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An easy characterization of PG$(4, n)$

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Abstract. In this paper, a characterization of the projective space PG$(4, n)$ in terms of finite planar spaces is given.

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1. Introduction

In order to make the reading of this article self-contained we recall some well-known definitions. A (*finite*) *linear space* is a pair $(\mathcal{S}, \mathcal{L})$, where $\mathcal{S}$ is a (finite) non-empty set whose elements are called *points*, and $\mathcal{L}$ is a family of proper subsets of $\mathcal{P}$ each of size at least two, whose elements are called *lines* such that

(i) *any two distinct points belong to exactly one line.*

A subset of $\mathcal{S}$ is a *subspace* if it contains the line through any two of its distinct points.

A (*finite*) *planar space* is a triple $(\mathcal{S}, \mathcal{L}, \mathcal{P})$, where $(\mathcal{S}, \mathcal{L})$ is a (finite) linear space and $\mathcal{P}$ is a family of proper subspaces called *planes* such that:

(ii) *Every plane contains at least three non-collinear points.*

(iii) *Every triple of non-collinear points is contained in a single plane.*

Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be a finite planar space. Let us denote by v, b and c respectively the sizes of $\mathcal{S}$, $\mathcal{L}$ and $\mathcal{P}$. The number $[p]$ of all lines passing through a fixed point p of $\mathcal{S}$ is the *degree* of p. The set of the lines through a point p of $\mathcal{S}$ and contained in a plane π through p is called a *pencil of lines with center* p and it is denoted with $\mathcal{F}_p$. If $n + 1$ is the maximum size of the pencils of lines, then

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the integer n is the *order* of the planar space. The set of all planes through a given line ℓ is called a *pencil of planes with axis* ℓ. The *length* of a line ℓ is its size and it is denoted with $|\ell|$.

Two lines ℓ and ℓ' are *parallel* either if $\ell = \ell'$ or if they are coplanar and $\ell \cap \ell' = \emptyset$. Two distinct non-coplanar lines are called *skew*. A line and a plane are *parallel* either if the line is contained in the plane or if their intersection is the empty-set.

Let n and q be two integers, $n \geq 2$, a finite planar space is (n, q)-*regular* if it satisfies the following properties:

I. *Every pencil of lines has size* $n + 1$.

II. *Every pencil of planes has size* $q + 1$.

Namely, from Property I it follows that $q \geq n$ and

Proposition 1.1. $|\ell| \leq n + 1$ *for every line* ℓ *of the planar space.*

A line of length $n + 1$ is called a *projective line*. Clearly, by Property I, a projective line is intersected by any of its coplanar line.

In this note, we will assume that the planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is (n, q)-regular and it satisfies the following two properties:

a. $q > n$

b. *The intersection of any two planes is non-empty.*

Every projective space $\mathrm{PG}(r, n)$, $r \geq 3$, is an (n, q)-regular planar space. If $r = 3$, then Property a. is not satisfied. If $r \geq 5$, then Property b. is not satisfied. Thus, the only projective space satisfying both properties is $\mathrm{PG}(4, n)$. Moreover, the planar space obtained from $\mathrm{PG}(4, n)$ by deleting one point is (n, q)-regular and it satisfies Property a. but not Property b.

The authors believe that it is possible to propose the following:

Conjecture. An (n, q)-regular planar space satisfying properties a. and b. is necessarily $\mathrm{PG}(4, n)$.

In a previous paper [2], the authors prove that the conjecture is true when the (n, q)-regular planar space satisfies the following extra condition:

c. *It there exists at least one projective line.*

In the appendix, for completeness it is given an easy proof of the following

Theorem I. *An* (n, q)*-regular planar space* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *is* PG$(4, n)$ *if and only if it satisfies properties a., b. and c.*

In this paper, the authors support their conjecture by proving the following

Theorem II. *An* (n, q)*-regular planar space* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *is* PG$(4, n)$ *if and only if it satisfies the following two properties:*

a'. $q > n^2 - n - 5 \geq n$
b. *The intersection of any two planes is not empty.*

Thus, the conjecture is true when q is large enough compared to n.

