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Ordered metric geometry

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Abstract. Metric geometry in the sense of Hjelmslev and Bachmann studies metric planes of a very general kind without any assumption about order, continuity and the existence and uniqueness of joining lines. An order structure can be defined in an additional step by introducing a relation of betweenness which satisfies the axioms of order of Hilbert's *Grundlagen der Geometrie*, i.e., one-dimensional axioms which characterize the linear order of collinear points and a single plane order axiom which was proposed by Pasch. The Pasch axiom however is based on the assumption that any two points have a unique joining line. This is not necessarily satisfied by Cayley–Klein geometries (e.g. by Minkowskian planes) and even in plane absolute geometry the Pasch axiom is not a necessary condition for an ordering of the associated field of coordinates (see Sect. 5). The aim of this article is to introduce an order structure for the widest class of metric planes (without any assumption about the existence of joining lines, free mobility or some form of a parallel axiom) and to show that the correspondence between geometrical and algebraical order structures, which is well-known in affine and projective geometry, can be extended to plane absolute geometry. The article closes with a discussion of the role of the Pasch axiom in ordered metric geometry. An axiomatization of ordered metric planes in a first-order language is provided in an Appendix.

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1. Introduction

Metric geometry in the sense of Hjelmslev [9,10] and Bachmann [1,2] is independent of any assumption about order and continuity. A relation of order is introduced in the literature in an additional step by means of an undefined notion of betweenness which satisfies one-dimensional axioms (which characterize the linear order of collinear points) and a single plane order axiom which was proposed by Pasch [19] and Hilbert [8] and is commonly referred to as the *Pasch axiom* (see Pejas [20] and [21], Bachmann [1], Hessenberg and Diller [7], Ewald [5]).

However, in the plane absolute geometry (in the sense of Bachmann [1]; we call the models Bachmann planes) the Pasch axiom holds if and only if the associated field of coordinates (a) is orderable and (b) induces a convex order of the plane (i.e., the set of points of the plane is a convex subset of the set of points of the associated ideal plane; see Pejas [21]). The smallest subplane of the real Euclidean plane which is a Bachmann plane is a 'natural' example of a metric plane with an orderable field of coordinates which does not satisfy the Pasch axiom (see Sect. 5).

Moreover, the Pasch axiom is formulated on the basis of Hilbert's plane axioms of incidence (the axioms I, 1–3 in [8]) which postulate the existence and uniqueness of a joining line. These assumptions are satisfied by Bachmann planes but not necessarily by Cayley–Klein geometries (see H. Struve and R. Struve [23] and [24]). Galilean, Minkowskian, cominkowskian and coeuclidean planes, for example, contain points which have more than one resp. no joining line at all (see Bachmann [2]).

It is the aim of this article

(a) to show how an order relation can be introduced for the widest class of metric geometries, which do not have to satisfy any additional axioms such as free mobility or some form of a parallel axiom or assumptions about the existence of joining lines;
(b) to show that the order of a Bachmann plane corresponds to an order of the associated field of coordinates (which extends the correspondence between geometrical and algebraical order structures, which is well-known in affine and projective geometry,[1] to plane absolute geometry);
(c) to discuss the role of the Pasch axiom in ordered metric geometry.

In Sect. 2 we define the axiomatic basis and introduce metric planes[2] of a very general kind where no assumptions are made about order or free mobility and about the existence and uniqueness of joining lines. Plane metric geometry in this sense is a common generalization of Hjelmslev's Allgemeine Kongruenzlehre [10] and the plane absolute geometry of Bachmann [1].

To describe the order structure of a metric plane we follow an idea of Sperner [22] and call a metric plane *orderable* if every line is orderable and if every line admits a partition into sides which is compatible with the linear order of lines (see Definition 3.9).

