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Convexity without convex combinations

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Abstract. Separation theorems play a central role in the theory of Functional Inequalities. The importance of Convex Geometry has led to the study of convexity structures induced by Beckenbach families. The aim of the present note is to replace recent investigations into the context of an axiomatic setting, for which Beckenbach structures serve as models. Besides the alternative approach, some new results (whose classical correspondences are well-known in Convex Geometry) are also presented.

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1. Introduction

Separation theorems play a crucial role in many fields of Analysis. They have basic applications in Nonsmooth Analysis [9], Functional Analysis [10], and Convex Analysis [27]. However, besides these applications, they can be interesting on their own sight: Let us recall here the theorem of Baron, Matkowski, and Nikodem [2], the main motivation of the forthcoming investigations.

Theorem. *Let $I \subset \mathbb{R}$ be an interval. There exists a convex function separating the given ones $f, g\colon I \to \mathbb{R}$ if and only if, for all $x, y \in I$ and $\lambda \in [0, 1]$, the next inequality holds:*

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda g(x) + (1-\lambda)g(y).$$

The sufficient part of the statement is a straightforward calculation, while proving necessity is highly nontrivial: The classical Carathéodory Theorem [8], one of the most important tool of Convex Geometry, has to be applied.

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The Baron–Matkowski–Nikodem Theorem initiated a blooming research in the last few decades. Its counterpart, the characterization of the existence of affine separators was obtained by Nikodem and Wasowicz [23]. The proof is based on the Helly Theorem [13], an other key tool of Convex Geometry. This stream concluded in the complete solution of the separation problem via interpolation families. For polynomial systems [24], the characterization is due to Wasowicz [31], and Balaj and Wasowicz [1]. For Chebyshev systems [16], analogue results were presented by Bessenyei and Páles [5]. The general case, when the underlying system is a convex-closed Beckenbach family, was studied and solved by Bessenyei and Szokol [7].

During the investigations, the importance of Convex Geometry has become clear. This phenomenon motivated the pioneer work of Krzyszkowski (see [18, 19]), who introduced convexity structures using two parameter Beckenbach families and extended the Carathéodory Theorem. As an application, he proved convex and affine separation theorems, as well. This work was continued by Páles and Nikodem [22]. A systematic study of convexity structures induced by two parameter Beckenbach families is presented in [4].

The aim of the present note is to enlighten a more effective and more general treatment of the topic. This treatment imposes no algebraic and analytic structure, and hence the usual tools do not work. Therefore, alternative approach and new ideas have to be applied. Following the idea of [30], first we give a set-theoretic view of convexity. Then, keeping certain parts of on Hilbert's System [14], an axiomatic method provides to establish the correspondences of some fundamental results of Convex Geometry. Finally we show, that Beckenbach families serve as models of our structure. Hence most of the former results are immediate consequences of the results to be presented.

2. Hull operators and pretopologies

In this section, we give a brief overview of some basic set theoretical facts. We start with presenting the most important properties of hull operators defined below (compare with the paper by Rådström [26]).

Definition. If X is a given set, then a mapping $\Phi\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is termed to be a *hull operator*, if it is increasing, extensive and idempotent. That is,

(i) $\Phi(A) \subset \Phi(B)$ whenever $A \subset B$ and $A, B \in \mathcal{P}(X)$;
(ii) $H \subset \Phi(H)$ for all $H \in \mathcal{P}(X)$;
(iii) $\Phi^2 = \Phi$, where $\Phi^2 := \Phi \circ \Phi$.

The next lemma characterizes increasing mappings in term of superadditivity. It shows that each hull operator, in particular, needs to be superadditive.

Lemma 1. *A mapping $\Phi\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is increasing if and only if it is superadditive; that is, for any $\mathcal{H} \subset \mathcal{P}(X)$, we have*

$$\bigcup_{H\in\mathcal{H}} \Phi(H) \subset \Phi\Big(\bigcup\mathcal{H}\Big).$$

Furthermore, an increasing mapping is extensive if and only if singletons belong to their image.

