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Synthetic foundations of cevian geometry, I: fixed points of affine maps

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Abstract. We give synthetic proofs of new results in triangle geometry, focusing especially on fixed points of certain affine maps which are defined in terms of the cevian triangles of a point P and its isotomic conjugate P', with respect to a given triangle ABC. We give a synthetic proof of Grinberg's formula for the cyclocevian map in terms of the isotomic and isogonal maps, and show that the complement Q of the isotomic conjugate P' has many interesting properties. If T_P is the affine map taking ABC to the cevian triangle DEF for P, it is shown that Q is the unique ordinary fixed point of T_P when P does not lie on the sides of triangle ABC, its anticomplementary triangle, or the Steiner circumellipse of ABC. This paper forms the foundation for several more papers to follow, in which the conic on the 5 points A, B, C, P, Q is studied and its center is characterized as a fixed point of the map $\lambda = T_{P'} \circ T_P^{-1}$.

Mathematics Subject Classification. 51A05, 51A20, 51M99, 51N10.

1. Introduction

The cyclocevian mapping for a triangle ABC, which we will denote by ϕ, is the mapping that takes a point P to a point $\phi(P)$ for which the traces of P and $\phi(P)$ on the extended sides of triangle ABC, meaning the intersections

$$\begin{aligned} &AP \cdot BC, \quad A\phi(P) \cdot BC,\\ &BP \cdot AC, \quad B\phi(P) \cdot AC,\\ &CP \cdot AB, \quad C\phi(P) \cdot AB, \end{aligned}$$

lie on a common circle. By early 2004, the second author had independently discovered and proved the fact that

$$\phi = \iota \circ \gamma' \circ \iota \tag{1}$$

where ι is the isotomic mapping for triangle ABC and γ' is the isogonal mapping for the anticomplementary triangle of ABC (see [1,4]). The proof made extensive use of computer-assisted algebra and absolute barycentric coordinates. The coordinates of $\phi(P)$ are 8th degree rational functions in the barycentric coordinates (u, v, w) of P, but when ϕ is conjugated by the isotomic mapping there is a remarkable drop in degree: $\iota \circ \phi \circ \iota$ becomes a 2nd degree rational function in (u, v, w), which turns out to be the same as $\gamma'(P)$. The formulas that occur in this proof can be given a nice form, but are difficult to verify by hand.

Let K denote the complement mapping with respect to triangle ABC: K is the affine mapping which maps P to a point $K(P)$ on line PG (G the centroid), for which the signed length $K(P)G = \frac{1}{2}GP$. During the previous year, unbeknownst to us, Grinberg, in [5], had announced the equivalent formula

$$\phi = \iota \circ K^{-1} \circ \gamma \circ K \circ \iota, \tag{2}$$

(γ is now the isogonal map for ABC itself) which he derived using the concept of the *isotomcomplement* of a point P with respect to ABC (also called the *inferior* of the isotomic conjugate of P, see [15].) This is the point

$$Q = K \circ \iota(P).$$

Grinberg derived his formula with the help of homogeneous barycentric coordinates, and noted that he had found synthetic arguments for all but one of the main facts he had used in his proof of (2), which is Theorem 1 in his message [5], given here as Theorem 2.4 in Sect. 2 below. This theorem is also contained in Paul Yiu's message [15] in slightly disguised form.

The starting point for this paper is to show how this theorem of Grinberg and Yiu can be proved synthetically, thus filling in the synthetic gap in Grinberg's argument for (2). The key step is the Midpoint Perspectivity Theorem (Theorem 2.3). For completeness we give the proof of Grinberg's formula (2) using the cross-ratio in Theorem 2.5.

In the process of finding this proof, we discovered that we could synthetically prove many of the other facts concerning the isotomcomplement that have been noted in the Hyacinthos messages. See [3,5–9]. For example, we prove synthetically Ehrmann's observation [3] that the point $Q = K \circ \iota(P)$ is a fixed point of the affine mapping T_P which maps triangle ABC to the cevian triangle DEF of P with respect to ABC (see Theorem 3.2). We show in addition that Q is the only fixed point of T_P in the finite plane (under suitable hypotheses; see Theorem 3.12) and that $T_P \circ K(P) = P$ (Theorem 3.7).

We also show that if $P' = \iota(P)$ is the isotomic conjugate of P with respect to ABC, then the affine mapping $T_P \circ T_{P'}$ has a fixed point X which is the P-ceva conjugate of Q, defined to be the perspector of the cevian triangle of P and the anticevian triangle of the isotomcomplement Q of P (both with respect to ABC). With this notation the anticevian triangle of Q with respect to ABC, which is the unique triangle for which ABC is the cevian triangle of Q, turns out to be simply $T_{P'}^{-1}(ABC)$. Moreover, the set of points for which $P' = \iota(P)$ is on the line at infinity, which is by definition the Steiner circumellipse, can be characterized in terms of the mappings T_P and $T_{P'}$ as the set of ordinary points P for which $T_P \circ T_{P'} = K^{-1}$ equals the inverse of the complement map for triangle ABC. Along the way, we prove several results that are of independent interest, including the Collinearity Theorem (Theorem 3.5), which gives relationships holding between the points $P, Q, P', Q' = K(P)$, and X defined above and their traces on the sides of ABC.

What results is a completely synthetic treatment of many new results in the theory of cevian triangles. This turns out to be an extended and entertaining exercise in classical projective geometry. Moreover, our development shows that many important points in triangle geometry can be synthetically characterized as fixed points of specific affine maps.

