J. Geom. 107 (2016), 125–135

0047-2468/16/010125-11
published online June 26, 2015
DOI 10.1007/s00022-015-0279-x

Journal of Geometry

Holonomy and contact geometry

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Abstract. On a Riemannian manifold, any parallel form is preserved by the flow of any Killing vector field with constant magnitude. As a consequence, on a 2n+1-dimensional K-contact manifold, there are no nontrivial parallel forms except of degrees 0 and 2n+1. Flat contact metrics on 3-manifolds are characterized by reducible holonomy.

Mathematics Subject Classification. 53D10.

Keywords. Holonomy, parallel form, K-contact, Sasakian.

1. Introduction

This paper is an extension and a clarification of paper [5]. In [5], it was not clear how the identity $L_Z\mu = 0$ holds at the beginning of the proof of Proposition 2.1 therein. We show in here that any parallel form on a, not necessarily compact, Riemannian manifold is left invariant by the flow of any Killing vector field with constant magnitude. Using this result, we show that there are no nontrivial parallel forms on any 2n+1-dimensional K-contact manifold, except in degrees 0 and 2n+1. The global version of this result is well known for closed Sasakian manifolds [2].

2. Preliminaries

A **contact form** on a $2n+1$-dimensional manifold M is a 1-form η such that $\eta \wedge (d\eta)^n$ is a volume form on M. There is always a unique vector field Z, the characteristic vector field of η, which is determined by the equations $\eta(Z) = 1$ and $d\eta(Z, X) = 0$ for arbitrary X. The distribution $D_p = \{V \in T_pM : \eta(V) = 0\}$ is called the contact distribution of η. Clearly, D is a symplectic vector bundle with symplectic form $d\eta$.

On a contact manifold (M, η, Z), there is also a nonunique Riemannian metric g and a partial complex operator J adapted to η in the sense that the identities $g(Z, Z) = 1$ and

$$2g(X, JY) = d\eta(X, Y),\ J^2X = -X + \eta(X)Z$$

hold for any vector fields X, Y on M. We have adopted the convention for exterior derivative so that

$$d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta([X,Y]).$$

The tensors η, Z, J and g are called contact metric structure tensors and the manifold M with such a structure will be called a **contact metric manifold** [1]. We will use the notation (M, η, Z, J, g) to denote a contact metric manifold M with specified structure tensors. Assuming that (M, g) is a complete Riemannian manifold, let ψ_t, $t \in \mathbb{R}$ denote the 1-parameter group of diffeomorphisms generated by Z. The contact form η is invariant under the 1-parameter group ψ_t, that is, $\psi_t^*\eta = \eta$. If ψ_t is also a 1-parameter group of isometries of g, then the contact metric manifold is called a **K-contact manifold**. By ∇, we shall denote the Levi-Civita covariant derivative operator of g. On a K-contact manifold, one has the identity

$$\nabla_X Z = -JX$$

valid for any tangent vector X. On a general contact metric manifold, the identity

$$\nabla_X Z = -JX - JhX$$

is satisfied, where $hX = \frac{1}{2}(L_Z J)X$. If the identity

$$(\nabla_X J)Y = g(X,Y)Z - \eta(Y)X$$

is satisfied for any vector fields X and Y on M, then the contact metric structure (M, η, Z, J, g) is called a **Sasakian** structure.

3. Holonomy on Riemannian manifolds

Given a path $x_t = x(t)$ on a Riemannian manifold (M, g), we denote by τ_b^a the parallel translation along the path from x_a to x_b. The proof of the lemma below can be found literally in [3]. We provide it here for completeness.

Lemma 3.1. *Let (M, g) be a Riemannian manifold and X an arbitrary Killing vector field on M. Let $C_t = \tau_0^t \circ (\phi_t)_*$ where ϕ_t is the (local) 1-parameter group of isometries generated by X and τ_0^t is parallel translation from $x_t = \phi_t(x)$ to $x_0 = x$ along the flow line of X through x. Then C_t is a (local) 1-parameter group of linear transformations of T_xM, that is $C_{s+t} = C_t \circ C_s$ and $C_t = exp(-t(A_X)_x)$; where $A_X = L_X - \nabla_X$.*