Sperner [22][3] defines a partition into sides of a line g as a partition of the set of points of the plane. We prefer to define a partition into sides of g as a partition

[1]See Karzel and Kroll [11] and Pambuccian [17].
[2]These planes are called in [2] non-elliptic Hjelmslev planes.
[3]cf. Ellers and Karzel [3, Ch. 6] and Karzel et al. [12].

of the set of translations. This corresponds to the intuitive idea that the half-planes of g correspond to sets of translations in 'opposite directions' (which contain with two elements their product—if the product is a translation—but with an element $\tau \neq 1$ not the inverse element τ^{-1}).

In the group-theoretical approach of Hjelmslev and Bachmann this way is preferable since the set T of translations of a metric plane has much more group-theoretical structure than the set of points: T is a *partial group*[4] (i.e., an invariant subset of a group G which contains the identity element of G and with an element τ the inverse element τ^{-1}) with an algebraical structure which turns out to be very rich.

In an orderable metric plane the betweenness relation for collinear points satisfies the 1-dimensional universal properties of linear order (see Pambuccian [16] and [17]) and the order structure is compatible with the incidence and metric of the underlying plane, i.e., the following additional properties hold: The relation of betweenness for collinear points A, B, C is independent of the choice of the joining line of A, B, C; lines which have two common points A and B are incident with all points which lie between A and B (they have a common segment); the relation of betweenness is invariant under orthogonal projections; motions preserve the relation of betweenness.

An orderable metric plane does not necessarily satisfy the Pasch axiom, but a weak form of the Pasch axiom holds (if A, B, C are points which are not incident with a line g and if A, B are on different sides of g then A, C or B, C lie on different sides of g).

In Sect. 4 we discuss the role of the full Pasch axiom in ordered metric geometry. It is shown that in a metric plane the Pasch axiom holds if and only if two axioms hold, namely the axiom of the existence of a joining line and an axiom that states that lines a, g have no point of intersection if and only if all points of a lie on the same side of g.

In Sect. 5 we extend the relationship between ordered geometrical structures and ordered algebraical structures to plane absolute geometry and show that the ordering of a Bachmann plane corresponds to an ordering of the associated field of coordinates.

We prove that the ideal plane of an ordered Bachmann plane is an ordered affine plane and this implies—as is well-known—that the field of coordinates is orderable. Our proof method provides a substitute for the consideration of germs of orderings (see Hessenberg and Diller [7, §61]) and for the construction of a singular pseudo-metric of a metric plane (see Kunze [13]).

We close Sect. 5 by an example of a Pasch-free orderable Bachmann plane and by further examples of Cayley–Klein geometries which are orderable if and only if their field of coordinates is orderable.

[4] This term was coined by Birkhoff [15, p. 18].

In an Appendix we axiomatize ordered metric planes in a first-order language. By this we do not claim that our axiom system is simple or preferable to its competitors, but simply that the theory can be expressed in a first-order language. For reasons of simplicity we consider planes with the property that any pair of distinct points has a unique joining line (Bachmann planes).

The article generalizes the results of [25]—where singular[5] metric planes were considered—to include the non-singular case (all definitions and theorems hold for singular and non-singular planes). We refer to proofs in [25] whenever possible, i.e., if no modifications are necessary since the assumption that the underlying metric plane is singular is not used.

2. Metric geometry

Hjelmslev [9,10], Bachmann [1] et. al. showed that metric geometry can be formulated in the group of motions and that the calculus of reflections allows it to axiomatize and to coordinatize the classical Euclidean and non-Euclidean geometries over fields of characteristic $\neq 2$.

We follow this approach and choose as starting point of our investigations the group-theoretic axiom system of Bachmann [2] for plane metric geometry which is a common generalization of Hjelmslev's Allgemeine Kongruenzlehre [10] and the plane absolute geometry treated in Bachmann [1]. No assumptions are made about order or free mobility and about the existence and uniqueness of joining lines.

Basic assumption Let G be a group which is generated by an invariant set S of involutory elements.

Notation The elements of S will be denoted by lower case latin letters. The set of involutory elements of S^2 will be denoted by P and their elements by upper case letters $A, B, \ldots$ The 'stroke relation' $\alpha \mid \beta$ is an abbreviation for the statement that α, β and $\alpha\beta$ are involutory elements. The statement $\alpha \mid \delta$ and $\beta \mid \delta$ is abbreviated by $\alpha, \beta \mid \delta$.

