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Absolute geometry proofs of two geometric inequalities of Chisini

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Abstract. Two results, proved synthetically in plane Euclidean geometry by Chisini in 1924—stating that: (i) if MAB is an isosceles triangle, with $MA \equiv MB$, inscribed in a circle $\mathcal{C}$, P_1 and P_2 are two points on $\mathcal{C}$ such that $\{B, P_i\}$ separates $\{A, M\}$ for $i \in \{1, 2\}$, and $\{B, P_2\}$ separates $\{M, P_1\}$, then $AP_1 + BP_1 < AP_2 + BP_2$, and (ii) of all triangles inscribed in a given circle the equilateral triangle has the greatest perimeter—are proved inside Hilbert's absolute geometry.

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1. Introduction

Following the example of his mentor Federigo Enriques (1871–1946), whom he met as an engineering student at the University of Bologna, Oscar Chisini (1889–1967) was actively involved in the teaching of mathematics at the secondary school level. He held the position of editorial secretary (*segretario di redazione*) of the journal *Il Periodico di Matematiche* in 1921, when Enriques became its managing editor (*direttore*), and after Enriques' death in 1946 Chisini held the managing editorship until his death in 1967.

In 1925, another mathematician with an interest in elementary mathematics, Alessandro Padoa, published a note [6] (and later an addendum [7]) in the *Periodico*, in which he proved by using elementary inequalities that of all triangles with a given perimeter, the equilateral triangle has the greatest radius of the circumscribed circle. In the note [1], appearing right after Padoa's paper, Chisini provides a purely geometric proof of the validity of the theorem in plane Euclidean geometry over the real numbers, rephrased as

Theorem 1. *Of all triangles inscribed in a given circle, the equilateral triangle has the greatest perimeter.*

To prove it, Chisini first proved that the following

Lemma 1. *If MAB is an isosceles triangle, with $MA \equiv MB$, inscribed in a circle of center O, $\widehat{AM}$ denotes the shorter of the two arcs joining A and M, and P_1 and P_2 are two points on $\widehat{AM}$ with $AP_1 < AP_2$, then*

$$AP_1 + BP_1 < AP_2 + BP_2. \tag{1.1}$$

holds in plane Euclidean geometry over the real numbers.

After the passing of time allowed [1] to be forgotten, Lemma 1, in the special case in which $P_2 = M$, was proved by different means, also purely geometrically, in [5, pp. 16–17], and later, in two different ways by two Dutch mathematicians, in [4, pp. 21–24]. The first of the two later proofs, in [4, pp. 22–23], attributed to Jan van Yzeren, has the distinction of being valid in Hilbert's absolute geometry $\mathcal{A}$ (whose axioms are the plane axioms of groups I, II, and III of Hilbert's *Grundlagen der Geometrie*, or equivalently the axioms A1–A9 in [9]). Lemma 1 itself was reproved, with three different proofs, all valid only in the Euclidean setting, in [3], where only [4] and [5] are cited as references.

The aim of this paper is to show that both Theorem 1 and Lemma 1 hold in Hilbert's absolute geometry. Almost as an afterthought, we will also show that

Theorem 2. *Let n be a natural number ≥ 3, and let $\mathfrak{H}$ be a model of absolute geometry in which there exists a regular n-gon. Then, of all convex n-gons inscribed in a given circle, the regular n-gon has the greatest perimeter.*

Chisini's proof of Lemma 1 uses specifically Euclidean features and cannot be used in the absolute setting. Our proof will expand on van Yzeren's idea.

