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The principle of duality in Euclidean and in absolute geometry

Rolf Struve

Abstract. In Euclidean geometry and in absolute geometry fragments of the principle of duality hold. Bachmann (Aufbau der Geometrie aus dem Spiegelungsbegriff, 1973, §3.9) posed the problem to find a general theorem which describes the extent of an allowed dualization. It is the aim of this paper to solve this problem. To this end a first-order axiomatization of Euclidean (resp. absolute) geometry is provided which allows the application of Gödel's Completeness Theorem for first-order logic and the solution of Bachmann's problem.

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1. Introduction

The principle of duality is a guiding principle in geometry which allows the deduction of new theorems from old ones and the discovery of relationships between certain ideas and concepts. It was established in projective geometry by Poncelet and Gergonne (see Gray [4]), but it also holds in metric geometries, such as in elliptic planes (see Bachmann [1]) and in Galilean planes (see Yaglom [11]).

In Euclidean geometry and in absolute geometry fragments of the principle of duality hold (see Schmidt [7]). Bachmann [1, §3.9] posed the problem to find a general theorem which describes the extent of an allowed dualization. It is the aim of this paper to solve this problem for the Euclidean case and for plane absolute geometry (in the sense of [1]).

The principle of duality is a theorem of the meta-theory of a geometric theory. For projective planes, for example, it can be formulated in the following way (see Hughes and Piper [5]): *If* Ω *is a theorem about projective planes and if* Ω^*

is the statement obtained by interchanging the words 'points' and 'lines', then Ω^* *is also a theorem about projective planes.* Expressions of a meta-theory, like 'theorem about projective planes', refer to the language of the underlying theory which can be specified—for elementary theories—by a first-order axiomatization (cp. Schwabhäuser et al. [8]; Pambuccian [6]).

In Sect. 2 we introduce the group-theoretical approach of Bachmann [1] to plane absolute geometry and the associated group-theoretical language. In Sects. 2.1 and 2.2 we present an axiom system $\mathcal{A}$ which contains with each axiom the dual one and show that the models of $\mathcal{A}$ are exactly the models of Bachmannn's plane absolute geometry and their dual models.

Section 2.3 provides a first-order version of $\mathcal{A}$. This step is essential not only for a precise definition of the principle of duality but also for a solution of Bachmann's problem, since Gödel's Completeness Theorem for first-order logic is the key to the following theorem of Sect. 2.4:

If Ω *is a statement of plane absolute geometry and* Ω^* *the dual one* (*obtained by interchanging the words 'point' and 'line'*) *then the following holds:*

- *If* Ω *can be deduced from axiom system* $\mathcal{A}$ *then* Ω and Ω^* *hold in plane absolute geometry.*
- *If* Ω *and* Ω^* *hold in plane absolute geometry then* Ω *can be deduced from* $\mathcal{A}$.

In Sect. 3 the Euclidean case is considered. We proceed in an analogous way and introduce an axiom system $\mathcal{E}$ which contains with each axiom the dual one, show that the models of $\mathcal{E}$ are exactly the Euclidean planes over fields of characteristic $\neq 2$ and their dual models (which are called co-Euclidean planes; see H. Struve and R. Struve [9]), provide a first-order axiomatization of $\mathcal{E}$ and prove the following completeness theorem:

Let Ω *and* Ω^* *be dual statements of plane Euclidean geometry.*

- *If* Ω *can be deduced from axiom system* $\mathcal{E}$ *then* Ω *and* Ω^* *hold in plane Euclidean geometry.*
- *If* Ω *and* Ω^* *hold in plane Euclidean geometry then* Ω *can be deduced from* $\mathcal{E}$.

The development of an 'Euclidean theory of duality' (i.e., of the theory defined by axiom system $\mathcal{E}$) is left to a forthcoming paper.

2. The principle of duality in plane absolute geometry

2.1. The axiom system $\mathcal{A}$

Following the group-theoretical approach of Bachmann [1, §3.2 and §20.2] we start with the following Basic Assumption.

Basic Assumption. Let G be a group and S and P invariant subsets of involutions of G such that $S \cup P$ generates G.