Proof. If Φ is increasing, then $H \subset \cup\mathcal{H}$ implies $\Phi(H) \subset \Phi(\cup\mathcal{H})$. Therefore, Φ is superadditive. The converse implication follows immediately choosing $\mathcal{H} = \{A, B\backslash A\}$, where $A \subset B$. In case of extensive and increasing mappings, singletons clearly belong to their image. The converse statement is a direct consequence of superadditivity. □

It turns out that hull operators can be characterized via a representation formula. This result shows the deeper reason why classical hulls are introduced in the usual way. For the fixed points of a self-map Φ, we shall use the notation $\mathrm{Fix}(\Phi)$.

Lemma 2. *A mapping* $\Phi\colon \mathcal{P}(X) \to \mathcal{P}(X)$ *is a hull operator if and only if* $\mathrm{Fix}(\Phi) \neq \emptyset$ *and, for all* $H \subset X$*, we have*

$$\Phi(H) = \bigcap\{K \in \mathrm{Fix}(\Phi) \mid H \subset K\}.$$

Moreover, if $\mathcal{F} \in \mathcal{P}(X)$ *is arbitrary, then any hull operator* Φ *fulfills the inclusion*

$$\bigcup\{\Phi(F) \subset X \mid F \subset H,\, F \in \mathcal{F}\} \subset \Phi(H).$$

Proof. For our convenience, let $\mathcal{K} := \{K \in \mathrm{Fix}(\Phi) \mid H \subset K\}$. Assume first that Φ is hull operator. Since Φ is extensive, $X \subset \Phi(X)$ holds. Thus $X \in \mathrm{Fix}(\Phi)$. If $K \in \mathcal{K}$, then $H \subset K$ holds; hence $\Phi(H) \subset \Phi(K) = K$ follows. That is, $\Phi(H) \subset \bigcap\mathcal{K}$. For the converse inclusion, observe that $H \subset \Phi(H)$ and consequently $\Phi(H) \subset \Phi^2(H)$. These together imply $\Phi(H) \in \mathcal{K}$. Therefore, $\bigcap\mathcal{K} \subset \Phi(H)$.

For the converse implication, we shall prove that the representation formula defines an increasing, extensive, and idempotent mapping. Take $A, B \in \mathcal{P}(X)$ such that $A \subset B$, and define

$$\mathcal{A} := \{K \in \mathrm{Fix}(\Phi) \mid A \subset K\}, \quad \mathcal{B} := \{K \in \mathrm{Fix}(\Phi) \mid B \subset K\}.$$

Then $\mathcal{B} \subset \mathcal{A}$ and hence $\bigcap\mathcal{A} \subset \bigcap\mathcal{B}$, yielding $\Phi(A) \subset \Phi(B)$. The property $H \subset \Phi(H)$ is obvious. Finally, if $K \in \mathrm{Fix}(\Phi)$ is such that $H \subset K$, then $\Phi(H) \subset \Phi(K)$ by monotonicity. On the other hand, if $\Phi(H) \subset K$, then $H \subset K$ due to the extensiveness. Therefore,

$$\{K \in \mathrm{Fix}(\Phi) \mid H \subset K\} = \{K \in \mathrm{Fix}(\Phi) \mid \Phi(H) \subset K\},$$

yielding $\Phi(H) = \Phi^2(H)$.

For the last statement, let $\mathcal{L} := \{\Phi(F) \subset X \mid F \subset H,\, F \in \mathcal{F}\}$. If $K \in \mathcal{K}$ and $F \in \mathcal{L}$, then $F \subset K$; by monotonicity, $\Phi(F) \subset \Phi(K) = K$. This means $\Phi(F) \subset \bigcap\mathcal{K} = \Phi(H)$, yielding $\bigcup\mathcal{L} \subset \Phi(H)$ as it was desired. □

A weaker version of Lemma 2 can be found in [4]. Note also that, in view of the Tarski Fixed Point Theorem, an increasing mapping always has a fixed point.

Those hull operators that possess finitely generated inner representations play a distinguished role. The next lemma gives a sufficient condition to have this property. To motivate it, assume that Φ is a hull operator and $K \in \mathrm{Fix}(\Phi)$. Then $\Phi(\{x,y\}) \subset \Phi(K) = K$ for all elements x, y of K. That is,

$$\mathrm{Fix}(\Phi) \subset \{K \subset X \mid \Phi(\{x,y\}) \subset K,\, x,y \in K)\}$$

remains true for any hull operator Φ. It turns out, that if the sets above coincide, then Φ has a finitely generated inner representation.