2. Proving Lemma 1

First, notice that, if M and O lie on the same side of AB, if C denotes the other endpoint of the diameter through B, and if (i) both P_1 and P_2 lie on the same side of BC as A, or (ii) P_2 is C and P_1 lies on the same side of BC as A, then (1.1) follows from the fact that $AP_1 < AP_2$ (by hypothesis) and $BP_1 < BP_2$ (since the perpendicular bisector b of P_1P_2 passes through O and thus $BP_1 < BC$, which is $BP_1 < BP_2$ in case (ii), and, since, in case (i), the ray $\overrightarrow{BP_2}$ lies between the rays $\overrightarrow{BC}$ and $\overrightarrow{BP_1}$, b must intersect the side BP_2 of triangle BP_1P_2).[1]

Thus, if M and O lie on the same side of AB, it is enough to prove (1.1) in the case in which both P_1 and P_2 lie on the same side of BC as M or in which $P_1 = C$ and P_2 lies on the same side of BC as M.

[1] It follows from the triangle inequality that if the perpendicular bisector of side XY in triangle XYZ intersects side XZ, then $ZY < ZX$.

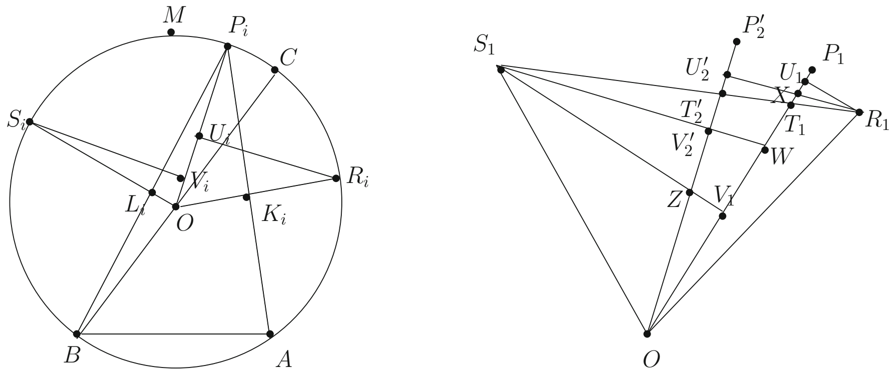

FIGURE 1 The triangles OS_iR_i after a rotation

For $i = 1, 2$, let K_i and L_i denote the feet of the perpendiculars from O to P_iA and P_iB respectively (i.e., the midpoints of the sides P_iA and P_iB). Let R_i and S_i denote the intersections of the perpendiculars dropped from O to P_iA and P_iB with the arc $\overparen{AMB}$ respectively (these points are thus the midpoints of the arcs $\overparen{AP_i}$ and $\overparen{BP_i}$ (the ones included in $\overparen{AMB}$, see Fig. 1). Here we define the midpoint of the arc $\overparen{XY}$ as the point of intersection of the perpendicular from O, the center of the circle $\mathcal{C}$, with the arc $\overparen{XY}$. Let U_i and V_i be the feet of the perpendiculars from R_i and S_i to OP_i respectively. Since the triangles OP_iR_i and OP_iS_i are isosceles, the altitudes P_iK_i and R_iU_i, and P_iL_i and S_iV_i must be congruent. Thus $R_iU_i + S_iV_i = P_iK_i + P_iL_i = (P_iA + P_iB)/2$.

To prove the lemma it is thus enough to show that $R_1U_1 + S_1V_1 < R_2U_2 + S_2V_2$. Since the altitudes OL_i and OK_i in the isosceles triangles OBP_i and OAP_i are angle bisectors as well, the angles $\widehat{S_iOR_i}$, for $i = 1, 2$ are congruent, their measure being half of $\widehat{AOB}$ if M and O lie on different sides of AB, or the supplement of half of $\widehat{AOB}$ in case O and M lie on the same side of AB. The triangles R_1OS_1 and R_2OS_2 are thus congruent, and, by the assumption on the relative positions of P_1 and P_2, we have $\widehat{R_1OP_1} < \widehat{R_2OP_2}$. If we rotate triangle R_2OS_2 about O, so that the point R_2 is mapped into R_1, then the two triangles R_1OS_1 and R_2OS_2 will coincide, and the point P_1 will lie inside the angle $\widehat{R_1OP_2'}$, where P_2' is the image of P_2 under the rotation about O that mapped R_2 into R_1. In other words, we have laid off $\widehat{R_2OP_2}$ on the side of ray $\overrightarrow{OR_1}$ on which P_1 lies, to get an angle $\widehat{R_1OP_2'}$ congruent to $\widehat{R_2OP_2}$. Given our assumption on the positions of P_1 and P_2 in case M and O lie on the same side of AB, we have that $\widehat{S_1OP_1}$ is $\leq 90°$. Since the points P_i are on $\overparen{AM}$, we also have that $\widehat{R_1OP_2'}$ is $< 90°$. Thus, $\widehat{S_1OP_2'}$, $\widehat{R_1OP_2'}$, $\widehat{R_1OP_1}$ are all acute and $\widehat{S_1OP_1} \leq 90°$. Since, for any point F between R_1 and S_1, we have