Axiom A1. *For A, b there exists c with $A, b \mid c$.*

Axiom A2. *If $A, b \mid c, d$ then $c = d$.*

Axiom A3. *If $a, b, c \mid e$ then $abc \in S$.*

Axiom A4. *If $a, b, c \mid E$ then $abc \in S$.*

Axiom X. *There exist a, b with $a \mid b$.*

The axiom system is satisfied by the group G of motions of Euclidean, hyperbolic, Galilean or Minkowskian planes and of other classical geometries (see [1] and [2]), with the set S of line-reflections and the set P of point-reflections

[5] A metric plane is called *singular* if the translations form a group (or equivalently if in any quadrilateral with three right angles the fourth angle is a right one).

and the stroke relation which describes the orthogonality of lines (if restricted to $S \times S$) and the incidence of points and lines (if restricted to $S \times P$ resp. $P \times S$).

We do not consider elliptic planes (see [2, §1.7]) since in this special case an order relation on a set of collinear points is a cyclic order and not a linear one (cp. Pambuccian [17, section 2.5]).

According to *Axiom A1* and *Axiom A2* there is a unique perpendicular from a point to a line and according to *Axiom A3* and *Axiom A4* the *theorem of three reflections* holds: If three lines have a common point or a common perpendicular, then the product of the reflections in these lines is a line reflection.

We call the geometrical structure described by this axiom system a *metric plane*. In a metric plane there are points, lines, motions and relations such as incidence, orthogonality etc. defined which satisfy the axioms given above. In the group-theoretical terminology a pair (G, S) which satisfies the basic assumption and the axioms *A1, A2, A3, A4* and X is called a (non-elliptic) *Hjelmslev group*.

A metric plane can contain points with more than one joining line. A quadruple (A, B, c, d) with $A, B \mid c, d$ and $A \neq B$ and $c \neq d$ is called a *double incidence*. The metric plane is called a *Bachmann plane* if the axioms *A1* and *A2* are replaced by the existence and uniqueness of a joining line (*For* A, B *with* $A \neq B$ *there exists a unique element* c *with* $A, B \mid c$) and if there exist three lines a, b, c in a general position (*There exist* a, b, c *such that* $a \mid b$ *and neither* $a \mid c$ *nor* $b \mid c$ *nor* $ab \mid c$). The pair (G, S) is called a (non-elliptic) *Bachmann group*.

A product Ab with $A \in P$ and $b \in S$ is called a *glide reflection*. If A, b are not incident then Ab has a unique fixed line (the perpendicular from A to b) which is called the *axis* of the glide reflection. Every product abc with $a, b, c \in S$ is a glide reflection (see Bachmann [2, §3.2]).

The products AB with $A, B \in P$ are called *translations*. According to *Axiom A3*, the set $T_g = \{AB : A, B \mid g\}$ of translations along a line g forms an abelian group. If the set $T = \{AB : A, B \in P\}$ of all translations contains with two translations their product then $(T, \cdot)$ is a group and (G, S) is called *singular*.

In the general case the product of two translations need not to be a translation and $(T, \cdot)$ is a substructure of G with a partially defined binary associative operation (the restriction of the group operation of G to $T \times T$). The set T is an invariant subset of G which contains the identity element 1 of G and with a translation AB the inverse element BA.

Birkhoff introduced the concept of a *partial algebra* (to study subsets of universal algebras rather than subalgebras) and coined the term of a *partial group* [15, p. 18].

Definition 2.1. Let T be a subset of a group G which contains the identity element 1 of G and with an element α the inverse element α^{-1}. Then $(T, *)$ with the restriction $*$ of the group operation of G to $T \times T$ is called a *partial group*.

In this article a partial group T always denotes the *partial group of translations* of a Hjelmslev group (G, S). The next definition generalizes a notion of the theory of ordered groups (see Blyth [4] or Fuchs [6]).