Lemma 3. *If* $\Phi\colon \mathcal{P}(X) \to \mathcal{P}(X)$ *is nonempty valued, increasing and idempotent mapping such that*

$$\mathrm{Fix}(\Phi) = \{K \subset X \mid \Phi(\{x,y\}) \subset K,\, x,y \in K)\},$$

then

$$\Phi(H) = \bigcup\{\Phi(F) \subset X \mid F \subset H,\, \mathrm{card}(F) < \infty\}.$$

Proof. Define $\mathcal{L} := \{\Phi(F) \subset X \mid F \subset H,\, \mathrm{card}(F) < \infty\}$. Note that, under the assumptions, singletons belong to $\mathrm{Fix}(\Phi)$. In particular, Φ is necessarily a hull operator, and $H \subset \bigcup\mathcal{L}$ always remains true. In view of the last statement of Lemma 2, it suffices to show only the inclusion $\Phi(H) \subset \bigcup\mathcal{L}$. Let $x, y \in \bigcup\mathcal{L}$. Then, there exist finite sets $F(x)$ and $F(y)$ such that $x \in \Phi(F(x))$ and $y \in \Phi(F(y))$. Then, using monotonicity (or equivalently: superadditivity) and idempotency, we arrive at

$$\Phi(\{x,y\}) \subset \Phi\big(\Phi(F(x)) \cup \Phi(F(y))\big) \subset \Phi\big(\Phi(F(x) \cup F(y))\big) = \Phi\big(F(x) \cup F(y)\big).$$

Here $F(x) \cup F(y)$ is finite, providing $\Phi(\{x,y\}) \subset \bigcup\mathcal{L}$. Therefore $\bigcup\mathcal{L}$ is such that it belongs to $\mathrm{Fix}(\Phi)$ and contains H. Hence in view of Lemma 2, we get $\Phi(H) \subset \bigcup\mathcal{L}$, which was to proved. □

Finally, we formulate a variant of the classical Riesz lemma concerning compactness. To do this, we shall need the concepts of pretopological notions.

Definition. Under a *pretopology* on a nonempty set X we mean a family of $\mathcal{P}(X)$ containing the empty set and X. A set is called *open* if it belongs to the pretopology; *closed* if its complement belongs to the pretopology. We say that a set is *compact*, if its any open covering contains a finite open covering.

Lemma 4. *Any closed and centered family in a pretopology that contains a compact member has nonempty intersection.*

Proof. Denote the family by $\mathcal{F}$ and assume to the contrary that $\bigcap\mathcal{F} = \emptyset$. Then the family $\mathcal{H}$ containing the complements of $\mathcal{F}$ fulfills $X = \bigcup\mathcal{H}$ according to the De Morgan identities. Therefore $\mathcal{H}$ is an open covering system for the compact member F_0. Hence

$$F_0 \subset H_1 \cup \cdots \cup H_n$$

holds with suitable members $H_1, \ldots, H_n$ of $\mathcal{H}$. This implies $F_0 \subset (X \backslash F_1) \cup \cdots \cup (X \backslash F_n)$, where F_k stands for the complement of H_k. In other words, $F_0 \cap F_1 \cap \cdots \cap F_n = \emptyset$, contradicting to the centered property of $\mathcal{F}$. □

3. Basic axioms and their consequences

In order to introduce convexity in lack of algebraic manipulations, we use some of the axioms of geometry proposed by Hilbert [14]. More precisely, we shall need the axioms of incidence, the axioms of betweenness, and the axiom of half-plane. For the Reader's convenience, let us sketch here these axioms in the nice and simplified way as it is presented in the book of Hartshorne [12].

Axioms of incidence and betweenness. Assume that X is a nonempty set, whose elements are called *points* and we also consider certain subsets of X whose elements are termed *lines*. Besides the usual relations of set theory, we postulate a relation called *betweenness* among collinear points a, b, c abbreviated by (abc). We require that the next axioms are satisfied.