Proof. Since ϕ_t maps the flow line segment (x_0, x_s) into the flow line segment (x_t, x_{t+s}) and since ϕ_t is compatible with parallel translation, one has:

$$(\phi_t)_* \circ \tau_0^s = \tau_t^{t+s} \circ (\phi_t)_*.$$

Hence:

$$\begin{aligned} C_t \circ C_s &= \tau_0^t \circ (\phi_t)_* \circ \tau_0^s \circ (\phi_s)_* \\ &= \tau_0^t \circ \tau_t^{t+s} \circ (\phi_t)_* \circ (\phi_s)_* \\ &= \tau_0^{t+s} \circ (\phi_{s+t})_* = C_{t+s} \end{aligned}$$

Thus, there is a linear endomorphism say A of T_xM such that $C_t = exp(tA)$. Our claim is that

$$A = -(A_X)_x.$$

To prove the claim, we will show that

$$\lim_{t\to 0}\frac{1}{t}(C_tY_x - Y_x) = -(A_X)_xY_x$$

for any $Y_x \in T_xM$. We first look at the case $X_x \neq 0$. There is a local coordinates system $(x^1, \ldots, x^m)$ where $x_t = (t, 0, \ldots, 0)$ for small values of t. We extend Y_x into a local vector field Y on M such that $\phi_{t*}(Y_x) = Y_{x_t}$ for those small values of t. Then $(L_XY)_x = 0$. Moreover, one has:

$$\begin{aligned}-(A_X)_xY_x &= (\nabla_XY)_x - (L_XY)_x = (\nabla_XY)x\\ &= \lim_{t\to 0}\frac{1}{t}(\tau_0^tY_{x_t} - Y_x)\\ &= \lim_{t\to 0}\frac{1}{t}(\tau_0^t \circ ((\phi_{t*})Y_x - Y_x))\\ &= \lim_{t\to 0}\frac{1}{t}(C_tY_x - Y_x).\end{aligned}$$

Secondly, we consider the case $X_x = 0$. In this case, ϕ_t leaves x fixed and τ_0^t is the identity of T_xM. Thus, $(\nabla_XY)_x = 0$ and

$$\begin{aligned}-(A_X)_xY_x &= (\nabla_XY)_x - (L_XY)_x\\ &= -(L_XY)_x\\ &= -\lim_{t\to 0}\frac{1}{t}(Y_x - \phi_{t*}Y_x)\\ &= \lim_{t\to 0}\frac{1}{t}(C_tY_x - Y_x)\end{aligned}$$

which completes the proof. □

The next proposition can be found in [3] for compact manifolds and arbitrary infinitesimal affine transformations. Our proof is an adaptation to not necessarily compact Riemannian manifolds, with a restriction to Killing vector fields with constant magnitude.

Proposition 3.2. *Let X be a Killing vector field with constant magnitude on a Riemannian manifold (M, g). Then for each $x \in M$, the endomorphism $(A_X)_x$ belongs to the Lie algebra $\mathfrak{g}(x)$ of the linear holonomy group $\Psi(x)$.*

Proof. In the Lie algebra $\mathfrak{E}(x)$ of skew symmetric endomorphisms of T_xM, we define a positive definite inner product:

$$(A, B) = -trace(AB).$$

Let $\mathfrak{B}(x)$ be the orthogonal complement of $\mathfrak{g}(x)$ in $\mathfrak{E}(x)$, with respect to the above inner product. Let $A_X = S_X + B_X$, where $S_X \in \mathfrak{g}(x)$, $B_X \in \mathfrak{B}(x)$, $x \in M$. In order to continue with the proof, we need the following lemma:

Lemma 3.3. *In the orthogonal decomposition* $A_X = S_X + B_X$*, the tensor field* B_X *is parallel.*

Proof. Let c be an arbitrary curve from x to y in M. The parallel translation τ along c gives an isomorphism $\tau(\mathfrak{E}(x)) = \mathfrak{E}(y)$ which maps $\mathfrak{g}(x)$ into $\mathfrak{g}(y)$. Since τ preserves the inner products, it also maps $B(x)$ into $B(y)$. That is, for any vector field Y on M, $\nabla_Y(S_X) \in \mathfrak{g}(x)$ whereas $\nabla_Y(B_X) \in \mathfrak{B}(x)$ at each point $x \in M$. On the other hand, $\nabla_Y(A_X) = R(X,Y)$ (valid for any affine infinitesimal transformation X) implies that $\nabla_Y(A_X) \in \mathfrak{g}(x)$ at each $x \in M$.