In further papers we will explore this connection more fully. In Parts II and III of this paper [10,11] we will study the conic $\mathcal{C}_P = ABCPQ$ on the five points A, B, C, P, Q; along with the center Z of $\mathcal{C}_P$, which is the pole of the line at infinity with respect to $\mathcal{C}_P$; and the generalized orthocenter H of P, defined to be the intersection of the lines through the vertices A, B, C which are parallel, respectively, to the lines QD, QE, QF. We will give a synthetic proof that Z is the generalized Feuerbach point, the point where the nine point conic $\mathcal{N}_H$ of the quadrangle $ABCH$ (with respect to the line at infinity l_∞) is tangent to the inscribed conic $\mathcal{I}$ of ABC, the conic which is tangent to the sides at the points D, E, F. Moreover, the point Z can be characterized as the center of the map

$$\Phi_P = T_P \circ K^{-1} \circ T_{P'} \circ K^{-1};$$

i.e., the unique ordinary fixed point of the homothety Φ_P, if Z is ordinary, and the direction of the translation Φ_P when Z is on the line at infinity. See [12].

Notation. We use the results and notation of Coxeter's book [2], which gives a synthetic development of projective geometry. See also [4] for many of the elementary geometrical results and concepts that we use, including directed distance; and [13] or [14] or [16] for definitions of terms in triangle geometry.

More specifically, we use the following notation. If P is any point not on the sides of the ordinary triangle ABC and not on the sides of its anticomplementary triangle $K^{-1}(ABC)$, we have the cevian triangles listed in Table 1.

For example, $D_0E_0F_0$, the cevian triangle of the centroid G (i.e. the medial triangle of ABC) is defined by $D_0 = AG \cdot BC, E_0 = BG \cdot AC, F_0 = CG \cdot AB$. Here X is the fixed point (center) of $\mathcal{S} = T_P \circ T_{P'}$; see Theorems 3.5 and 3.8.

Table 1 Cevian triangles

Centroid G	P	$Q = K(\iota(P))$	$P' = \iota(P)$	$Q' = K(P)$	X
$D_0E_0F_0$	$D_1E_1F_1$	$D_2E_2F_2$	$D_3E_3F_3$	$D_4E_4F_4$	$D_5E_5F_5$

Also, we set $A_i = T_P(D_i), B_i = T_P(E_i), C_i = T_P(F_i)$, for $0 \leq i \leq 5$. We will occasionally replace P by the point P', and put primes on the above listed points, to indicate that they correspond to the point P', so that, for example, $D_1' = D_3$, $A_i' = T_{P'}(D_i')$, etc.

2. The Grinberg–Yiu theorem

In this section we explore some basic properties of the isotomcomplement of a point. We always take the vertices of the triangle ABC to be ordinary. We assume throughout this section that P does not lie on the extended sides of triangle ABC or its anticomplementary triangle, so that the vertices of its cevian triangle DEF are always ordinary points. Usually we will assume P is also ordinary, but the isotomic point $\iota(P) = P'$ of P may be infinite, if P lies on $\iota(l_\infty)$, the Steiner circumellipse for ABC. (Cf. Theorem 3.14; l_∞ is the line at infinity). Note, however, that most of our proofs work when P is infinite (in that case P' is ordinary). As in the introduction, K denotes the complement map with respect to ABC and $Q = K(P') = K \circ \iota(P)$ is the isotomcomplement of P.

Further, we let $D_0 = K(A), E_0 = K(B)$, and $F_0 = K(C)$ denote the midpoints of sides $a = BC, b = AC$, and $c = AB$, respectively.

Theorem 2.1 (Theorem 3 in [5]). *Let ABC be a triangle and D, E, F the traces of point P on the sides opposite A, B, and C. Let D_0, E_0, F_0 be the midpoints of the sides opposite A, B, C, and let M_d, M_e, M_f be the midpoints of AD, BE, CF. Then D_0M_d, E_0M_e, F_0M_f meet at the isotomcomplement $Q = K \circ \iota(P)$ of P.*

Proof. (See Fig. 1). We may assume $P \neq$ the centroid G of ABC. Without loss of generality, assume P is not on AG. Let D_3 be the trace of P' on side BC.

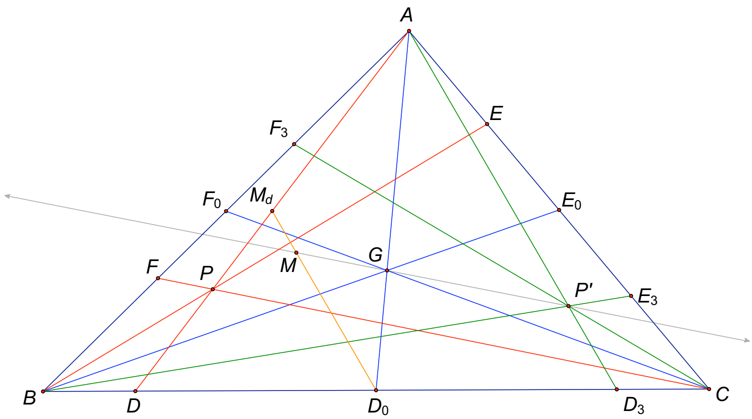

FIGURE 1 Proof of Theorem 2.1

Then D_0M_d is the midline of ΔDAD_3 and is thus parallel to $AD_3 = AP'$. Draw $P'G$, and let M be the intersection $D_0M_d \cdot P'G$. Since D_0M_d is parallel to AP', the triangles GMD_0 and $GP'A$ are similar; so $AG = 2GD_0$ implies $GP' = 2GM$. Thus, $M = K(P') = Q$ and D_0M_d intersects GP' at Q; similarly, so do E_0M_e and F_0M_f. If P lies on BG it cannot lie on CG; in that case, $Q = D_0M_d \cdot F_0M_f$, and P on BG implies that P', Q, and M_e also lie on BG, giving the assertion. This argument applies as long as the point M is ordinary. If $M = P'G \cdot l_\infty$ is infinite, then $P'G \parallel D_0M_d \parallel P'A$ implies, since $P \neq G$, that P' is infinite and $M = P' = K(P') = Q$. Since BP' and CP' are now parallel to AP', the lines D_0M_d, E_0M_e, F_0M_f meet at $P' = Q$. □

Corollary 2.2. *$D_0M_d = D_0Q$ is parallel to AP' and $K(D_3) = M_d$.*

The above theorem is in Altshiller-Court [1] (p. 165, Supp. Ex. 10), except for the identification of the intersection of the lines D_0M_d, E_0M_e, F_0M_f as the isotomcomplement of P.