(i) Any two distinct points determine a unique line containing them.
(ii) Each line has at least two points.
(iii) There exist three noncollinear points (i.e., being not on the same line).
(iv) If (abc), then a, b, c are pairwise distinct and collinear; further, (cba).
(v) For distinct points a, b, there exists c such that (abc).
(vi) If (abc), then (acb) and (bac) do not hold.

For a line ℓ determined by the distinct points a and b, we use the notation $\ell = \ell(a, b)$. Betweenness makes possible to introduce the notion of *line segment* $[a, b]$ spanned by the points a, b as follows. If $a = b$, then $[a, b] := \{a\}$; otherwise,

$$[a, b] := \{t \in X \mid (atb)\} \cup \{a, b\}.$$

Once having segments, concepts of convexity can be defined in the next way. To distinguish it from the classical setting and also to indicate the role of the axioms above, we shall use the notation $\mathcal{A}$-convexity.

Definition. We say that a set $K \subset X$ is $\mathcal{A}$-*convex* if $[a, b] \subset K$ holds for all $a, b \in K$. The $\mathcal{A}$-*convex hull* of $H \subset X$ is the intersection of those $\mathcal{A}$-convex sets that contain H. The $\mathcal{A}$-convex hull of H is denoted by $\operatorname{conv}_{\mathcal{A}}(H)$.

The following lemma subsumes the most important properties of $\mathcal{A}$-convexity. These properties are similar to the standard ones.

Lemma 5. *Keeping the notation and axioms above,*

(i) *segments are $\mathcal{A}$-convex sets;*

(ii) *the intersection of $\mathcal{A}$-convex sets is $\mathcal{A}$-convex;*
(iii) *the union of nested $\mathcal{A}$-convex sets is $\mathcal{A}$-convex;*
(iv) *a set H is $\mathcal{A}$-convex if and only if* $\operatorname{conv}_{\mathcal{A}}(H) = H$*;*
(v) *the mapping* $\operatorname{conv}_{\mathcal{A}} : \mathcal{P}(X) \to \mathcal{P}(X)$ *is a hull operator.*

Proof. We concentrate only to the proof of last two assertions. For (iv), assume first that H is $\mathcal{A}$-convex. Then $H \subset \operatorname{conv}_{\mathcal{A}}(H)$. For the converse inclusion, let $p \in \operatorname{conv}_{\mathcal{A}}(H)$. Then by definition, p belongs to every $\mathcal{A}$-convex set containing H; in particular, $p \in H$. That is,

$$H = \operatorname{conv}_{\mathcal{A}}(H).$$

Conversely, assume that $\operatorname{conv}_{\mathcal{A}}(H) = H$ holds. Using assertion (ii), we get that $\operatorname{conv}_{\mathcal{A}}(H)$ is $\mathcal{A}$-convex set which implies that H need to be $\mathcal{A}$-convex, as well.

For (v), observe that singletons belong to $\operatorname{Fix}(\operatorname{conv}_{\mathcal{A}})$, therefore the mapping $\operatorname{conv}_{\mathcal{A}}$ is a hull operator due to its definition and Lemma 2. □

Further consequence of the axioms of incidence and betweenness that the operator $\operatorname{conv}_{\mathcal{A}}$ has finitely generated inner representation, analogously to standard convex hulls:

Lemma 6. *If X satisfies the axioms of incidence and betweenness, and $H \subset X$ is a nonempty set, then*

$$\operatorname{conv}_{\mathcal{A}}(H) = \bigcup\{\operatorname{conv}_{\mathcal{A}}\{p_0, \ldots, p_n\} \mid p_0, \ldots, p_n \in H,\ n \in \mathbb{N}\}.$$

Proof. Assertion (iv) of Lemma 5 provides that $\operatorname{Fix}(\operatorname{conv}_{\mathcal{A}}) = \{K \subset X \mid \operatorname{conv}_{\mathcal{A}}\{x, y\} \subset K\}$. Thus Lemma 3 completes the proof. □

As Moore pointed out [20], Hilbert's System in its original form is redundant. Moreover, only one primitive notion, the notion of point is enough for establishing geometry. However, the axiom of half planes still remains a basic tool. It will be important also for us to obtain more delicate results on $\mathcal{A}$-convexity.