$$\nabla_Y(A_X) = \nabla_Y(B_X) + \nabla_Y(S_X),$$

we see that $\nabla_Y(B_X)$ also belongs to $\mathfrak{g}(x)$, hence $\nabla_Y(B_X) = 0$. □

We will further show that $B_X = 0$. On one hand, $A_X X = L_X X - \nabla_X X = 0$ implies that $B_X X = 0$. On the other hand, given an orthonormal frame E_i, $i = 1, \ldots, m$ and using the identity $A_X Y = -\nabla_Y X$ (see [3]), one has

$$\begin{aligned}
0 = div B_X X &= \sum_{i=1}^{m} g(\nabla_{E_i}(B_X X), E_i) \\
&= \sum g(B_X \nabla_{E_i} X, E_i) \\
&= -\sum g(B_X A_X E_i, E_i) \\
&= -\sum g(B_X(B_X + S_X)E_i, E_i) \\
&= -\sum g(B_X B_X E_i, E_i) \\
&= -trace(B_X B_X) \geq 0
\end{aligned}$$

Hence $B_X = 0$. □

In [8], Wang established the invariance of parallel tensors by the identity component of the isometry group on a compact manifold. We generalize this result to any Riemannian manifold, but restricting to invariance by isometries generated by Killing vector fields with constant magnitude.

Theorem 3.4. *Let* (M, g) *be a Riemannian manifold. Then every parallel tensor field* T *on* M *is invariant by the flow of any Killing vector field with constant magnitude on* M*.*

Proof. Let X be a Killing vector field with constant magnitude on (M, g) and ϕ_t the (local) 1-parameter group of isometries generated by X. By Lemma 3.1 and Proposition 3.2, the 1-parameter group $C_t = \tau_0^t \circ (\phi_t)_*$ of linear transformations of $T_x M$ is contained in the linear holonomy group $\Psi(x)$. When C_t is extended to a 1-parameter group of automorphisms of the tensor algebra over $T_x M$, it leaves T invariant. Thus,

$$\phi_t^{*-1}(T_x) = \tau_t^0 T_x = T_{x_t}$$

for every t, where $x_t = \phi_t(x)$. It follows that ϕ_t leaves T invariant. □

4. Parallel forms of even-degrees

In [7], Tachibana, proved that if μ is a harmonic p-form with $1 \leq p \leq n$ on a compact Sasakian manifold $M^{2n+1}, n \geq 1$, then $\mu(Z,\dots) = 0$, where Z is again the Reeb vector field. For parallel forms of even degree, the compactness condition can be dropped and the Sasakian condition weakened so that the same conclusion extends to 2p-forms, $1 \leq p \leq n$, on K-contact manifolds.

Proposition 4.1. *On a K-contact manifold $(M^{2n+1}, \eta, Z, J, g)$, $n \geq 1$; if μ is a parallel 2p-form, $1 \leq p \leq n$, then μ is orthogonal to Z, that is,*

$$\mu(Z,\dots) = 0.$$

The proof of this proposition uses the following lemma:

Lemma 4.2. *On a K-contact manifold (M, η, Z, J, g), the K-contact flow admits local J-parallelisms $\{Z, E_i, JE_i\}$ such that $[Z, E_i] = 0 = [Z, JE_i]$ for $i = 1, 2\dots, n$.*

Proof. Every point on the manifold admits a neighborhood with Darboux local coordinates z, x_i, y_i, $i = 1, 2, \dots, n$, in which the contact form takes the expression:

$$\eta = dz - \sum_{i=1}^{n} y_i dx_i$$

and the Reeb field is $Z = \frac{\partial}{\partial z}$. Let $E_i = \frac{\partial}{\partial y_i}$. Clearly $[Z, E_i] = 0$ and $[Z, JE_i] = 0$ if the contact form is K-contact. □

Proof of the Proposition

Proof. Let μ be a parallel 2p-form on a K-contact manifold M of dimension 2n+1. Let also Z denote the characteristic vector field of the K-contact form η on M. By Theorem 3.4, μ is invariant under the one-parameter group of isometries generated by Z, hence the Lie derivative of μ in the direction of Z, $L_Z\mu$, is identically zero, i.e, the identity

$$L_Z\mu = 0$$

is satisfied everywhere on M.