$OF < OR_1$ (as $\widehat{OFR_1} > \widehat{OS_1F}$ (as external angle) and thus $\widehat{OFR_1} > \widehat{OR_1F}$, and across the greater angle lies the greater side) the segments OP_1 and OP_2' intersect the segment R_1S_1 in two points T_1 and T_2' respectively. Given that a side of a right triangle is less than the hypotenuse, this implies that the feet of the perpendiculars from R_1 and S_1 to OP_1 and OP_2' all lie on the closed segments OP_1 and OP_2'. These feet are U_1, V_1 and U_2' and V_2' respectively. Notice that R_1U_2' and S_1V_2' are congruent with R_2U_2 and S_2V_2 respectively. Given that the other angles of a right triangle must be acute, we have the following order on the lines OP_1 and OP_2': $OV_1WT_1XU_1P_1$ and $OZV_2'T_2'U_2'P_2'$ (where we may have $V_2' = T_2' = U_2'$, a situation which occurs only if $P_2 = M$), where W denotes the intersection of the ray $\overrightarrow{S_1V_2'}$ with the segment V_1T_1, X is the point of intersection of the segment OP_1 and R_1U_2', and Z is the point of intersection of the segment S_1V_1 with the segment OV_2'.

Now, given that the hypotenuse is greater than the side, we have (α) $S_1W = S_1V_2' + V_2'W > S_1V_1$, (β) $R_1X > R_1U_1$. Given that—as can be seen by applying the fact that the base of a Saccheri quadrilateral is parallel to its summit (see [2, p. 177] for a definition of the base and the summit of a Saccheri quadrilateral) and the Pasch axiom—if F is between E and G, and g is a line through E that does not go through F, then the distance from F to g is less than that from G to g, we also have (γ) $XU_2' \geq V_2'W$ (equality holds only if $U_2' = V_2'$, which occurs only if $P_2 = M$, i.e., if we are in the case solved by van Yzeren). Thus, applying in this order (γ), (α), and (β), we get $R_2U_2 + S_2V_2 = R_1U_2' + S_1V_2' = S_1V_2' + R_1X + XU_2' \geq S_1V_2' + R_1X + V_2'W > S_1V_1 + R_1X > S_1V_1 + R_1U_1$.

3. Proving the Theorems

To prove our theorems, we'll first prove the following in $\mathcal{A}$:

Lemma 2. *Let $n \geq 2$ be a natural number, let P_0 and P_n be two points on a circle $\mathcal{C}$, not necessarily different, let $\widehat{P_0P_n}$ denote the arc, considered in some fixed direction, bordered by P_0 and P_n (if $P_0 = P_n$, the arc is the entire circle), and let α denote its measure. Let $P_1, \dots P_{n-1}$ be, in this order, points on the arc $\widehat{P_0P_n}$, dividing it into equal parts, i. e., such that the measure of $\widehat{P_iP_{i+1}}$ is $\frac{\alpha}{n}$, for all $i \in \{0, 1, \dots, n-1\}$. Let $P_1', \dots P_{n-1}'$ be, in this order, arbitrary points on the arc $\widehat{P_0P_n}$. Then, by setting $P_i' = P_i$ for $i \in \{0, n\}$, we have $\sum_{i=0}^{n-1} P_i'P_{i+1}' \leq \sum_{i=0}^{n-1} P_iP_{i+1}$, equality holding if and only if $P_i' = P_i$ for all i.*[2]