Definition 2.2. A subset C of the partial group T of translations is called a *cone* of T if the following properties hold:

(1) If $\alpha, \beta \in C$ and $\alpha\beta \in T$ then $\alpha\beta \in C$.
(2) If $\alpha \in C$ and $\alpha \neq 1$ then $\alpha^{-1} \notin C$.
(3) $1 \in C$.

If the set T of translations is a group then any two translations commute and a cone of T is the positive cone of a partial ordering of the abelian group T.

If C^+ and C^- are cones of a partial group T with $C^+ \cap C^- = \{1\}$ and $C^+ \cup C^- = T$, then we say that $T = C^+ \cup C^-$ is a *partition into cones*. If X is a subset of a group G then we shall use the notation $X^{-1} = \{\alpha^{-1} : \alpha \in X\}$.

Lemma 2.3. *If $T = C^+ \cup C^-$ is a partition into cones of a partial group T then C^+ and C^- are inverse cones, i.e., $C^+ = (C^-)^{-1}$.*

Proof. If $T = C^+ \cup C^-$ is a partition into cones then $\{\alpha^{-1} : \alpha \in C^+\} \subseteq C^-$ and $\{\alpha^{-1} : \alpha \in C^-\} \subseteq C^+$ (since C^+ and C^- contain no invertible elements $\neq 1$) and hence $C^+ = (C^-)^{-1}$. □

3. Ordered metric planes

In H. Struve and R. Struve [25] it is shown that the order structure of a *singular* metric plane corresponds to an order structure of the group of translations of the plane. We generalize this approach to include metric planes which are non-singular.

We follow the lines of argumentation of [25]. Most theorems of this section are proved for the singular case in [25, Section 3]. We refer to proofs in [25] whenever possible, i.e., if no modifications are necessary since the assumption that the underlying metric plane is singular is not used.

For the reader's convenience we reproduce the following definitions.

Definition 3.1. A line g of a metric plane is called *orderable* if the group T_g of translations along g can be linearly ordered, i.e. if T_g admits a partition into cones.

Let g be an orderable line and $T_g = T_g^+ \cup T_g^-$ the associated partition into cones of the group of translations along g.

Definition 3.2. Let O be a point of an orderable line g. The set of points $L_g^+ = \{A : OA \in T_g^+ \text{ and } A \neq O\}$ and $L_g^- = \{A : OA \in T_g^- \text{ and } A \neq O\}$ are called the *halflines* of g with origin O.

Definition 3.3. If A, B, C are points of g then B lies *between* A and C if $AB, BC \in T_g^+$ or $AB, BC \in T_g^-$. If B lies between A and C we write $(A.B.C)_g$ or just $(A.B.C)$ if it is obvious which line g is considered.

Since $1 \in T_g^+$ and $1 \in T_g^-$ the points A, B, C are not supposed to be distinct. If $A = B$ or $B = C$ then $(A.B.C)_g$. We refer to the variant that A, B, C are assumed to be distinct points as *strict betweenness* (cp. Pambuccian [17]).

A partition $T_g = T_g^+ \cup T_g^-$ induces two dual binary order relations on g which can be defined for points A, B of g by $A \leq B$ if and only if $AB \in T_g^+$ and $A \geq B$ if and only if $AB \in T_g^-$.

Given three distinct points on g one and only one of them lies between the other two and the 1-dimensional universal properties of linear order hold (see [25, Section 3.2]).

Remark on a betweenness relation for points without a joining line. Definition 3.3 can be generalized. We call a set $\overline{P}$ of points (linearly) orderable if the associated set $\{AB : A, B \in \overline{P}\}$ of translations generates a group $\overline{T}$ of translations which is linearly orderable (i.e., $\overline{T}$ admits a partition into cones $\overline{T}_1$ and $\overline{T}_2$). If $\overline{T} \cap S^2 = \{1\}$ then points A, B, C with $AB, BC \in \overline{T}$ have no joining line (see [2, Lemma 3.9]) but a betweenness relation can be defined by $(A.B.C)$ if and only if $AB, BC \in \overline{T}_1$ or $AB, BC \in \overline{T}_2$. For an example we refer to an ideal line $\mathbf{x}$ of a Minkowskian plane (see [25, Section 4]) which has the property that the set of translations along $\mathbf{x}$ forms an abelian group $T_{\mathbf{x}}$ with $T_{\mathbf{x}} \cap S^2 = \{1\}$.