Let P be an arbitrary point in M. On a neighborhood of P one can find a frame field $\{Z, E_1, \dots, E_{2n}\}$, such that $[Z, E_i] = 0,\ for\ i = 1, 2, \dots, 2n$. (Pick a local J-parallelism, as provided by Lemma 4.2.)

From $(L_Z\mu)(E_1, \dots, E_{2p-1}, Z) = 0$, one has

$$\mu(-JE_1, E_2, \dots, E_{2p-1}, Z) + \cdots + \mu(E_1, \dots, -JE_{2p-1}, Z) = 0 \qquad (1)$$

Covariantly differentiating with respect to Z, one obtains:

$$-(2p-1)\mu(E_1, .., E_{2p-1}, Z) + \sum_{i \neq j} \mu(.., JE_i, .., JE_j, .., Z) = 0 \qquad (2)$$

Thus,

$$\mu(E_1, .., E_{2p-1}, Z) = \frac{1}{2p-1} \sum_{i \neq j} \mu(.., JE_i, .., JE_j, .., Z) \tag{3}$$

Covariantly differentiating with respect to Z again shows that the Left Hand Side terms in (3) become the Left Hand Side terms of (1), thus vanish. Therefore,

$$0 = - \sum_{i \neq j \neq k} \mu(.., JE_i, .., JE_j, .., JE_k, .., Z) + (2p-2) \sum_i \mu(.., JE_i, .., Z) \tag{4}$$

The second sum in (4) vanishes by (1), so that

$$\sum_{i \neq j \neq k} \mu(.., JE_i, .., JE_j, .., JE_k, .., Z) = 0. \tag{5}$$

Another covariant derivative with respect to Z shows that

$$\begin{aligned} &(2p-3) \sum_{j \neq k} \mu(.., JE_j, .., JE_k, .., Z) \\ &\quad - \sum_{i \neq j \neq k \neq l} \mu(.., JE_i, .., JE_j, .., JE_k, .., JE_l, .., Z) = 0 \end{aligned}$$

and hence, using (3),

$$\begin{aligned} &\sum_{i \neq j \neq k \neq l} \mu(.., JE_i, .., JE_j, .., JE_k, .., JE_l, .., Z) \\ &\quad = (2p-3) \sum_{j \neq k} \mu(.., JE_j, .., JE_k, .., Z) \\ &\quad = (2p-3)(2p-1)\mu(E_1, .., E_{2p-1}, Z). \end{aligned}$$

Continuing this way, one sees that when the number of J's in the argument of μ is odd, the summation $\sum \mu(.., Z)$ satisfies:

$$\sum \mu(\ldots, Z) = 0,$$

and when the number of J's in the argument of μ is even, then the summation is a constant multiple of μ without any J in its argument:

$$\sum \mu(\ldots, Z) = [constant]\mu(E_1, \ldots, E_{2p-1}, Z).$$

Eventually, one obtains

$$\mu(JE_1, \ldots, JE_{2p-1}, Z) = 0.$$

In other words, μ is orthogonal to Z. $\square$

5. Vanishing of parallel forms

In [2], Blair and Goldberg showed that on a compact Sasakian manifold M^{2n+1}, there are no nonzero parallel p-forms for $1 \leq p \leq 2n$. Dropping the compactness and weakening the Sasakian conditions, we generalize the above result to all K-contact manifolds.

Theorem 5.1. *On a K-contact manifold M^{2n+1} with K-contact form η and Reeb field Z, there are no nonzero parallel p-forms for $1 \leq p \leq 2n$.*

Proof. We first prove that a parallel 2p-form on a K-contact manifold must be trivial for $1 \leq p \leq n$. It will follow that parallel forms of odd degrees are also trivial since Hodge's star $*\mu$ is parallel whenever μ is parallel and it is known that $*$ is an isomorphism.