[2]The reference to the *measure* of an angle is to be understood here as a figure of speech. That a certain arc $\widehat{X_0X_1}$ is the nth part of another arc $\widehat{UV}$ in the same circle $\mathcal{C}$ can be expressed by stating that $UV \equiv X_0X_n$, where X_i are defined as different points on $\mathcal{C}$ with $X_{i-1}X_i \equiv X_iX_{i+1}$, for $i \in \{1, 2, \dots, n-1\}$.

Proof. The proof will proceed by induction. The $n = 2$ case is Lemma 1 with $P_2 = M$, which was proved by van Yzeren. Suppose the statement is true for some $n \geq 2$. We will prove that the statement holds for $n+1$ as well. Unless $P_i' = P_i$ for $i \in \{0, \ldots n+1\}$, there must be two adjacent arcs among the $\widehat{P_i'P_{i+1}'}$, say $\widehat{P_m'P_{m+1}'}$ and $\widehat{P_{m+1}'P_{m+2}'}$, such that one (say, the first) has measure less than $\frac{\alpha}{n}$ and the other (say, the second) has measure greater than $\frac{\alpha}{n}$. Let M be the midpoint of the arc $\widehat{P_m'P_{m+2}'}$. If the measure of $\widehat{P_m'P_{m+2}'}$ is greater than $\frac{2\alpha}{n}$, then the point R for which $\widehat{P_m'R}$ has measure $\frac{\alpha}{n}$ is on the arc $\widehat{P_m'M}$, included in $\widehat{P_m'P_{m+2}'}$, so, by Lemma 1, we have $P_m'R + RP_{m+2}' > P_m'P_{m+1}' + P_{m+1}'P_{m+2}'$, and thus, by transporting the segment $P_m'R$, such that P_m' becomes P_0 and R becomes R_1', we can transport the remaining segments of the polygonal line $P_0P_1' \ldots P_n'P_{n+1}'$ from R' on, obtaining a polygonal line $P_0R_1' \ldots R_n'P_{n+1}$, whose segments represent a rearrangement of those in the polygonal line $P_0P_1' \ldots P_n'P_{n+1}'$, and thus have the same length. However, the length of the polygonal line $R_1' \ldots R_n'P_{n+1}$ is known by the induction hypothesis to be $\leq n$ times P_0R_1', and that equality occurs if and only if the R_i' are equal division points. This proves the statement for $n+1$ under the assumption that the measure of $\widehat{P_m'P_{m+2}'}$ is greater than $\frac{2\alpha}{n}$. If the measure of $\widehat{P_m'P_{m+2}'}$ is less than $\frac{2\alpha}{n}$, then the point R for which $\widehat{P_{m+2}'R}$ has measure $\frac{\alpha}{n}$ is on the arc $\widehat{MP_{m+2}'}$, included in $\widehat{P_m'P_{m+2}'}$, so, by Lemma 1, we have $P_m'R + RP_{m+2}' > P_m'P_{m+1}' + P_{m+1}'P_{m+2}'$, and from here we reason as in the case treated above. If the measure of $\widehat{P_m'P_{m+2}'}$ is $\frac{2\alpha}{n}$, then, by appealing to the case proved by van Yzeren of Lemma 1, we get that $P_m'M + MP_{m+2}' > P_m'P_{m+1}' + P_{m+1}'P_{m+2}'$, and from here we reason as in the case treated in detail above. This ends the induction and thus the proof of Lemma 2. □

Now Theorem 1 follows from the existence, proved in [8], of the angle of 120° in $\mathcal{A}$ and Lemma 2, whereas Theorem 2 is a direct consequence of Lemma 2.

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