In an ordered metric plane the halfplanes of a line g correspond to sets of translations in 'opposite directions' which contain with two elements their product (if the product is a translation) but with an element $\tau \neq 1$ not the inverse element τ^{-1} (i.e., the sets of translations are cones).

Definition 3.4. A line g admits a *partition into sides* if there exist cones C_g^+ and C_g^- of the partial group T of translations of the metric plane such that $T = T_g \cup C_g^+ \cup C_g^-$ and $C_g^+ \cap C_g^- = T_g \cap C_g^+ = T_g \cap C_g^- = \{1\}$.

It is easily seen that C_g^+ and C_g^- are inverse cones, i.e., $C_g^+ = (C_g^-)^{-1}$

Theorem 3.5. *Let $T = T_g \cup C_g^+ \cup C_g^-$ be a partition into sides of a line g and O, Q two different points of g and $A \neq O, Q$. Then $OA \in C_g^+$ if and only if $QA \in C_g^+$ and $OA \in C_g^-$ if and only if $QA \in C_g^-$.*

Proof. See H. Struve and R. Struve [25, Theorem 3.12]. □

Theorem 3.5 allows the definition of the *sides of a line* g.

Definition 3.6. Let g be a line which admits a partition $T = T_g \cup C_g^+ \cup C_g^-$ into sides.

(a) The set of points $H_g^+ = \{A : OA \in C_g^+ \text{ and } O \mid g \text{ and } A \neq O\}$ and $H_g^- = \{A : OA \in C_g^- \text{ and } O \mid g \text{ and } A \neq O\}$ are called the *sides* of g (or the *halfplanes* determined by g).

(b) The *closed halfplanes* are the sets of points $\bar{H}_g^+ = H_g^+ \cup P_g$ and $\bar{H}_g^- = H_g^- \cup P_g$ (if P_g denotes the set of points of g).
(c) If $A, B \in H_g^+$ or $A, B \in H_g^-$ we say that A, B lie *on the same side* of g which we denote by $g \upharpoonright A, B$.

According to the next theorem the reflection in a line g and the reflection in a point of g interchange the sides of g.

Theorem 3.7. *Let $T = T_g \cup C_g^+ \cup C_g^-$ be a partition into sides of a line g and $O|g$ and $h|O, g$. Then the following holds:*

(a) *Let $\tau = OA$ and $\tau \neq 1$. If $\tau \in C_g^+$ and $\tau' \in T_g$ then $\tau\tau' \in C_g^+$.*
(b) If $OA \in C_g^+$ then $(OA)^O \in C_g^-$ and $(OA)^g \in C_g^-$ and $(OA)^h \in C_g^+$.

Proof. (a) Let $OA \in C_g^+$ with $A \neq O$ and $OQ \in T_g$ with $O, Q|g$. Then $OA^O = AO = (OA)^{-1} \in C_g^-$ (since C_g^+ and C_g^- are inverse cones) and $QA^O \in C_g^-$ (according to Theorem 3.5). Hence $OA \cdot OQ = A^O Q = (QA^O)^{-1} \in C_g^+$ (since C_g^+ and C_g^- are inverse cones).

(b) See H. Struve and R. Struve [25, Theorem 3.13, (b)]. □

The next theorem summarizes some properties of the relation $g \upharpoonright A, B$.

Theorem 3.8. *Let g be a line which admits a partition into sides. Then the following holds:*

(a) *If $g \upharpoonright A, B$ then A, B are not incident with g.*
(b) *The relation $\upharpoonright$ is an equivalence relation on the set of points which are not incident with g.*
(c) *If A, B are points of a line h which has a common perpendicular with g and if $h \neq g$ then $g \upharpoonright A, B$.*

Proof. (a) and (b) are immediate consequences of Definition 3.6.