Axiom of half-planes. Let ℓ be a line of X. Then there exists an $\mathcal{A}$-convex partition $\{H_1, H_2\}$ of $X \backslash \ell$ such that $[p_1, p_2] \cap \ell$ is nonempty whenever $p_1 \in H_1$ and $p_2 \in H_2$.

This axiom is crucial for the Pash and Peano properties below. These properties provide the validity of drop representation, which is important in proving a separation theorem via complementary convex sets.

Lemma 7. *Assume that a, b, c are not collinear points of X, where X fulfills the axioms of incidence, betweenness, and half-plane.*

(i) *If (byc) and (azy), then $\ell(c, z) \cap [a, b]$ is nonempty.*
(ii) *If (axb) and (byc), then $[a, y] \cap [c, x]$ is nonempty.*

Proof. Due to the Half-plane axiom, there exists an $\mathcal{A}$-convex partition $\{H_1, H_2\}$ of $X \backslash \ell(c, z)$ such that $[h_1, h_2] \cap \ell(c, z)$ is nonempty whenever $h_1 \in H_1$ and

$h_2 \in H_2$. Since $z \in [a, y] \cap \ell(c, z)$, then $a \in H_1$ and $b \in H_2$ or conversely, which means that $[a, b] \cap \ell(c, z)$ is nonempty.

Similarly, due to the half-plane axiom, (axb) implies that $\ell(c, x) \cap [a, y] \neq \emptyset$ and (byc) follows that $\ell(a, y) \cap [c, x] \neq \emptyset$. On the other hand, $[a, y] \subset \ell(a, y)$ and $[c, x] \subset \ell(c, x)$, hence $\ell(a, y) \cap \ell(c, x)$ is nonempty. Finally, using the fact that two distinct lines can have at most one point in common, we get that the unique member of the previous intersection belongs to both $[a, y]$ and $[c, x]$. □

Lemma 8. *Assume that X satisfies the axioms of incidence, betweenness, and half-plane. If $A \subset X$ is an $\mathcal{A}$-convex set and $p \in X$, then*

$$\mathrm{conv}_{\mathcal{A}}(\{p\} \cup A) = \{[p, a] \mid a \in A\}.$$

Proof. If $x \in \{[p, a] \mid a \in A\}$ is arbitrary, then there exists $a \in A$ such that $x \in [p, a]$. If K is an $\mathcal{A}$-convex set such that $\{p\} \cup A \subset K$, then $[p, a] \subset K$ yielding $x \in K$. Therefore,

$$\{[p, a] \mid a \in A\} \subset \mathrm{conv}_{\mathcal{A}}(\{p\} \cup A).$$

For the converse inclusion, observe first that $\{p\} \cup A \subset \{[p, a] \mid a \in A\}$ holds evidently. Therefore it suffices to prove that this latter set is $\mathcal{A}$-convex. Fix elements q_1, q_2 of $\{[p, a] \mid a \in A\}$. Then there exist $a_1, a_2 \in A$ such that $q_1 \in [p, a_1]$ and $q_2 \in [p, a_2]$ remain true. If q belongs to the segment $[q_1, q_2]$, then the Pash property guarantees that $\ell(p, q) \cap [a_1, a_2] = \{r\}$ and (pqr) is valid. Since A is $\mathcal{A}$-convex, therefore $r \in [a_1, a_2] \subset A$, yielding $q \in \{[p, a] \mid a \in A\}$. In other words, the set $\{[p, a] \mid a \in A\}$ is $\mathcal{A}$-convex, indeed. Hence

$$\mathrm{conv}_{\mathcal{A}}(\{p\} \cup A) \subset \{[p, a] \mid a \in A\}$$

holds, completing proof. □

4. The main results

In what follows, some basic theorems of Convex Geometry and Convex Analysis are discussed. The first one corresponds to the result of Kakutani [15] and Stone [28]. This extension remains true in such geometries, where planes are also postulated and possess the axiom of half-plane.