To show that a parallel $2p$-form μ is identically zero, consider an orthonormal tangent frame $\{E_1, \ldots, E_{2n-1}, Z\}$ at a point P of M and let Y be an arbitrary tangent vector at P. Extend the frame into a frame field along the geodesic tangent to JY at P by parallel translation. Since by Proposition 4.1, $\mu(\ldots, Z) = 0$, covariantly differentiating with respect to JY yields:

$$\begin{aligned} 0 = JY\mu(E_1, \ldots, E_{2p-1}, Z) &= \mu(E_1, , \ldots, E_{2p-1}, -J^2Y) \\ &= \mu(E_1, \ldots, E_{2p-1}, Y - \eta(Y)Z) \\ &= \mu(E_1, \ldots, E_{2p-1}, Y) \end{aligned}$$

Since Y was arbitrary, it follows that any parallel 2p-form μ is identically zero. □

Remarks. The K-contact condition is necessary in Theorem 5.1. Indeed, nonzero parallel 1-forms and 2-forms are known to exist on contact metric structures that are not K-contact. For example, consider the standard flat contact metric structure on the torus $\mathbb{T}^3$ with coordinates $\theta^i,\ i = 1, 2, 3$ and contact form

$$\cos\theta^3 d\theta^1 + \sin\theta^3 d\theta^2.$$

The 1-forms

$$d\theta^i,\ i = 1, 2, 3$$

and 2-forms

$$d\theta^i \wedge d\theta^j,\ i, j = 1, 2, 3$$

are parallel nontrivial. Observe that for $i, j = 1, 2, 3$,

$$d\theta^i \wedge d\theta^j$$

are parallel 2-forms which are not even basic, illustrating the necessity of the K-contact condition in Proposition 4.1 also.

Parallel, nonzero 1-forms are also found on any flat, 3-dimensional contact metric manifold. It has been shown in [4] that any closed, flat contact metric 3-manifold carries a nontrivial, parallel 1-form.

For the case of degree 2 alone, non-existence of parallel nonzero 2-forms on K-contact manifolds follows also from a result of Sharma in [6].

6. Contact metric holonomy

The discussion in this section should be seen as a complement to the Houston Journal paper [4]. A Riemannian manifold (M, g) is said to be locally reducible if every point in M has a neighborhood U such that

$$U = Q \times N$$

is a metric product. In particular, each factor is a totally geodesic submanifold. A Riemannian manifold is reducible in this sense when for instance its Riemannian holonomy representation is reducible. The tangent bundle TM admits an orthogonal decomposition as $TM = TQ \oplus TN$. Now, suppose (M, g, η) is a contact metric manifold and let J denote any compatible almost complex structure on the contact bundle $ker(\eta)$.

Lemma 6.1. *With the above notations, none of the factor tangent subbundles TQ or TN is J invariant.*

Proof. Suppose TQ is J invariant, hence TN is also J invariant. It follows that the Reeb field Z of η must be tangent to one of the factors, say Q. This in turn implies that Q is at least 2-dimensional, since the contact sub-bundle is non-integrable. For any vector field X tangent to N, and any vector field Y tangent to Q, one has, since Q is totally geodesic:

$$0 = g(\nabla_Y Z, X) = g(-JY - JhY, X) = g(hY, JX). \tag{6}$$

It follows from (6) that the tangent sub-bundle TQ is h invariant and so is the tangent sub-bundle TN.

We also have, due to the h invariance of TN:

$$g(\nabla_X Z, Y) = g(-JX - JhX, Y) = 0. \tag{7}$$

If W is any other vector tangent to N, then

$$g(-JX - JhX, W) = g(\nabla_X Z, W) = -g(Z, \nabla_X W) = 0. \tag{8}$$

The combination of identities (7) and (8) implies that

$$-JX - JhX = 0$$

or equivalently,

$$hJX = JX$$

for any X tangent to N. But this leads to a contradiction, as

$$hX = -hJ^2X = JhJX = J^2X = -X$$

for any X tangent to N. □

Theorem 6.2. *Let (M, g, η) be a contact metric, 3-dimensional manifold. Then (M, g) is locally reducible if and only if (M, g) is flat.*