(c) Let g and h be two distinct lines with a common perpendicular e and $E|e, h$ and $O|e, g$ and $OE \in C_g^+$. It is sufficient to show that $A|h$ with $A \neq E$ implies $g \upharpoonright A, E$, i.e., $OA \in C_g^+$.

We consider the cases $EA \in T_g$ and $EA \in C_g^+$ and $EA \in C_g^-$. If $EA \in T_g$ then $OA = OE \cdot EA \in C_g^+$ according to Theorem 3.7, (a) and hence $g \upharpoonright A, E$.

If $EA \in C_g^+$ then $OA = OE \cdot EA \in C_g^+$ (since a cone contains with two elements their product) and hence $g \upharpoonright A, E$.

If $EA \in C_g^-$ then $AE \in C_g^+$ and $OE \cdot AE = OA^E \in C_g^+$ and $(OA^E)^e \in C_g^+$ (according to Theorem 3.7, (b)). Hence $(OA^E)^e = O^e \cdot A^{Ee} = O \cdot A^h = OA \in C_g^+$ which proves $g \upharpoonright A, E$. □

A line g splits the set of points, which are not incident with g, into two disjoint classes (the halfplanes determined by g). Hence the following weak form of the Pasch axiom holds:

($\star$) **Weak Pasch axiom** *Let A, B, C be points which are not incident with a line g which admits a partition into sides. If A, B are on different sides of g then A, C or B, C lie on different sides of g.*

We call the partition into sides of a line g and the linear order of a line a *compatible* if a closed halfplane of g which contains one point of a halfline L_a^+ contains all points of L_a^+. This is according to Definition 3.2 and Definition 3.6 equivalent with the following condition:

(*) *If $T_a^+ \cap C_g^+ \neq \{1\}$ then $T_a^+ \cap C_g^- = \{1\}$, i.e., $T_a^+ \subseteq C_g^+ \cup T_g$.*

Please note that (*) is equivalent with the contraposition: If $T_a^+ \cap C_g^- \neq \{1\}$ then $T_a^+ \cap C_g^+ = \{1\}$, i.e., $T_a^+ \subseteq C_g^- \cup T_g$. Since T_a^+ and T_a^- resp. C_g^+ and C_g^- are inverse cones $T_a^+ \cap C_g^+ \neq \{1\}$ implies $T_a^- \cap C_g^- \neq \{1\}$ and hence $T_a^+ \subseteq C_g^+ \cup T_g$ implies $T_a^- \subseteq C_g^- \cup T_g$.

Definition 3.9. A metric plane is called *orderable* if the following conditions hold:

(1) Every line is orderable.
(2) Every line admits a partition into sides.
(3) For any two lines a and g, the linear order of a is compatible with the partition into sides of g.

We now show that the order structure of a metric plane is compatible with the incidence structure and start with the theorem that the relation of betweenness for collinear points A, B, C is independent of the choice of the joining line of A, B, C. This justifies the notation $(A.B.C)$ if B lies between A and C.

Theorem 3.10. *If $A, B, C \,|\, g, h$ and $(A.B.C)_g$ then $(A.B.C)_h$.*

Proof. See H. Struve and R. Struve [25, Theorem 3.15]. □

Lines which have two common points A and B are incident with all points which lie between A and B, i.e., they have a common segment.

Theorem 3.11. *Let $A, B, C \,|\, g$ and $(A.B.C)$. If $A, C \,|\, h$ then $B \,|\, h$.*

Proof. See H. Struve and R. Struve [25, Theorem 3.16]. □

The next two theorems describe the relationship between the betweenness relation and the partition into sides of lines of the metric plane.