Theorem 1. *Assume that X satisfies the axioms of incidence, betweenness, and half-plane. If $A, B \subset X$ are nonempty, disjoint, $\mathcal{A}$-convex sets then there exist A^* and B^* $\mathcal{A}$-convex partition of X such that $A \subset A^*$ and $B \subset B^*$.*

Proof. Let $\mathcal{P}$ be the set of all pairs (C, D) where C, D are nonempty disjoint $\mathcal{A}$-convex sets such that $A \subset C$ and $B \subset D$ hold. Clearly, $(A, B) \in \mathcal{P}$ showing that $\mathcal{P} \neq \emptyset$. For pairs (C_1, D_1) and (C_2, D_2) of $\mathcal{P}$, we write $(C_1, D_1) \preceq (C_2, D_2)$ if and only if $C_1 \subset C_2$ and $D_1 \subset D_2$ remain true. It is immediate to see that $(\mathcal{P}, \preceq)$ is a partial ordered set. Assume that $\mathcal{L} = \{(C_\gamma, D_\gamma) \mid \gamma \in \Gamma\}$ is a chain in $\mathcal{P}$. Define

$$C = \bigcup_{\gamma \in \Gamma} C_\gamma, \quad D = \bigcup_{\gamma \in \Gamma} D_\gamma.$$

Then, C and D are $\mathcal{A}$-convex sets, since they are obtained as the nested union of $\mathcal{A}$-convex sets. Moreover, C and D are disjoint. Indeed, if $p \in C \cap D$, then there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $p \in C_{\gamma_1}$ and $p \in D_{\gamma_2}$ hold. The chain property of $\mathcal{L}$ ensures that one of the relations below is satisfied:

$$(C_{\gamma_1}, D_{\gamma_1}) \preceq (C_{\gamma_2}, D_{\gamma_2}); \quad (C_{\gamma_2}, D_{\gamma_2}) \preceq (C_{\gamma_1}, D_{\gamma_1}).$$

In the first case, $C_{\gamma_1} \subset C_{\gamma_2}$ implies $p \in C_{\gamma_2}$. That is, $p \in C_{\gamma_2} \cap D_{\gamma_2}$, contradicting to the disjointness. The other case leads to contradiction in a similar way. Therefore, (C, D) is an upper bound for $\mathcal{L}$ in $\mathcal{P}$.

The Kuratowski–Zorn lemma guarantees that there exists a maximal element (A^*, B^*) of $\mathcal{P}$. We are going to verify that A^*, B^* are proper choices. Assume to the contrary that there exists some $p \in X$ such that $p \notin A^* \cup B^*$. Then the maximality forces $\operatorname{conv}_{\mathcal{A}}(A^* \cup \{p\}) \cap B^* \neq \emptyset$. In other words, taking into consideration Lemma 8, there exist $a^* \in A^*$ and $b \in B^*$ such that $b \in [p, a^*]$. Similarly, there exist $b^* \in B^*$ and $a \in A^*$ such that $a \in [p, b^*]$. Then Lemma 7 gives $[a, a^*] \cap [b, b^*] \neq \emptyset$. Hence, using also the convexity, $A^* \cap B^*$ turns out to be nonempty, which is a contradiction. □

The next result is an extension of the Carathéodory Theorem (see the original paper [8]).

Theorem 2. *Assume that X satisfies the axioms of incidence, betweenness, and half-plane. If $H \subset X$ is a nonempty set and $p \in \operatorname{conv}_{\mathcal{A}}(H)$, then there exist $h_0, h_1, h_2 \in H$ such that $p \in \operatorname{conv}_{\mathcal{A}}\{h_0, h_1, h_2\}$.*

Proof. If $p \in \operatorname{conv}_{\mathcal{A}}(H)$, then there exist points $p_0, \dots, p_n$ of H such that $p \in \operatorname{conv}_{\mathcal{A}}\{p_0, \dots, p_n\}$ in view of Lemma 6. We may assume that $p_k \notin \operatorname{conv}_{\mathcal{A}}(\{p_0, \dots, p_n\} \setminus \{p_k\})$ holds for all indices k. Then, the vertices can be labeled so that the line $\ell(p_0, p_k)$ separate the sets $\{p_0, \dots, p_{k-1}\}$ and $\{p_{k+1}, \dots, p_n\}$, respectively. Then,

$$\operatorname{conv}_{\mathcal{A}}\{p_0, \dots, p_n\} = \bigcup_{k=1}^{n-1} \operatorname{conv}_{\mathcal{A}}\{p_0, p_k, p_{k+1}\}.$$