Theorem 2.3 (Midpoint Perspectivity Theorem). *In triangle ABC, let E and F be points on sides AC and AB. Let:*

E_0, F_0 be the midpoints of $b = AC$ and $c = AB$,
A_b, A_c be the midpoints of AE and AF,
M_e, M_f be the midpoints of BE and CF.

Then the triangles $A_bE_0M_e$ and $A_cF_0M_f$ are perspective.

Proof. (See Fig. 2). We want to show that A_bA_c, E_0F_0, M_eM_f are concurrent at a point O. Using quadrangle $D_0M_eQM_f$, we have: $QM_f \cdot D_0M_e = F_0$ by Theorem 2.1 and the fact that M_e lies on the midline D_0F_0. Similarly, $QM_e \cdot D_0M_f = E_0$ and $D_0Q \cdot E_0F_0 = M_d$. Defining $O = M_eM_f \cdot E_0F_0$, we

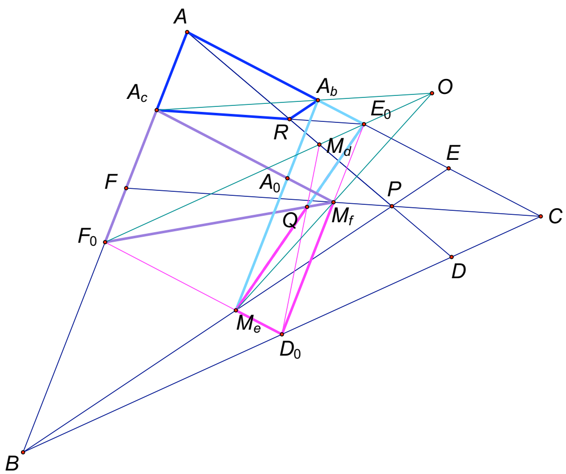

FIGURE 2 Midpoint Perspectivity Theorem

obtain the harmonic relation $H(E_0F_0, M_dO)$, by the definition of a harmonic set. See [2].

Now we use quadrangle AA_bRA_c, where $R = A_bF_0 \cdot A_cE_0$. Since A_cE_0 is the median of AFC, we have $A_cE_0 \cdot AP =$ midpoint of AP and similarly $A_bF_0 \cdot AP =$ midpoint of AP, so A_cE_0, A_bF_0, AP are concurrent at R. This implies that $AA_b \cdot A_cR = E_0$; $AA_c \cdot A_bR = F_0$; and $AR \cdot E_0F_0 = AP \cdot E_0F_0 = M_d$ in quadrangle AA_bRA_c. But this says that $A_bA_c \cdot E_0F_0$ is the harmonic conjugate of M_d with respect to E_0F_0, which is O by the first part of the argument. Therefore, A_bA_c, E_0F_0, M_eM_f are concurrent at O. □

Theorem 2.4 (Grinberg–Yiu [5,15]). *With D, E, F as before, let A_0, B_0, C_0 be the midpoints of EF, DF, and DE, respectively. Then the lines AA_0, BB_0, CC_0 meet at the isotomcomplement Q of P.*

Proof. Using the notation of Theorem 2.1, $A_bM_e \cdot A_cM_f = A_0$ because A_bM_e is a midline of ΔAEB and so passes through the midpoint of EF; similarly, A_cM_f is a midline of ΔAFC and also passes through the midpoint of FE. Thus we have $A_bE_0 \cdot A_cF_0 = A$; $A_bM_e \cdot A_cM_f = A_0$; and $E_0M_e \cdot F_0M_f = Q$, by Theorem 2.1. By Theorem 2.3 and Desargues' theorem, these three points are collinear. Similarly, B, B_0, and Q are collinear, and C, C_0, and Q are collinear. □

Part of the statement of this proposition is in Altshiller-Court [1] (p. 165, Supp. Ex. 8).

We now prove the following theorem of Grinberg using Theorem 2.4. We give a simple proof using the cross-ratio. For Grinberg's proof see [6].

Theorem 2.5 (Theorem 8 in [5]). *Suppose P_1 and P_2 are cyclocevian conjugates with respect to triangle ABC. Then their isotomcomplements Q_1 and Q_2 are isogonal conjugates with respect to ABC. Equivalently, we have*

$$P_2 = \phi(P_1) = \iota \circ K^{-1} \circ \gamma \circ K \circ \iota(P_1),$$

where ι is the isotomic map and γ is the isogonal map.