Elements $a, b, c, \dots$ of S are called *lines* and elements $A, B, C, \dots$ of P *points.* If $A = b$ then A, b are *polar* to each other and the point A is called the *pole* of b and the line b the *polar* of A.

The 'stroke relation' $\alpha \,|\, \beta$ is an abbreviation for the statement that α, β and $\alpha\beta$ are involutory elements (i.e., group elements of order 2). Hence $\alpha\beta = \beta\alpha$ so the relation is symmetric. The statement $\alpha, \beta \mid \delta$ is an abbreviation of $\alpha \,|\, \delta$ and $\beta \,|\, \delta$.

A point A and a line b are *incident* if $A \,|\, b$. Lines $a, b \in S$ are *orthogonal* if $a \,|\, b$. Points $A, B \in P$ are *polar* if $A \mid B$. Two lines are called *parallel* if they have no point of intersection. Dually, two points are called *parallel* if they have no joining line.

A pair (A, b) is a *flag* if A, b are incident. Flags (A, a) and (B, b) are called *parallel* if $Aa = Bb$.

A *quadrangle* is a set of four points A, B, C, D and four lines a, b, c, d with $a \mid A, B$ and $b \mid B, C$ and $c \mid C, D$ and $d \mid D, A$.

Let $\mathcal{A}$ be the axiom system which consists of the Basic Assumption and the following axioms **A1**–**A8**.

- **A1.** If $a \,|\, b$ then $ab \in P$ and if $A \,|\, B$ then $AB \in S$.
- **A2.** If $A \,|\, b$ then $Ab \in P \cup S$.
- **A3.** For every pair (A, b) there exists (a, B) with $a \,|\, A$ and $B \,|\, b$ and $Aa = bB$ and if $A \neq b$ then (a, B) is unique.
- **A4.** If $A, B \,|\, c, d$ then $A = B$ or $c = d$.
- **A5.** If $A, B, C \,|\, d$ then $ABC \in P$ and if $a, b, c \,|\, D$ then $abc \in S$.
- **A6.** If $A \,|\, a$ and $B \,|\, b$ and $C \,|\, c$ and $Aa = Bb = Cc$ then $ABC \in P$ and $abc \in S$.
- **A7.** If $A \,|\, a$ and $B \,|\, b$ then there exists e with $e \,|\, A, B$ or E with $E \,|\, a, b$.
- **A8.** There exists a quadrangle.

The axioms make the following statements: According to **A1** orthogonal lines a, b intersect in the point ab and polar points A, B are incident with the line AB. Axiom **A2** states that if (A, b) is a flag then (A, Ab) or (Ab, b) is also a flag. **A3** states that if A is a point and b a line, then there exists a line a through A and a point B on b with $Aa = bB$ (a 'perpendicular' from A to b with foot B) and if $A \neq b$ then (a, B) is unique. According to **A4** two distinct points have at most one joining line and two distinct lines have at most one common point. **A5** states that if A, B, C are collinear points then ABC is a point (the *fourth reflection point*) and that if a, b, c are copunctual lines then abc is a line (the *fourth reflection line*). **A6** states that parallel flags (A, a), (B, b) and (C, c) have a fourth reflection point ABC and a fourth reflection line abc. **A7** states that if (A, a) and (B, b) are flags then A and B have a joining line or a and b have a point of intersection. According to **A8** there exists a quadrangle.

If H and K are subsets of G then we denote the set of involutions of H by $I(H)$ and define $H \cdot K := \{\alpha\beta : \alpha \in H \text{ and } \beta \in K\}$. According to **A1** and **A2** the inclusions $I(S^2) \subseteq P$ and $I(P^2) \subseteq S$ and $I(PS) \subseteq P \cup S$ hold.

Axiom system $\mathcal{A}$ contains with each axiom the dual statement (all axioms are self-dual). Hence the principle of duality holds. Let (G, S, P) be a model of $\mathcal{A}$. By interchanging points and lines we get the *dual model* (G, S', P') with $S' = P$ and $P' = S$ which again satisfies $\mathcal{A}$.