Therefore, $p \in \operatorname{conv}_{\mathcal{A}}\{p_0, p_k, p_{k+1}\}$ with some suitable index k, and the proof is completed choosing $h_0 = p_0$, $h_1 = p_k$ and $h_2 = p_{k+1}$. □

The last result contains two generalized versions of the Helly Theorem. Besides the original paper [13], let us quote here [25] and [17] for alternative approaches, and also [11] for interesting and important historical details. To formulate the statement, we need a concept which plays the role of topology.

Definition. We say that $p \in H$ is an $\mathcal{A}$-*interior point*, if $[p, x] \cap (H \setminus \{p\}) \neq \emptyset$ for all $x \in X$. A subset of X is called to be $\mathcal{A}$-*open*, if its any point is $\mathcal{A}$-interior point; $\mathcal{A}$-*closed*, if its complement is $\mathcal{A}$-open. We say that a set is $\mathcal{A}$-*compact*, if its any $\mathcal{A}$-open covering contains a finite $\mathcal{A}$-open covering.

Theorem 3. *Assume that X satisfies the axioms of incidence, betweenness, and half-plane. If $\mathcal{K}$ is a finite collection of $\mathcal{A}$-convex sets of which three member subcollections are intersecting, then $\bigcap \mathcal{K} \neq \emptyset$. Moreover, if $\mathcal{K}$ is a family of*

$\mathcal{A}$-convex, $\mathcal{A}$-closed sets whose three member subfamilies are intersecting, and $\mathcal{K}$ contains an $\mathcal{A}$-compact member, then $\bigcap \mathcal{K} \neq \emptyset$.

Proof. Assume first that $\mathcal{K} = \{K_0, K_1, K_2, K_3\}$. Then, for all index $k \in \{0, 1, 2, 3\}$, there exist some element p_k belonging to $\bigcap\{K_j \mid j \neq k\}$. If p_0, p_1, p_2, p_3 are collinear, then we may assume that $[p_1, p_2]$ is contained by $[p_0, p_3]$. This yields

$$[p_1, p_2] \subset [p_0, p_3] \subset K_1 \cap K_2.$$

That is, $p_1, p_2 \in \bigcap \mathcal{K}$ holds. Assume that there exist at least three noncollinear points. Then two of the points p_0, p_1, p_2, p_3 determine a line ℓ that separates the other two points. For simplicity, assume that $\ell = \ell(p_0, p_1)$. Define $p = \ell(p_0, p_1) \cap \ell(p_2, p_3)$. If (p_0, p_1, p) holds, then $p_1 \in \bigcap \mathcal{K}$; otherwise (p_0, p, p_1) implies $p \in \bigcap \mathcal{K}$.

Assume that the statement remains true for any n-element collection of $\mathcal{A}$-convex sets whose three element subcollections are intersecting. Take a collection of $(n + 1)$ sets $\{K_0, \dots, K_n\}$ satisfying the requirements of the theorem and consider the n-member family $\{K_0 \cap K_1, K_2 \dots, K_n\}$. The previous part of the proof ensures that its three element subcollections are intersecting. Therefore, the entire intersection is nonempty, as well.

For the second statement, note that $\mathcal{K}$ has the finite intersection property due to the first part. Therefore, Lemma 4 completes the proof choosing the pretopology as the $\mathcal{A}$-open sets in X. □

5. Applications

Beckenbach families are continuous functions having unique interpolation property. These families were introduced and studied by Beckenbach [3] and Popoviciu [24]. One of the most important result concerning pointwise convergence of Beckenbach functions is due to Tornheim [29]. Applying Beckenbach families, the relation betweenness can be introduced in the next way.

Definition. A set $\mathcal{B}$ of real valued continuous functions defined on an interval I is called a *B*eckenbach family if, for all points p_1, p_2 of $I \times \mathbb{R}$ with distinct first coordinates, there exists unique Beckenbach line φ_{p_0,p_1} of $\mathcal{B}$ interpolating the points.