Proof. Let P_1 and P_2 be cyclocevian conjugates in triangle ABC, D, E, F the traces of P_1, and D', E', F' the traces of P_2, on sides BC, AC, AB, respectively, so that all six traces lie on a circle (see [14] and [16]). Also, on side EF of triangle DEF let V be the trace of the angle bisector of $\angle BAC = \angle FAE$. On the one hand, the cross-ratio of the lines AB, AC, AQ_1, and AV is given by

$$A(BC, Q_1V) = \frac{\sin BAQ_1}{\sin Q_1AC} \div \frac{\sin BAV}{\sin VAC} = \frac{\sin BAQ_1}{\sin Q_1AC}.$$

Next, consider the isogonal conjugate $\gamma(Q_1) = Q_1^\gamma$. Since $\angle BAQ_1^\gamma \cong \angle CAQ_1$ and $\angle CAQ_1^\gamma \cong \angle BAQ_1$, we have

$$A(BC, Q_1^\gamma V) = \frac{\sin BAQ_1^\gamma}{\sin Q_1^\gamma AC} = \frac{\sin Q_1AC}{\sin BAQ_1} = \frac{1}{A(BC, Q_1V)}.$$

On the other hand, by Theorem 2.4 and the fact that A_0 is the midpoint of EF we have

$$A(BC, Q_1V) = (FE, A_0V) = \frac{FA_0}{A_0E} \div \frac{FV}{VE} = \frac{AE}{AF},$$

since $\frac{FV}{VE} = \frac{AF}{AE}$ in triangle AFE. In the same way, we have

$$A(BC, Q_2V) = \frac{AE'}{AF'} = \frac{AF}{AE} = \frac{1}{A(BC, Q_1V)} = A(BC, Q_1^{\gamma}V);$$

the second equality holding because E, E', F, F' lie on a circle, so that the products $AE' \cdot AE = AF' \cdot AF$ are equal. This implies that AQ_2 is precisely the reflection of AQ_1 across the angle bisector, i.e., $AQ_2 = AQ_1^{\gamma}$. Applying the same argument to the vertices B and C, we see that Q_2 is the isogonal conjugate of Q_1. □

3. Cevian triangles and affine maps

In this section we consider the affine transformation T_P which maps the triangle ABC to the cevian triangle DEF of point P, so that $T_P(A) = D$, $T_P(B) = E$, $T_P(C) = F$. We also consider some important points related to the mapping T_P on the sides of DEF. We first give a basic lemma in order to prove geometrically that the fixed point of the affine transformation T_P is Q.

Lemma 3.1. *Let X be on EF such that the signed distances satisfy $\frac{FX}{XE} = \frac{BD}{DC}$. Then $DX \parallel AA_0$.*

Proof. (See Fig. 3). Draw lines through E and F parallel to AA_0, and let them intersect BC at L and M, respectively, and let AA_0 intersect BC at K_1. Draw a line through D parallel to AA_0 and let it intersect EF at Y. We must show that $X = Y$. The parallel lines give us the equalities

$$\frac{CE}{EA} = \frac{CL}{LK_1}, \qquad \frac{AF}{FB} = \frac{K_1M}{MB}, \qquad \frac{FY}{YE} = \frac{MD}{DL}.$$

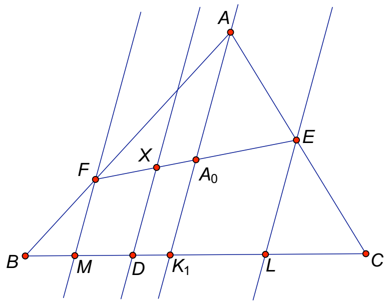

FIGURE 3 Proof of Lemma 3.1

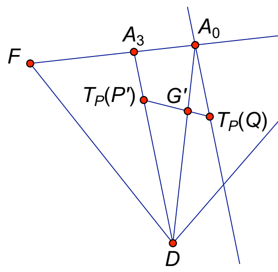

FIGURE 4 Proof of Theorem 3.2

By Ceva's theorem, $1 = \dfrac{AF}{FB}\dfrac{BD}{DC}\dfrac{CE}{EA} = \dfrac{K_1M}{MB}\dfrac{BD}{DC}\dfrac{CL}{LK_1}$.

Since A_0 is the midpoint of EF, K_1 is the midpoint of LM, so $LK_1 = K_1M$ implies that

$$1 = \frac{CL}{MB}\frac{BD}{DC}, \text{ so } \frac{BM}{LC} = \frac{MB}{CL} = \frac{BD}{DC} = \frac{BM + MD}{DL + LC} = \frac{BM + MD}{LC + DL}.$$

This last equality implies that

$$\frac{BM}{LC} = \frac{MD}{DL} = \frac{FY}{YE}, \text{ i.e. } \frac{FX}{XE} = \frac{BD}{DC} = \frac{BM}{LC} = \frac{FY}{YE}.$$

But there is exactly one point X on EF such that the signed ratio $\dfrac{FX}{XE}$ equals $\dfrac{BD}{DC}$, so $X = Y$. □

Theorem 3.2 (Ehrmann). *If T_P is the unique affine mapping which takes ABC to DEF, then $T_P(Q) = Q$.* (This holds even when the point P lies on l_∞).

Proof. (See Fig. 4). We show that AA_0 passes through $T_P(Q)$. It will follow similarly that BB_0 and CC_0 also pass through $T_P(Q)$. This implies the result because these lines intersect at Q.

First, if K is the complement map with respect to triangle ABC and K' the complement map with respect to triangle DEF, then $T_P \circ K = K' \circ T_P$. This is because T_P preserves ratios; so if $Y_1 = K(Y)$, then Y_1 is collinear with G and Y, and $YG = 2 \cdot GY_1$ implies $T_P(Y)T_P(G) = 2 \cdot T_P(G)T_P(Y_1)$; hence $T_P(Y_1) = K'(T_P(Y))$, since $G' = T_P(G)$ is the centroid of DEF.