A triplet (G, S, P) which satisfies the Basic Assumption and the axioms **A1**, **A3**, **A4**, **A5**, **A6**, **A8** is a *Cayley–Klein group* (see R. Struve [10]).

The groups of plane absolute geometry (which are called *Bachmann groups*) are the Cayley–Klein groups with the property that any two points have a joining line (see [10, Theorem 4.1]). Hence plane absolute geometry can be axiomatized by an axiom system which contains the Basic Assumption, **A1**, **A3**, **A4**, **A5**, **A6**, **A8** and the axiom of the existence of a joining line (*For* A, B *there exists* c *with* $A, B \mid c$). We denote this axiom system by $\mathcal{B}$.

2.2. The models of axiom system $\mathcal{A}$

Theorem 2.1. *Let (G, S, P) be a model of $\mathcal{A}$. Then any two points have a joining line or any two lines have a point of intersection.*

Proof. Let (G, S, P) be a model of $\mathcal{A}$ and (A, b) a flag. Then $Ab \in S \cup P$ (according to **A2**). We show that if $Ab \in S$ then any two points have a joining line. This proves the theorem since the principle of duality implies that if $Ab \in P$ then any two lines have a point of intersection.

Let $A \mid b$ and $Ab \in S$. Then b and Ab are orthogonal lines of the Cayley–Klein group (G, S, P). The existence of orthogonal lines implies $P = I(S^2)$ and $I(P \cdot S) = S$ (see [10, Theorem 3.5 and Theorem 3.9]).

We can assume $P \cap S = \varnothing$ since otherwise (G, S, P) is an elliptic Cayley–Klein group and any two points have a joining line and any two lines have a point of intersection (see [10, Theorem 4.3 and Theorem 4.5]). Under this assumption **A2** states the existence and uniqueness of a perpendicular.

Now let D and E be two distinct points and d and e lines with $d \mid D$ and $e \mid E, Dd$ (the line e is the perpendicular from E to the line Dd). Then (D, d) and (E, e) are flags with lines d and e which have a common perpendicular Dd. The uniqueness of a perpendicular implies that d and e have no common point. By **A7** there exists a joining line of D and E. □

According to this theorem a model of $\mathcal{A}$ is a Cayley–Klein group with the property that any two points have a joining line or any two lines have a point of intersection. These Cayley–Klein groups are exactly the groups of plane absolute geometry (Bachmann groups) and their dual groups (see [10, Theorem 4.1 and Theorem 5.1]).

Theorem 2.2. *The models of $\mathcal{A}$ are exactly the groups of plane absolute geometry and the associated dual groups.*

2.3. A first-order version of axiom system $\mathcal{A}$

In this section we provide a first-order version of axiom system $\mathcal{A}$ (i.e., a first-order axiomatization of the theory defined by $\mathcal{A}$). The axiom system, which we denote by $\mathcal{D}$, can be expressed with one sort of individual variables (elements $\alpha, \beta, \gamma, \ldots$ of a set G), two unary predicates, which correspond to subsets P and S of G, and a binary operation $\cdot$ on G. The elements of S are to be interpreted as 'lines' and are denoted by lowercase Latin variables $a, b, c, \ldots$. The elements of P are to be interpreted as 'points' and are denoted by uppercase variables $A, B, C, \ldots$.

To improve the readability of the axioms, we introduce the following abbreviations:

$\varepsilon(\alpha) \;\Leftrightarrow\; \alpha \cdot \alpha = \alpha$ (to be interpreted as α is an *idempotent element*)
$\iota(\alpha) \;\Leftrightarrow\; \varepsilon(\alpha \cdot \alpha) \wedge \neg\, \varepsilon(\alpha)$ (to be interpreted as α is an *involution* of $(G, \cdot)$)
$\alpha | \beta \;\Leftrightarrow\; \iota(\alpha) \wedge \iota(\beta) \wedge \iota(\alpha \cdot \beta)$ (we write $\alpha, \beta \mid \gamma$ if $\alpha \mid \gamma \wedge \beta \mid \gamma$)

We present the axioms in informal language (their formalization being straightforward).