2. First properties of an (n, q)-regular planar space with $q > n$

Proposition 2.1. *Every point* p *has constant degree*

$$[p] = (q + 1)n + 1. \tag{2.1}$$

Proof. Let p be a point and ℓ be a line through p. The lines through p and different from ℓ are divided in groups of n in the $q + 1$ planes through ℓ and so (2.1) follows. □

Proposition 2.2. *Through every point* p *there is a constant number* c_p *of planes, with*

$$c_p = \frac{[p]([p] - 1)}{n(n + 1)}. \tag{2.2}$$

Proof. Given a point p, counting in double way the pairs $((\ell, \ell'), \pi)$, where ℓ and ℓ' are two distinct lines through p and π is the plane containing both ℓ and ℓ' it gives:

$$[p]([p] - 1) = c_p n(n + 1) \tag{2.3}$$

from which (2.2) follows. □

Proposition 2.3. *Let* π *be a plane and* p *be a point of* π*. The number* I_p *of planes through* p *intersecting* π *exactly in* p *is given by*

$$I_p = c_p - 1 - q(n + 1). \tag{2.4}$$

Proof. The number of planes through p different from π and intersecting π in a line through p, is $q(n + 1)$ and so the assertion is proved. □

Hence, for any given plane π and for any point $p \in \pi$, the number of planes intersecting π exactly in p is a constant depending neither from π nor from the choosen point $p \in \pi$.

Let π be a plane and let b_π and v_π denote the number of lines and points of π, respectively.

Since the planes are pairwise intersecting it follows that

$$qb_\pi + v_\pi I_p = c - 1. \tag{2.5}$$

Thus, the following assertions are equivalent.

(a) *Every plane has a constant number of lines.*

(b) *All the planes have the same size.*

(c) *All the lines have the same length.*

From (2.5) the equivalence between (a) and (b) follows. If all the lines have constant length $k+1$ then every plane has size $(n+1)k+1$. If every plane has size h for any line ℓ one has

$$v = |\ell| + (q+1)(h - |\ell|).$$

Fundamental for our purposes is the following.

Lemma 2.1. *Every plane of size at most $q+1$ is a projective plane of order n.*

Proof. Let π be a plane and assume that $|\pi| \leq q+1$. It will be sufficient to show that every line of π has length $n+1$. Assume, to the contrary that π has a line ℓ of length at most n. Let π' be a plane through ℓ, different from π. Let p be a point of π' not in ℓ. Since p has degree $n+1$, there is at least one line t of π' through p and parallel to ℓ. For every point x of $\pi - \ell$ let $\pi_x = \langle x, t\rangle$ be the plane containing x and t. Since ℓ has at least two points, the points of $\pi - \ell$ are at most $q-1$ and so the planes through t intersecting π are at most q. Thus, there is at least one plane through t which does not intersect π, which is a contradiction. □

3. (n, q)-Regular planar spaces with $q > n^2 - n - 5$ and with no disjoint planes

In this section, we are going to prove Theorem II. Thus, we will suppose that $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a (n, q)-regular planar spaces with no disjoint planes. We will prove that if $q > n^2 - n - 5 \geq n$, then the planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is $\mathrm{PG}(4, n)$.

By Theorem I it is sufficient to show that there exists at least one line of length n + 1. Assume on the contrary that no line has length $n+1$. Hence, by Lemma 2.1 it follows that every plane has size at least $q+2$.

Now, we prove that no line has length n. Assume, by way of contradiction, that there exists a line ℓ of length n. Since every point has planar degree $n+1$, the line ℓ, together with its parallel lines, gives rise to a line partition $\mathcal{F}$ of the planar space. By adding a common point ∞ to every line of the partition one obtains a planar space $(\mathcal{S}', \mathcal{L}', \mathcal{P}')$ which is again q-regular and with a line

$\ell' = \ell \cup \infty$ of length $n+1$. Let π be a plane through ℓ and let p be a point of $\pi - \ell$ different from ∞. Let t be a line through p and not contained in π. Since p has degree $n+1$, there are $q+1$ planes through ℓ and since $q > n$ there exists at least one plane π_0 through t intersecting π exactly in the point p. The plane π_0 does not contain ∞ otherwise it should contain the line connecting p and ∞ and such a line lies in π. Then, it follows that every plane through ℓ intersects π_0 in exactly one point and so π_0 has size $q+1$. Since the plane π_0 does not contain ∞ it is a plane of $\mathcal{P}$ and since its size is $q+1$ by Lemma 2.1 every line of π_0 has length $n+1$.

