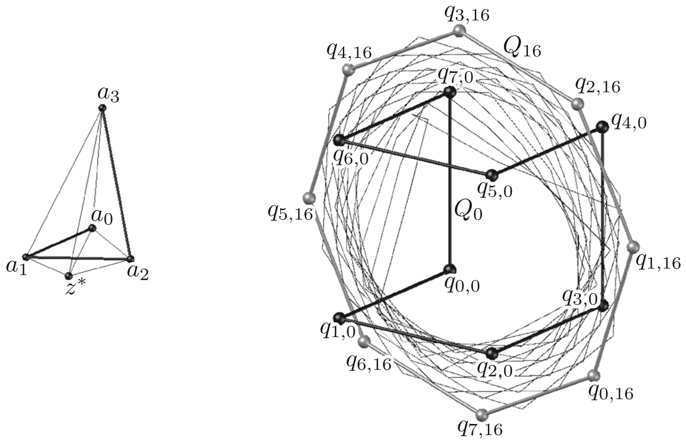

FIGURE 2 An example with $n = 8$, $m = d = 3$ with $\mathbb{J}_1 = \{1, 7\}$

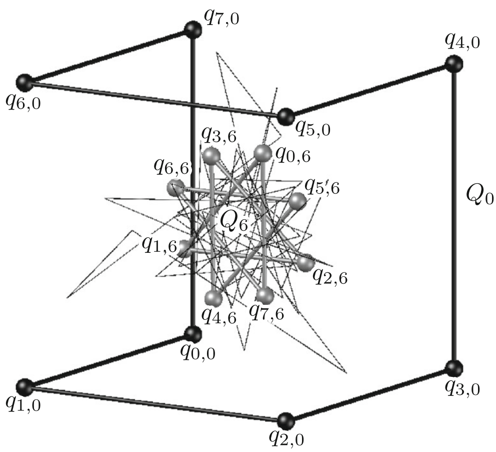

FIGURE 3 An exceptional example for $n = 8$, $m = d = 3$ with $b_4 = \mathbf{o}_d$ and $\mathbb{J}_1 = \{4\}$. For this specific n-gon Q_0 the algorithm works as if it was affinely regularizing of kind 3

Figure 2 shows an example for the same initial octogon Q_0 as in Fig. 1 ($n = 8, m = d = 3$). Here the barycentrics of z^* are $(u_0, u_1, u_2, u_3) = (0.4, 0.5, 0.3, -0.2)$. We get $n_1 \approx 1.03, n_2 \approx 0.71, n_3 \approx 0.13, n_4 \approx 0.4$ and therefore we have $\mathbb{J}_1 = \{1, 7\}$. The algorithm is regularizing of kind 1 (case B). The figure shows Q_0 and the following generations up to Q_{16}.

7. Remarkable exceptions

For specific initial polygons Q_0 the algorithm may deliver unexpected results. If the coefficient vectors b_ν of the regarded eigenvalues λ_ν for $\nu \in \mathbb{J}_1$ in (6.4) vanish the respective eigenvalues have no influence on the regularizing process. So, for such a specific n-gon Q_0, the algorithm works in the same way as if in (3.10) these eigenvalues λ_ν had been replaced by $\lambda_\nu = 0$. The remaining eigenvalues deliver another maximum norm $N^* < N$ and a different set $\mathbb{J}_1$. Now our classification (Sect. 6) reveals the affine shape of the series Q_k.

Figure 3 shows such an example for $n = 8, m = d = 3$ where we have $(u_0, u_1, u_2, u_3) = (0.5, -0.25, 0.5, 0.25)$. We get $n_1 \approx 0.52, n_2 = 0.5, n_3 \approx 0.99 < 1, n_4 = 1$. Hence $N = 1$ and we conclude that the algorithm is affinely regularizing of kind 4; Q_k is expected to approach the shape of the prototype is T_4 which is a line segment. The special initial octogon Q_0, however, yields $b_4 = \mathbf{o}_d$; we put $\lambda_4 := 0$ and perform a new case study. The affine shape of Q_k tends towards the prototype T_3. Figure 3 displays Q_0 and the following generations up to Q_6.

8. Conclusion

We studied affine iterations transforming an initial n-gon Q_0 in $\mathbb{E}^d$ $(d > 1)$ into successive generations of n-gons Q_k. The Affine Regularization Theorem in this paper does not only extend the results in [5] to dimensions $d > 2$; surprisingly, even for dimensions $d > 2$ the regularization leads to planar, regular prototypes no matter which generic input n-gon Q_0 we start with. For very specific input n-gons Q_0, though, the same algorithm seems to regularize in a different way. The understanding of this phenomenon completes the results.

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