- **D1.** If $\alpha, \beta, \gamma \in G$ then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
- **D2.** If $\alpha \in S \cup P$ then $\iota(\alpha)$.
- **D3.** If $a, b \in S$ and $A, B \in P$ then $bab, BaB \in S$ and $bAb, BAB \in P$.
- **D4.** If $\alpha, \beta \in G$ and $\alpha \cdot \alpha = \alpha$ then $\alpha \cdot \beta = \beta = \beta \cdot \alpha$.
- **D5.** If $\alpha \in G$ then there are a, b with $\alpha = a \cdot b$ or A, B with $\alpha = A \cdot B$ or a, A with $\alpha = a \cdot A$.
- **D6.** If $a | b$ then $ab \in P$ and if $A | B$ then $AB \in S$.
- **D7.** If $A | b$ then $Ab \in P \cup S$.
- **D8.** For every pair (A, b) there is a unique pair (a, B) with $a \mid A$ and $B \mid b$ and $Aa = bB$.
- **D9.** If $A, B | c, d$ then $A = B$ or $c = d$.
- **D10.** If $A, B, C | d$ then $ABC \in P$ and if $a, b, c | D$ then $abc \in S$.
- **D11.** If $A|a$ and $B|b$ and $C|c$ and $Aa = Bb = Cc$ then $ABC \in P$ and $abc \in S$.
- **D12.** If $A | a$ and $B | b$ then there exists e with $e | A, B$ or E with $E | a, b$.
- **D13.** There exists a quadrangle (distinct points A, B, C, D and lines a, b, c, d with $a \mid A, B$ and $b \mid B, C$ and $c \mid C, D$ and $d \mid D, A$).

The axioms make the following statements: **D1** states that the binary operation $\cdot$ on G is associative. **D2** states that the elements of S and P are involutions of G. **D3** states that the subsets S and P of G are invariant with respect to transformations with elements of $S \cup P$. Axiom **D4** states that every idempotent element is a neutral element. **D5** states that every element of G is the product of two elements of $S \cup P$. The axioms **D6**–**D13** are the axioms **A1**–**A8** of Sect. 2.1 with the interpretation described in that section.

Theorem 2.3. *Axiom system D provides a first-order axiomatization of the theory which is defined by $\mathcal{A}$.*

Proof. A model of $\mathcal{A}$ obviously satisfies the axioms of $\mathcal{D}$. We show conversely that any model of $\mathcal{D}$ satisfies the axioms of $\mathcal{A}$.

According to **D1** the operation $\cdot$ is a binary associative operation on G. Let $\alpha \in S$. Then $\alpha^4 = \alpha^2$ (according to **D2**) and α^2 is a neutral element (according to **D4**) which we denote by 1. This definition does not depend on the choice of α: If $\beta \in S$ then $(\alpha\alpha)\beta = \beta$ (since $\alpha\alpha$ is a neutral element) and $((\alpha\alpha)\beta)\beta = \beta\beta$ which implies $\alpha\alpha = \beta\beta$ (since $\beta\beta$ is a neutral element).

We show that every element $\alpha \in G$ has an inverse element with respect to 1. If $\alpha = ab$ or $\alpha = AB$ or $\alpha = aA$ (see **D5**) then $\alpha^{-1} = ba$ resp. $\alpha^{-1} = BA$ resp. $\alpha^{-1} = Aa$ since $ab \cdot ba = a(bb)a = aa = 1$ resp. $aA \cdot Aa = aa = 1$. This proves that $(G, \cdot)$ is a group with 1 as identity element. Hence the elements $\alpha \in G$ with $\iota(\alpha)$ are the involutions of G. The sets S and P are invariant sets of involutions of G (according to **D2** and **D3**) which generate G (according to **D5**). Hence the Basic Assumption is satisfied.

The axioms **D6**–**D13** are the axioms **A1**–**A8** of Sect. 2.1 with the interpretation described in that section. $\square$

2.4. The completeness theorem

We now give an answer to the problem of Bachmann to find a general theorem which describes the extent of an allowed dualization of statements of plane absolute geometry.