Thus, the maximum line size is $n-1$. Therefore, there are two possibilities:

(i) *There is at least one line of length* $n-1$.

(ii) *All lines have lenght at most* $n-2$.

Let us examine Case (i).

Let ℓ be a line of length $n-1$ and let π be a plane containing ℓ. If all the lines of π have length $n-1$, counting the number of points of π via the lines through a given point of π gives $|\pi| = (n+1)(n-2)+1$.

Let b_π denote the number of lines of π. Counting in double way the point-line pairs (x,t) with $x \in t$ and t in π gives

$$[(n+1)(n-2)+1](n+1) = b_\pi(n-1).$$

Being b_π an integer, we have $n \leq 3$, while is $n \geq 4$ since $n^2 - n - 5 \geq n$.

Therefore, the plane π contains at least one line of length at most $n-2$, and so $|\pi| \leq n^2 - n - 2$.

Let π' be a plane through ℓ and different from π and let ℓ' be a line of π' different from ℓ and parallel to ℓ. The q planes through ℓ' different from π' intersect π in at least q different points not in ℓ, so

$$|\pi| \geq q + n - 1.$$

Since $q > n^2 - n - 5$, from

$$q + n - 1 \leq |\pi| \leq n^2 - n - 2$$

it follows that $n < 4$ contradicting $n \geq 4$.

Case (ii) *There is no line of length* $n-1$.

Let π be a plane. Counting the number of points of π via the lines through one of its points gives

$$q + 2 \leq |\pi| \leq (n+1)(n-3) + 1$$

from which it follows that $q \leq n^2 - 2n - 4$ against the assumption $q > n^2 - n - 5$.

In both cases there is a contradiction, and so the planar space contains a line of length $n+1$, and Theorem II is completely proved taking into account Theorem I.

Appendix: proof of Theorem I

In such a section we are going to prove

Theorem I. *An (n,q)-regular planar space $(\mathcal{S},\mathcal{L},\mathcal{P})$ with $q>n$, with no disjoint planes and having at least one line L of length $n+1$ is the projective space* $\mathrm{PG}(4,n)$.

Proof. We will proceed by steps.

STEP 1. There are planes that intersect in a single point.

Let π be a plane and p a point of π. Let ℓ be a line through p not contained in π. Each line of π through p, together with ℓ, gives rise to a plane intersecting π in a line. Thus, there are $n+1$ planes through ℓ intersecting π in a line through p. The remaining planes through p intersect π in exactlty the point p. Since $q>n$, the assertion follows.

Let L denote a projective line.

STEP 2. There are planes disjoint with the projective line L.

Let π be a plane through L and let p be a point of π not in L. Every plane intersecting π in exactly the point p is disjoint from L.

STEP 3. The planes disjoint from L have constant size $q+1$.

Let π be a plane disjoint from L. Every point x of π, together with L, determines a plane through L. Vice versa, since the planes have no empty intersection, a plane through L intersects π in a single point x, being L projective. Such a bijection shows the assertion.

STEP 4. Every plane disjoint from L is projective.

Let π be a plane disjoint from L and let r be a line of π. We prove that r has length $n+1$. Assume, by way of contradiction, that $|r|\leq n$. Let π' be a plane through r, different from π, and let x be a point of π' not in r. Since $|r|\leq n$ in π' there is at least one line t through x parallel to r. Since $|r|\geq 2$, in the plane π there are at least $q-1$ points not in r. Each of such points together with t gives rise to a plane through t. It follows that the planes through t intersecting π are at most q. So, there is at least one plane through t disjoint from π,a contradiction. If every lines of π has length $n+1$, then π is a projective plane.

STEP 5. Every line skew with L has length $n+1$.

Let r be a line skew with L. There are $q-n$ planes through r disjoint from L and so the assertion follows from Step 4.

STEP 6. Every line intersecting L has length $n+1$.

Let r be a line different from L and intersecting L in a point p. Let π be a plane through r not containing L. Let x be a point of π not belonging to r. The n lines through x, different from the line containing x and p, are skew with L and so, by Step 5, their length is $n+1$. Hence, each of such lines intersects r. It follows that $|r| = n+1$.

So, through Steps 5 and 6 we have proved that every line of the space has length $n+1$. it follows that every plane is a projective one and so the planar space is a projective space of dimension 4 since $q+1 = n^2+n+1$.

Thus, Theorem I is proved. □

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