Now, $T_P(Q) = T_P(K(P')) = K'(T_P(P'))$, so we really need to prove that AA_0 passes through the complement, in triangle DEF, of $T_P(P')$. Since P' lies on AD_3, $T_P(P')$ lies on $T_P(A)T_P(D_3) = DA_3$ and

$$\frac{BD}{DC} = \frac{D_3C}{BD_3} = \frac{A_3F}{EA_3} = \frac{FA_3}{A_3E}.$$

Lemma 3.1 now gives that $DA_3 \parallel AA_0$. If $P' = Q$ is an infinite point, then this implies that $T_P(P') = Q$, and so $T_P(Q) = K'(Q) = Q$ is fixed. If P' and Q are ordinary, then letting $G' = T_P(G)$ and $I = T_P(P')G' \cdot AA_0$ we have that $\Delta T_P(P')DG' \sim \Delta IA_0G'$. Since G' is the centroid of triangle DEF, we have $DG' = 2 \cdot G'A_0$, so also $T_P(P')G' = 2 \cdot G'I$, which means that $I = T_P(Q)$ is the complement of $T_P(P')$ and AA_0 lies on the point $T_P(Q)$. □

Corollary 3.3. *The point Q is the complement of $T_P(P')$ with respect to the triangle DEF.*

Lemma 3.4. *Let G be the centroid of ABC; E and F points on AC and AB, respectively, distinct from A, B and C; and $AG \cdot EF = A^*$. Then*

$$\frac{EA^*}{A^*F} = \frac{AE}{AF} \cdot \frac{AB}{AC}.$$

Proof. Let V be the trace on segment EF of the angle bisector of $\angle BAC = \angle FAE$, and let V' be its trace on BC. If (EF, A^*V) denotes the cross-ratio of these four points, we have

$$\frac{EA^*}{A^*F} \div \frac{AE}{AF} = (EF, A^*V) \stackrel{A}{=} (CB, D_0V') = \frac{BV'}{V'C} = \frac{AB}{AC}.$$

□

In order to prove the next theorem we will make use of the following involution on the line BC. (There are similar involutions for AB and AC). Let μ be the perspectivity taking a point on EF to a point on BC by projection from A. We define $\pi = \mu \circ T_P$. Since T_P maps BC to EF, π maps BC to itself. Thus, if Y is a point on BC,

$$\pi(Y) = \mu(T_P(Y)) = AT_P(Y) \cdot BC.$$

Since $\pi(B) = C$ and $\pi(C) = B$, π interchanges two points and is therefore an involution on BC. The significance of this mapping is that if a point Y on BC maps to $T_P(Y)$ on EF, then T_P maps the intersection $AT_P(Y) \cdot BC = Y'$ back to $AY \cdot EF = T_P(Y')$.

Now recall the definition of the points

$$D_0 = AG \cdot BC, \qquad D_1 = D = AP \cdot BC, \qquad D_2 = AQ \cdot BC,$$
$$D_3 = AP' \cdot BC, \qquad D_4 = AQ' \cdot BC, \qquad D_5 = AX \cdot BC.$$

Here G is the centroid of ΔABC, Q is the isotomcomplement of P, and Q' is the isotomcomplement of P'. Also, X is defined to be the intersection of the cevians AA_3, BB_3, and CC_3, where

$$A_3 = T_P(D_3), B_3 = T_P(E_3), C_3 = T_P(F_3).$$

This intersection exists because $A_3B_3C_3 = T_P(D_3E_3F_3)$ is the cevian triangle for $T_P(P')$ with respect to triangle $DEF = T_P(ABC)$, and is therefore perspective to ABC, by the cevian nest theorem [1] (p. 165, Supp. Ex. 7). Furthermore, $A_j = T_P(D_j)$ for $0 \leq j \leq 5$.

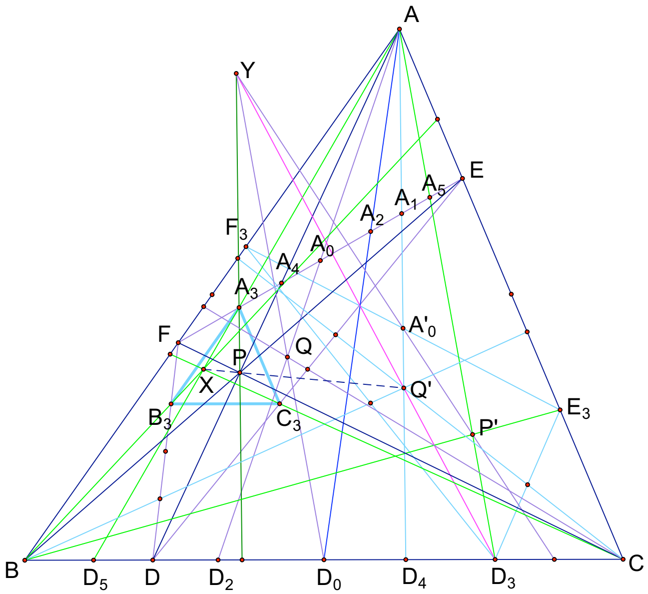

FIGURE 5 The points A_i and D_i, $i = 0, 1, 2, 3, 4, 5$

Theorem 3.5 (Collinearity Theorem). *The following sets of 4 points are collinear:* AA_0QD_2, $AA_1Q'D_4$, AA_2GD_0, AA_3XD_5, AA_4PD_1, $AA_5P'D_3$. (Similar statements hold for the other vertices B and C. This also holds when P or P' is infinite.)

Proof. (See Fig. 5). The collinearity of the four points AA_3XD_5 is immediate from the definition of the point X. Hence $\pi(D_3) = D_5$. Now $\pi(D_5) = D_3$ implies that $AP' \cdot EF = T_P(D_5) = A_5$ and so $AA_5P'D_3$ is a collinear set.

The collinearity AA_0QD_2 is immediate from Theorem 2.4, where we note that $T_P(D_0) = A_0$ since T_P preserves ratios along lines. Hence $\pi(D_0) = D_2$, which implies that $\pi(D_2) = D_0$, so AA_2GD_0 is also a collinear set, where $A_2 = T_P(D_2)$.