Theorem 2.4. *If Ω is a statement of plane absolute geometry (a statement of the theory axiomatized by $\mathcal{B}$)*[1] *and Ω^* the dual one then the following holds:*

(a) *If Ω can be deduced from axiom system $\mathcal{A}$ then Ω and Ω^* hold in plane absolute geometry.*
(b) *If Ω and Ω^* hold in plane absolute geometry then Ω can be deduced from $\mathcal{A}$.*

Proof. (a) A group (G, S, P) of plane absolute geometry (a Bachmann group) is a model of axiom system $\mathcal{A}$ (see Theorem 2.2) and hence satisfies every conclusion Ω of $\mathcal{A}$. Since $\mathcal{A}$ contains with each axiom the dual one, the dual statement Ω^* of Ω is also a conclusion of $\mathcal{A}$. Hence Ω and Ω^* hold in (G, S, P).

(b) According to Theorem 2.2 the models of $\mathcal{A}$ are exactly the groups of plane absolute geometry and the associated dual groups. According to Gödel's Completeness Theorem for first-order logic every universally valid statement of the theory axiomatized by $\mathcal{A}$ is formally provable, i.e., if a statement Ω and the dual one Ω^* hold in plane absolute geometry then they can be deduced from $\mathcal{A}$ (see Gödel [3]). $\square$

[1]See Sect. 2.1 for the axioms of $\mathcal{B}$.

3. The principle of duality in Euclidean geometry

3.1. The axiom system $\mathcal{E}$

Let $\mathcal{E}$ be the axiom system which consists of the Basic Assumption (introduced in Sect. 2.1) and the following axioms **E1**–**E9**.

- **E1.** If $a|b$ then $ab \in P$ and if $A|B$ then $AB \in S$.
- **E2.** If $A|b$ then $Ab \in P \cup S$.
- **E3.** For every pair (A, b) there is a unique pair (a, B) with $a|A$ and $B|b$ and $Aa = bB$.
- **E4.** If $A, B|c, d$ then $A = B$ or $c = d$.
- **E5.** If $A, B, C|d$ then $ABC \in P$ and if $a, b, c|D$ then $abc \in S$.
- **E6.** If $A|a$ and $B|b$ and $C|c$ and $Aa = Bb = Cc$ then $ABC \in P$ and $abc \in S$.
- **E7.** For A, b with $A \nmid b$ there is at most one line through A which has no common point with b.
- **E8.** For a, B with $a \nmid B$ there is at most one point on a which has no joining line with B.
- **E9.** There exists a quadrangle.

The axioms **E1, E2, E4, E5** and **E6** are the axioms **A1, A2, A4, A5** and **A6** of Sect. 2.1 with the interpretation described in that section. **E3** states that if A is a point and b a line, then there exists a unique pair (a, B) with $a \mid A$ and $B \mid b$ and $Aa = bB$ (and hence a unique 'perpendicular' from A to b). Please observe that **E3** is a stronger version of axiom **A3** since the uniqueness of a perpendicular is postulated without exception. **E7** is the Euclidean parallel axiom, stated in a form which is often called 'Playfair's axiom', even though it already appears in the commentary of Proclus (If A is not incident with b then there is at most one line through A which has no common point with b). **E8** is the dual statement. According to **E9** there exists a quadrangle.

Axiom system $\mathcal{E}$ contains with each axiom the dual statement. Hence the principle of duality holds.

Since the models of $\mathcal{E}$ satisfy the Basic Assumption and the axioms **E1**, **E3**, **E4**, **E5**, **E6** and **E9** they are Cayley–Klein groups (see R. Struve [10, Section 3.1]). If in addition any two points have a joining line they are non-elliptic Bachmann groups (see [10, Theorem 4.1])[2] which axiomatize plane Euclidean geometry if the following additional axiom holds (see [10, Theorem 4.7]):

- **E7*.** For A, b with $A \nmid b$ there is exactly one line through A which has no common point with b.

Axiom **E7*** is a stronger version of **E7** and implies that a Bachmann group is non-elliptic (see [1, §6.12]).

[2] A Bachmann group (G, S, P) is called *non-elliptic* if $P \cap S = \varnothing$. This condition is in Bachmann groups equivalent with the uniqueness of a perpendicular and with **E3** (see [1, §20.5] and [2, §1.7]).