Proof. Since every flat Riemannian manifold is locally reducible, we need only to show that a locally reducible, 3-dimensional contact metric manifold is flat. So suppose (M, g, η) is a 3-dimensional, locally reducible contact metric manifold. Then locally, (M, g) is a Riemann product $L \times N$, where L is 1-dimensional and N is 2-dimensional. The Reeb field Z of η is not tangent to L because the contact subbundle, which is orthogonal to Z is not integrable. We will show that under the above condition, Z is actually tangent to the factor N. Let X be any non-singular vector field tangent to N satisfying $\eta(X) = 0$. Denote by J the almost complex structure compatible with η and g on the contact subbundle. JX admits an orthogonal decomposition into a component tangent to N and one tangent to L:

$$JX = (JX)_N + (JX)_L.$$

Applying J on both sides of the decomposition, one obtains:

$$-X = J(JX)_N + J(JX)_L.$$

Since $J(JX)_L$ is tangent to N, it follows that $J(JX)_N = -X - J(JX)_L$ is tangent to N; hence, since TN is 2-dimensional and not J invariant, we deduce that, almost everywhere, $(JX)_N = 0$ and hence JX is tangent to L or $(JX)_N = fZ$ for some function f on M. In either case, we deduce that the Reeb field Z is tangent to N.

Claim. *Let E be a unit vector field tangent to L. Then E is parallel.*

Indeed,

$$g(\nabla_Z E, E) = 0,$$

and since N is totally geodesic,

$$g(\nabla_Z E, JE) = -g(E, \nabla_Z JE) = 0.$$
$$g(\nabla_Z E, Z) = -g(E, \nabla_Z Z) = 0.$$

This shows that $\nabla_Z E = 0$.

Also $\nabla_E E = 0$ since L is totally geodesic 1-dimensional. It remains to show that $\nabla_{JE} E = 0$ also. $g(\nabla_{JE} E, E) = -g(E, \nabla_{JE} E) = 0$ because E has constant magnitude, $g(\nabla_{JE} E, JE) = -g(E, \nabla_{JE} JE) = 0$ and $g(\nabla_{JE} E, Z) = -g(E, \nabla_{JE} Z) = 0$, both follow from the fact that$N$ is totally geodesic.

Another claim. The Lie bracket of Z and JE satisfies $[Z, JE] = 0$.

Indeed, since Z preserves the contact distribution and Z and JE are tangent to N, one has $[Z, JE]$ is tangent to N and therefore $[Z, JE] = aJE$ for some smooth function a on M.

In that case, on one hand:

$$d\eta([Z, JE], E) = d\eta(aJE, E) = 2ag(JE, JE) = 2a.$$

On the other hand:

$$\begin{aligned} d\eta([Z, JE], E) &= Zd\eta(JE, E) - L_Z d\eta(JE, E) - d\eta(JE, [Z, E]) \\ &= -2g(JE, J[Z, E]) = -2g(E, [Z, E]) \\ &= 2g(E, \nabla_E Z) \textit{ since } E \textit{ is parallel} \\ &= -2g(\nabla_E E, Z) = 0 \textit{ for the same reason as above.} \end{aligned}$$

It follows that $2a = 0$ and $[Z, JE] = 0$.

We now complete the proof of the theorem by showing that each of the sectional curvature $K(Z, E)$, $K(Z, JE)$ and $K(E, JE)$ vanishes. We denote by R the Riemann curvature tensor given by

$$R(X, Y)W = \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W.$$

with these notations, one has $K(Z, E) = g(R(Z, E)E, Z) = 0$ and $K(JE, E) = g(R(JE, E)E, JE) = 0$ both follow from the fact that E is parallel. Let us compute $K(Z, JE) = g(R(Z, JE)JE, Z)$. Using the fact that JE is parallel along Z, and $[Z, JE] = 0$, we obtain:

$$\begin{aligned} g(R(Z, JE)JE, Z) &= g(\nabla_Z \nabla_{JE} JE - \nabla_{JE} \nabla_Z JE - \nabla_{[Z,JE]} JE, Z) \\ &= g(\nabla_Z \nabla_{JE} JE, Z) \\ &= Zg(\nabla_{JE} JE, Z) = -Zg(JE, \nabla_{JE} Z) \\ &= -Zg(JE, \nabla_Z JE) = 0. \end{aligned}$$

□

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Received: February 26, 2015.