It remains to prove that the sets $AA_1Q'D_4$ and AA_4PD_1 are collinear. To do this we first redefine A_1 as the intersection $A_1 = AQ' \cdot EF$ and we show that $A_1 = T_P(D_1)$. This will imply the collinearity of the points AA_4PD_1 with $A_4 = T_P(D_4)$ using the map π. Using the cross-ratio and the fact that A_0 is the midpoint of segment EF we have that

$$\frac{EA_1}{A_1F} = \frac{A_1E}{FA_1} = (FE, A_0A_1) \stackrel{A}{=} (BC, D_2D_4) = \frac{BD_2}{D_2C} \div \frac{BD_4}{D_4C}.$$

Using Lemma 3.4 with $A^* = A_2$ on EF and $A^* = A_2'$ on E_3F_3, where we denote the analogues of the points D_i, A_i corresponding to P' by D_i', A_i' (so that $D_2' = D_4$ and $T_{P'}(B) = E_3$, etc.), we have

$$\frac{BD_2}{D_2C} = \frac{EA_2}{A_2F} = \frac{AE}{AF} \cdot \frac{AB}{AC} \text{ and } \frac{BD_4}{D_4C} = \frac{BD_2'}{D_2'C} = \frac{E_3A_2'}{A_2'F_3} = \frac{AE_3}{AF_3} \cdot \frac{AB}{AC}.$$

Hence, the above ratio EA_1/A_1F becomes

$$\frac{EA_1}{A_1F} = \frac{AE}{AF}\frac{AF_3}{AE_3} = \frac{AE}{AF}\frac{FB}{EC} = \frac{EA}{AF}\frac{FB}{CE} = \frac{BD}{DC},$$

by Ceva's theorem and the fact that (F, F_3) and (E, E_3) are isotomic pairs of conjugates on AB and AC, respectively. This implies that $A_1 = T_P(D) = T_P(D_1)$ and completes the proof of the theorem. □

The results of Theorem 3.5 can be phrased in the following way. There is a mapping δ_P, defined on arbitrary points X', for which

$$Y = \delta_P(X') = AT_P(AX' \cdot BC) \cdot BT_P(BX' \cdot AC),$$

and $\delta_P(Y) = X'$.

Corollary 3.6. *The mapping δ_P satisfies*

$$\delta_P(G) = Q, \quad \delta_P(P) = Q', \quad \delta_P(P') = X.$$

We will consider this mapping again in Part IV. The following fact is a simple corollary of Theorem 3.5, but is important enough in the following development to state as a theorem.

Theorem 3.7. $T_PK(P) = T_P(Q') = P$.

Proof. $T_P(Q') = T_P(AD_4 \cdot BE_4) = DA_4 \cdot EB_4 = DA \cdot EB = P$. □

Theorem 3.8 (Homothety theorem). *The affine mapping $T_PT_{P'}$ taking ABC to $A_3B_3C_3$ is either a homothety, whose center is the ordinary point $X = AA_3 \cdot BB_3 = AA_3 \cdot CC_3$ lying on the line PQ', or a translation in the direction of the line PQ'. Thus, triangles ABC and $A_3B_3C_3$ are either homothetic or congruent.*

Proof. Write T_1 for T_P and T_2 for $T_{P'}$, and let l_∞ be the line at infinity, as usual. We first show that T_1 and T_2 are inverse mappings on l_∞. Assume that the points P' and Q are ordinary. Note that T_2 maps the line AQ to $D_3P' = AP'$, since $T_2(Q) = P'$ (by Theorem 3.7). Hence, T_2 maps the point at infinity A_∞ on AQ to the point at infinity D_∞ on AP'. On the other hand, AP' is parallel to D_0Q (Corollary 2.2), so D_∞ lies on D_0Q. Moreover, $T_1(D_0Q) = A_0Q = AQ$ by Theorem 3.2 and Theorem 2.4. Therefore, $T_1(D_\infty) = A_\infty$. Arguing in the same way with $B_\infty = BQ \cdot l_\infty, C_\infty = CQ \cdot l_\infty$ and $E_\infty = BP' \cdot l_\infty, F_\infty = CP' \cdot l_\infty$, we see that on the line l_∞

T_2 induces the projectivity $A_\infty B_\infty C_\infty \barwedge D_\infty E_\infty F_\infty$, while
T_1 induces the projectivity $D_\infty E_\infty F_\infty \barwedge A_\infty B_\infty C_\infty$.

The fundamental theorem of projective geometry now implies that T_1T_2 is the identity map on l_∞. (Note that $A_\infty, B_\infty, C_\infty$ are distinct points since they lie on the concurrent lines AQ, BQ, CQ. If Q were on AB, say, then P' would lie on the anticomplementary triangle of ABC, so the trace $F_3 = CP' \cdot AB$

of P' would lie on l_∞, implying that $F = F_3$, and P would also lie on the anticomplementary triangle, contrary to the standing hypothesis about P). On the other hand, if $P' = Q$ is an infinite point, then P and Q' are ordinary, and we can apply the above argument to the map T_2T_1. Since this map is the identity on l_∞, so is T_1T_2.

Now it is clear that $T_1T_2(ABC) = T_1(D_3E_3F_3) = A_3B_3C_3$. By the above argument, the mapping $\mathcal{S} = T_1T_2$ fixes the point $AA_3 \cdot l_\infty$, so $\mathcal{S}(A) = A_3$ implies that AA_3 is an invariant line, as are BB_3 and CC_3 (and any line of the form $Y\mathcal{S}(Y)$). Hence $X = AA_3 \cdot BB_3$ is an invariant point of $\mathcal{S}$. If X is an ordinary point, then $\mathcal{S}$ is a projective homology [2], which must be a homothety since it takes any line to a parallel line. It follows that ABC and $A_3B_3C_3$ are homothetic from the center X. Since $\mathcal{S}(Q') = T_1T_2(Q') = T_1(Q') = P$, X lies on the line PQ'.