We denote the axiom system for Euclidean Bachmann groups (which contains the axioms of $\mathcal{B}$ and **E7***) by $\mathcal{B}^*$.

In the framework of Cayley–Klein groups the Euclidean Bachmann groups (G, S, P) are called *Euclidean Cayley–Klein groups* and their dual models (G, S', P') with $S' = P$ and $P' = S$ *co-Euclidean Cayley–Klein groups.*

3.2. The models of axiom system $\mathcal{E}$

We start the determination of the models of $\mathcal{E}$ with two theorems about the existence of joining lines and of points of intersections.

Theorem 3.1. *Let (G, S, P) be a model of $\mathcal{E}$ and let B be a point which is not incident with a line a. If B has a joining line with every point of a then the following holds:*

1. *There exist orthogonal lines.*
2. *If $A \mid b$ then $Ab \in S$.*
3. *Any two points have a joining line.*

Proof. Let (G, S, P) be a model of $\mathcal{E}$ and let B be a point which has a joining line with every point of a line a with $a \nmid B$.

Proof of (1). By axiom **E3** there exist (A, b) with $A \mid a$ and $b \mid B$ and $Aa = bB$. According to our assumption there exists a line e with $e \mid A, B$. Hence it is $e^{Aa} \mid A^{Aa}, B^{Aa}$. Since $A^{Aa} = A^a = A$ and $B^{Aa} = B^{Bb} = B^b = B$ it is $e, e^{Aa} \mid A, B$ and $e = e^{Aa}$ (according to **E4**; it is $A \neq B$ since $A \mid a$ but $B \nmid a$).

Since $e \mid A$ implies $e = e^{Aa} = e^a$ it is $e = a$ or $e \mid a$. Since $B \mid e$ but $B \nmid a$ it is $e \neq a$. This proves $e \mid a$ and statement (1).

Proof of (2). In Cayley–Klein groups (1) implies (2) (see [10, Theorem 3.5]).

Proof of (3). Suppose B and C are points which have no joining line. Let a be a line through C. According to **E3** and (2) there exists a perpendicular b from B to a with $b \mid B, a$. Hence C and C^b are points of a which have no joining line with B. According to **E8** it is $C = C^b$. Hence $C = b$ or $C \mid b$.

If $C = b$ then $B, C \mid Bb$ with $Bb \in S$ (according to (2)) which contradicts our assumption that B and C are points which have no joining line. If $C \mid b$ then $B, C \mid b$ which leads to the same contradiction. This proves that any two points have a joining line. □

Theorem 3.1 and axiom **E8** allow the distinction of two cases.

Theorem 3.2. *Let (G, S, P) be a model of $\mathcal{E}$. Then either any two points have a joining line or for every pair (a, B) with $a \nmid B$ there is exactly one point on a which has no joining line with B.*

The dual statement is formulated in the next theorem.

Theorem 3.3. *Let (G, S, P) be a model of $\mathcal{E}$. Then either any two lines have a point of intersection or for every pair (A, b) with $A \nmid b$ there is exactly one line through A which has no point of intersection with b.*

We now show that the models of $\mathcal{E}$ which satisfy the axiom of the existence of a joining line are exactly the Euclidean Cayley–Klein groups.

Theorem 3.4. *The models of $\mathcal{E}$ with the property that any two points have a joining line are the Euclidean Cayley–Klein groups.*

Proof. The Cayley–Klein groups with the property that any two points have a joining line are exactly the Bachmann groups (see [10, Theorem 4.1]). A Bachmann group is Euclidean if and only if the stronger version **E7*** of Playfair's axiom **E7** holds (for A, b with $A \nmid b$ there is one and only one line through A which has no common point with b; see Sect. 3.1). In a non-elliptic Bachmann group (G, S, P) the statements **E7** and **E7*** are equivalent since the line a with $a \mid A$ and $Aa \mid b$ (which exists according to **E3**) has no common point with b by the uniqueness of a perpendicular (also implied by **E3**). This proves the theorem. □

By dualization one gets the following result.