Clearly, the set of affine functions form a Beckenbach family and induce the notion of standard convexity. An other example is the line of the Moulton plane [21]. This construction has a particular importance in Projective Geometry, since it demonstrates that the Desargues property is independent on the axioms of projective plane.

Under the *segment* spanned by $p_0 = (x_0, y_0)$ and $p_1 = (x_1, y_1)$ we mean the set $[p_0, p_1]$ given by (distinguishing the cases $x_0 = x_1$ and $x_0 \neq x_1$)

$$[p_0,p_1] := \{(x,y) \in I \times \mathbb{R} \mid x_0 = x = x_1,\ \min\{y_0,y_1\} \le y \le \max\{y_0,y_1\}\};$$
$$[p_0,p_1] := \{(x,y) \in I \times \mathbb{R} \mid \min\{x_0,x_1\} \le x \le \max\{x_0,x_1\},\ y = \varphi_{p_0,p_1}(x)\}.$$

The next result is a direct consequence of the definition above. The details of the proof are left to the Reader. To prove the half-plane property, one should use the Bolzano Theorem.

Theorem 4. *Let $\mathcal{B}$ be a Beckenbach family over an open interval I. If $X = I \times \mathbb{R}$, lines are either (standard) vertical or Beckenbach lines, furthermore betweenness is determined via segments, then the structure obtained is an $\mathcal{A}$-convex structure.*

Using this theorem, direct and alternative proofs can be obtained for most of the result which concern the Convex Geometry of Beckenbach structures. The first corollary is a direct consequence of Theorem 1 and was presented first in [22]. The second one appears first in the works of Krzyszkowski (see [18,19]) and, together with the third one, in [4].

Corollary 1. *If $\mathcal{B}$ is a Beckenbach family on an open interval, $A, B \subset X$ are nonempty, disjoint, $\mathcal{B}$-convex sets then there exist A^* and B^* $\mathcal{B}$-convex partition of X such that $A \subset A^*$ and $B \subset B^*$.*

Corollary 2. *If $\mathcal{B}$ is a Beckenbach family on an open interval, $H \subset X$ is a nonempty set and $p \in \operatorname{conv}_{\mathcal{B}}(H)$, then there exist $h_0, h_1, h_2 \in H$ such that $p \in \operatorname{conv}_{\mathcal{B}}\{h_0, h_1, h_2\}$.*

Corollary 3. *If $\mathcal{B}$ is a Beckenbach family on an open interval, $\mathcal{K}$ is a finite collection of $\mathcal{B}$-convex sets of which three member subcollections are intersecting, then $\bigcap \mathcal{K} \neq \emptyset$. Moreover, if $\mathcal{K}$ is a family of $\mathcal{B}$-convex, closed sets whose three member subfamilies are intersecting, and $\mathcal{K}$ contains a compact member, then $\bigcap \mathcal{K} \neq \emptyset$.*

As a final application, the motivating result of the investigations is revisited. To formulate its extension to Beckenbach setting, we need the next concept of generalized convexity [3].

Definition. Let $\mathcal{B}$ be a Beckenbach family on an interval I. A function $h\colon I \to \mathbb{R}$ is said to be *c*onvex with respect to $\mathcal{B}$, if $\varphi(t) \ge h(t)$ holds on $[x_1, x_2]$ for all elements $x_1 < x_2$ of I, where $\varphi \in \mathcal{B}$ is determined by the interpolation properties $\varphi(x_1) = h(x_1)$ and $\varphi(x_2) = h(x_2)$.

A generalization of the Baron–Matkowski–Nikodem Theorem [2] now reads as follows. Its proof is based on Theorem 2 and can be found in [5,6] for linear Beckenbach families; in [4,22] for the general case.

Corollary 4. *Assume $\mathcal{B}$ is a Beckenbach family over a real interval I and $f, g\colon I \to \mathbb{R}$ are given functions. There exists a $\mathcal{B}$-convex function h separating f and g if and only if, for all elements $x_0 \le x_1 \le x_2$ of I, we have the inequality*

$f(x_1) \leq \varphi(x_1)$, where $\varphi \in \mathcal{B}$ is defined by the properties $\varphi(x_0) = g(x_0)$ and $\varphi(x_2) = g(x_2)$.

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