If X is a point at infinity, then AA_3, BB_3, CC_3 are parallel, so $\mathcal{S}$ must be a translation. Since $\mathcal{S}(Q') = P$, the translation is in the direction of the line PQ'. □

In order to further describe the point X we prove the following theorem of Grinberg.

Theorem 3.9 (Theorem 4 in [5]). *If parallel lines are drawn to the sides EF, DF, DE of the cevian triangle for P through the respective vertices A, B, and C, then the resulting triangle is the anticevian triangle of Q for triangle ABC.*

Proof. Consider the polarity corresponding to the conic $\mathcal{I}$ inscribed in ABC and lying on the points D, E, F. This is the unique conic which is tangent to the sides of ABC at D, E, and F, respectively. Here is a short argument to indicate why this conic exists. There is certainly a conic $\mathcal{C}$ through E and F tangent to AB at F, to AC at E, and tangent to BC at some point D'. (This is the dual of [2], 8.41, p. 78). We now appeal to the dual of Chasles' Theorem ([2], 7.31, p. 64): *If the poles of the sides of a triangle do not coincide with their respective opposite vertices, then their joins with the opposite vertices are concurrent.* Applying this to the triangle ABC, we get that the joins of A, B, C with the respective poles of BC, AC, AB are concurrent, i.e. AD', BE, CF are concurrent. Since $BE \cdot CF = P$, it follows that $AD' = AP$, so $D' = AP \cdot BC = D$. Therefore $d = BC$ is the tangent to $\mathcal{C}$ at D. Hence, the conic $\mathcal{I} = \mathcal{C}$ exists.

Thus, the lines $d = BC, e = AC, f = AB$ are the polars of the points D, E, F, while the polars of A, B, C are the lines $a = EF, b = DF, c = DE$. The lines through A, B, C parallel to these lines are the polars a_0, b_0, c_0 of the midpoints A_0, B_0, C_0 of the sides of DEF. This is because the polar of A_0 is the line through A and the harmonic conjugate of A_0 with respect to E, F, which is the point at infinity on EF. It follows from this that the pole of the line at infinity lies on AA_0, so Theorem 2.4 implies that this pole is Q. Let the vertices of the triangle formed by a_0, b_0, c_0 be $A' = b_0 \cdot c_0, B' = a_0 \cdot c_0, C' = a_0 \cdot b_0$. We must show that $A'Q$ lies on A, $B'Q$ lies on B, $C'Q$ lies on C. Using the polarity we see that $A'Q$ lies on A if and only if $B_0C_0 \cdot q = B_0C_0 \cdot l_\infty$ lies on EF. But this is obvious because B_0 and C_0 are midpoints in triangle DEF, so that B_0C_0 is parallel to EF. □

We can now prove

Theorem 3.10. *The fixed point $X = AA_3 \cdot BB_3$ of $T_PT_{P'}$ is the P-ceva conjugate of Q. The cevian triangle of P is homothetic to the anticevian triangle of Q for triangle ABC from the center X if X is an ordinary point, and is congruent to this triangle otherwise.*

Proof. Let $\mathcal{S} = T_PT_{P'}$. Consider the triangle $A'B'C' = \mathcal{S}^{-1}(DEF)$. Then the sides of $A'B'C'$ are parallel to the sides of DEF, since $\mathcal{S}$ fixes points at infinity. Furthermore, triangle $A_3B_3C_3$ is inscribed in triangle DEF, so $\mathcal{S}^{-1}(A_3B_3C_3) = ABC$ is inscribed in triangle $A'B'C'$. By Theorem 3.9, $A'B'C'$ must be the anticevian triangle of Q. The P-ceva conjugate of Q is by definition the perspector of DEF and $A'B'C'$, and by construction this point is the center $X = AA_3 \cdot BB_3$ of Theorem 3.8, whether $\mathcal{S}$ is a homothety or a translation. This proves the assertion. □

Corollary 3.11. (a) *The triangle $T_{P'}^{-1}(ABC)$ is the anticevian triangle of Q for ABC.*

(b) *The point Q' is the G-ceva conjugate of Q, so that the cevian triangle of G, namely $D_0E_0F_0$, is perspective to the anticevian triangle of Q from the center Q'.*

(c) *If X' is the X-point corresponding to P', then $T_P(X') = X$ and $T_{P'}(X) = X'$.*

(d) *X is an ordinary point if and only if X' is.*

Proof. (a) The anticevian triangle of Q is $A'B'C' = \mathcal{S}^{-1}(DEF) = T_{P'}^{-1}(ABC)$. (b) Applying the map $T_{P'}^{-1}$ to the collinear points A, A_0', Q' (Theorem 2.4) shows that $T_{P'}^{-1}(A), D_0$, and Q' are collinear. Similar statements for the other vertices and part (a) imply the assertion. (c) The perspector of $D_3E_3F_3$, the cevian triangle of P', and $T_P^{-1}(ABC)$, the anticevian triangle of Q', is X'. It follows that $T_P(X')$ is the perspector of triangles $A_3B_3C_3$ and ABC, hence $T_P(X') = X$. The second assertion in (c) follows on switching P and P'. Part (d) is immediate from (c). □

Theorem 3.12. *If P does not lie on $\iota(l_\infty)$* (the Steiner circumellipse for ABC), *then Q is the only fixed point of T_P in the finite plane.*

Remark. If the point P does lie on the Steiner circumellipse for ABC, then it can be shown that T_P has no ordinary fixed points, but does have the line $GT_P(G)$ as a fixed line, where G is the centroid of ABC. See the proofs of Lemma 2.5 and Theorems 2.4 and 4.3 in [10].