Theorem 3.5. *The models of $\mathcal{E}$ with the property that any two lines have a point of intersection are the co-Euclidean Cayley–Klein groups.*

The models of $\mathcal{E}$, which satisfy none of the assumptions of Theorems 3.4 and 3.5, satisfy instead (according to Theorems 3.2, 3.3) stronger versions of **E7** and **E8**:

E7*. If A is not incident with b then there is exactly one line through A which has no common point with b.

E8*. If a is not incident with B then there is exactly one point on a which has no joining line with B.

The Cayley–Klein groups which satisfy the additional axioms **E7*** and **E8*** are the Galilean Cayley–Klein groups (see R. Struve [10, Theorem 7.7]). Galilean Cayley–Klein groups are not models of axiom system $\mathcal{E}$ since they do not satisfy **E2**.

Our results are summarized in the following theorem.

Theorem 3.6. *The models of $\mathcal{E}$ are exactly the Euclidean and the co-Euclidean Cayley–Klein groups.*

3.3. A first-order version of axiom system $\mathcal{E}$

A first-order version of $\mathcal{E}$ (i.e., a first-order axiomatization of the theory defined by $\mathcal{E}$) can be obtained by a slight modification of the first-order axiomatization of absolute and dual absolute geometry provided in Sect. 2.3: We substitute **D12** of axiom system $\mathcal{D}$ by **E7** and **E8**. Let $\mathcal{D}^*$ denote this new axiom system.

Theorem 3.7. *Axiom system $\mathcal{D}^*$ provides a first-order axiomatization of the theory which is defined by $\mathcal{E}$.*

Proof. A model of $\mathcal{E}$ obviously satisfies the axioms of $\mathcal{D}^*$. We show conversely that any model of $\mathcal{D}^*$ satisfies the axioms of $\mathcal{E}$.

According to Theorem 2.3 a model of $\mathcal{D}^*$ satisfies the Basic Assumption and the axioms **A1**, **A2**, **A3**, **A4**, **A5**, **A6** and **A8** of $\mathcal{A}$ which correspond to the Basic Assumption and the axioms **E1**, **E2**, **E3**, **E4**, **E5**, **E6**, **E9** of $\mathcal{E}$. The axioms **E7** and **E8** are axioms of both axiom systems $\mathcal{E}$ and $\mathcal{D}^*$. □

3.4. The completeness theorem

We now give an answer to the problem of Bachmann to find a general theorem which describes the extent of an allowed dualization of statements of plane Euclidean geometry.

Theorem 3.8. *If* Ω *is a statement of plane Euclidean geometry* (*a statement of the theory axiomatized by* $\mathcal{B}^*$)[3] *and* Ω^* *the dual one then the following holds:*

(a) *If* Ω *can be deduced from axiom system* $\mathcal{E}$ *then* Ω *and* Ω^* *hold in plane Euclidean geometry.*
(b) *If* Ω *and* Ω^* *hold in plane Euclidean geometry then* Ω *can be deduced from* $\mathcal{E}$*.*

Proof. (a) A Euclidean Cayley–Klein group (G, S, P) is a model of axiom system $\mathcal{E}$ (see Theorem 3.4) and hence satisfies every conclusion Ω of $\mathcal{E}$. Since $\mathcal{E}$ contains with each axiom the dual one, the dual statement Ω^* of Ω is also a conclusion of $\mathcal{E}$. Hence Ω and Ω^* hold in (G, S, P).

(b) According to Theorem 3.6 the models of $\mathcal{E}$ are exactly the Euclidean Cayley–Klein groups and the associated dual groups (the co-Euclidean Cayley–Klein groups). According to Gödel's Completeness Theorem for first-order logic every universally valid statement of the theory axiomatized by $\mathcal{E}$ is formally provable, i.e., if a statement Ω and the dual one Ω^* hold in plane Euclidean geometry then they can be deduced from $\mathcal{E}$ (see Gödel [3]). □

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[3] See Sect. 3.1 for the axioms of $\mathcal{B}^*$.

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Rolf Struve
SIGNAL IDUNA Gruppe
Joseph-Scherer-Strasse 3
44139 Dortmund
Germany
e-mail: rolf.struve@arcor.de

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