Proof. Note, since P does not lie on $\iota(l_\infty)$, that the points P' and Q are ordinary points. We already know from Theorem 3.2 that T_P fixes Q. Suppose there is another finite fixed point R of T_P. Then $m = QR$ is an invariant line for

T_P. The line at infinity, l_∞, is also an invariant line since T_P is an affine transformation. Therefore, T_P fixes the point $M_\infty = m \cdot l_\infty$. Since T_P fixes three points on m, it fixes every point on m.

Suppose $m \cdot BC = S$. Then $S = T_P(S) = T_P(m \cdot BC) = m \cdot EF$, which implies $S = BC \cdot EF$. Similarly, $m \cdot AB = AB \cdot DE$ and $m \cdot AC = AC \cdot DF$, so m is the line of perspectivity of triangles DEF and ABC. Hence, m is the trilinear polar of the point P, and S is the harmonic conjugate of D with respect to B and C. Projecting line BC to line FE from A gives $(BC, DS) = -1 = (FE, A_4S) = (EF, A_4S)$ and since $S = T_P(S)$, the signed ratio of S along BC is the same as its ratio along $EF = T_P(BC)$:

$$\frac{BD}{DC} = -\frac{BS}{SC} = -\frac{ES}{SF} = \frac{EA_4}{A_4F}.$$

But the only point A^* on EF such that $\frac{BD}{DC} = \frac{EA^*}{A^*F}$ is $A_1 = T_P(D_1)$, so $A_1 = A_4$. By Theorem 3.5, $AA_1 = AQ'$ and $AA_4 = AP$, so we have $AQ' = AP$. Since $K(P) = Q'$, this implies that the centroid G lies on AP, so P is on AG. Similarly, P is on BG and CG, so $P = G$. But then $T_P = T_G = K$ and the line of perspectivity $m = QR$ is the line at infinity, yet $Q = G$ is not on $m = l_\infty$: a contradiction. □

Theorem 3.13. *If P is ordinary, the point $Q' = K(P)$ is the isotomcomplement of Q with respect to the anticevian triangle of Q for ABC.*

Proof. The unique affine mapping taking the vertices of the anticevian triangle $A'B'C' = T_{P'}^{-1}(ABC)$ of Q to the vertices of ABC is $T_{P'}$. Since the point P is ordinary, the point P' does not lie on the Steiner circumellipse for ABC, and therefore the point $Q = T_{P'}^{-1}(P')$ does not lie on the Steiner circumellipse for $A'B'C'$. (If ι' is the isotomic map for $A'B'C'$, then $T_{P'}^{-1} \circ \iota = \iota' \circ T_{P'}^{-1}$, and affine maps fix the line l_∞, so the Steiner circumellipse for ABC is mapped to the Steiner circumellipse for $A'B'C'$). It follows that the isotomcomplement of Q with respect to $A'B'C'$ is the unique ordinary fixed point of the mapping $T_{P'}$, by Theorem 3.12, so this point must be Q'. □

Next, we characterize the points P on the Steiner circumellipse in terms of the mapping T_P.

Theorem 3.14. *The point P ($\neq A, B,$ or C) lies on the Steiner circumellipse $\iota(l_\infty)$ of ABC if and only if $T_PT_{P'} = K^{-1}$.*

Proof. Assume P lies on $\iota(l_\infty)$, so that the point $P' = Q$ is an infinite point. Let A', B', C' be the midpoints of segments AD_3, BE_3, CF_3. We claim that $A'B'C'$ is the anticevian triangle of Q with respect to ABC. Note that $A' = M_d', B' = M_e', C' = M_f'$ in the notation of Theorem 2.1. By the corollary to that theorem, $K(DEF) = A'B'C'$. Thus, the sides of $A'B'C'$ are parallel to the sides of DEF. By Theorem 3.9 we just have to show that ABC is inscribed in $A'B'C'$. We note that the complete quadrangle $ABCP'$ has the diagonal triangle $D_3E_3F_3$. By the Collinearity Theorem (3.5) we know that $AA_4'P'D_3$

is a collinear set of points, where A_4' lies on E_3F_3. Thus, by the definition of a harmonic set, A and P' are harmonic conjugates with respect to A_4' and D_3. This implies that A is the midpoint of D_3A_4'. Now B' and C' are the midpoints of BE_3 and CF_3. Since B, C, and D_3 are collinear, as are E_3, F_3, and A_4', and furthermore the lines BE_3, CF_3, and $D_3A_4' = AP'$ are parallel, it is clear that the respective midpoints B', C', and A are collinear as well. Arguing the same with the other vertices shows that ABC is inscribed in $A'B'C'$. Hence, $A'B'C' = K(DEF) = KT_P(ABC)$ is the anticevian triangle of Q. From Corollary 3.11 we deduce that $KT_P(ABC) = T_{P'}^{-1}(ABC)$, whence the desired equation $T_PT_{P'} = K^{-1}$ follows.

Conversely, suppose that $T_PT_{P'} = K^{-1}$. Then $T_P(D_3E_3F_3) = A_3B_3C_3 = K^{-1}(ABC)$ is the anticomplementary triangle of ABC, from which it is clear that AA_3, BB_3, and CC_3 all pass through the centroid G of ABC. Theorem 3.5 implies that $A_3B_3C_3 = A_2B_2C_2$, hence $D_3E_3F_3 = D_2E_2F_2$, which gives that $P' = Q$. Hence, the point P' coincides with its complement and must be infinite (since P cannot be G), i.e., P lies on $\iota(l_\infty)$. □

Corollary 3.15. *If P lies on the Steiner circumellipse $\iota(l_\infty)$, then the triangle $A_2B_2C_2 = A_3B_3C_3$ is the anticomplementary triangle of ABC; the anticevian triangle of Q with respect to ABC is the triangle $K(DEF)$; and the anticevian triangle of Q' is the triangle $A_0'B_0'C_0'$.*

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