Theorem 3.12. *Let A, B, C be collinear points and $AB, BC \notin T_g$. Then $(A.B.C)$ if and only if $AB, BC \in C_g^+$ or $AB, BC \in C_g^-$.*

Proof. See H. Struve and R. Struve [25, Theorem 3.17]. □

Theorem 3.13. *Let A, B, C be three distinct points of a line a and $g \mid B$ and $g \nmid A, C$. Then $(A.B.C)$ holds if and only if A, C lie on different sides of g.*

Proof. See H. Struve and R. Struve [25, Theorem 3.18]. □

A consequence of the last theorem is that the betweenness relation for collinear points can as well be introduced by the relation '$g \restriction A, B$'.

If A, B, C are three different points of g then B lies between A and C if and only if A, C do not lie on the same side of any line through B.

Next, we show that the relation of betweenness is compatible with the metric of the plane.

Theorem 3.14. *The relation of betweenness is invariant under orthogonal projections.*

Proof. See H. Struve and R. Struve [25, Theorem 3.19]. □

Conversely the following theorem holds:

Theorem 3.15. *Let A, B, C be three distinct points of a line g and A', B', C' the feet of perpendiculars from A, B, C to a line h. If A', B', C' are three distinct points with $(A'.B'.C')$ then $(A.B.C)$.*

Proof. See H. Struve and R. Struve [25, Theorem 3.20]. □

A line a of a metric plane may have several common points with lines b, c with $b \mid c$ (see Bachmann [2, § 5.6]). In this case a is called a *winding line*. This phenomenon cannot occur if the metric plane is orderable.

Theorem 3.16. *There are no winding lines.*

Proof. See H. Struve and R. Struve [25, Theorem 3.21]. □

Theorem 3.16 is used as an additional axiom—called the 'grid-axiom'—by Hjelmslev [10, 3. Mitt. 11] and Bachmann [2, § 11.6]. In [2] the relationship between this axiom and the existence of rotations is studied. We now prove that the grid-axiom is equivalent to a statement about translations. This equivalence is essential for the generalization of the singular case to the non-singular one.

Theorem 3.17. *Let (G, S) be a Hjelmslev group. Then the following properties are equivalent:*

(a) *There are no winding lines.*
(b) *If a, b, g are lines with $a|b$ then $T_a \cap T_g = \{1\}$ or $T_b \cap T_g = \{1\}$.*

Proof. (b) ⇒ (a). We show the contraposition. Let g be a line which has several common points with lines a, b with $a \mid b$. If A, B, O are distinct points with $O = ab$ and $A|a$ and $B|b$ and $A, B, O|g$ then $AO \in T_a \cap T_g$ and $OB \in T_b \cap T_g$ which proves $T_a \cap T_g \neq \{1\}$ and $T_b \cap T_g \neq \{1\}$.

(a) ⇒ (b). We show the contraposition. Let a, b be lines with $a|b$ and $O = ab$. Suppose g is a line such that $T_a \cap T_g \neq \{1\}$ and $T_b \cap T_g \neq \{1\}$. Then there exist points U, V, X, Y with $U \neq V$ and $X \neq Y$ and $UV \in T_a \cap T_g$ and $XY \in T_b \cap T_g$. Hence $a^{UV} = a$ and $b^{XY} = b$. According to Bachmann [2, § 10.1, (3)] there exist points A', B', C', D' with $A', B' \mid a$ and $A'B' = UV$ and $C', D' \mid b$ and $C'D' = XY$.

Let $A = A'B'O$ and $B = OC'D'$ (*Axiom A3* of a Hjelmslev group implies that A, B are points of a resp. b). Then it is $A'B' = AO$ and $C'D' = OB$. Let c, d, e be the lines through A, B resp. O which are perpendicular to g.

Since g is a fixed line of the translations AO and OB there exist joining lines $a' \,|\, O, A$ and $b' \,|\, O, B$ with $Aa', Oa', Ob', Bb' \,|\, g$, i.e., $Aa' = c$ and $Bb' = d$ and $Oa' = e$ and $Ob' = e$ (according to Bachmann [2, § 10.1, (6)]). Hence $Oa' = Ob'$ and $a' = b'$ and $A, B, O \,|\, a'$. This shows that a' is a winding line. □

We note that the proof of Theorem 3.17 does not depend in any way on the orderability of the Hjelmslev group.

Theorem 3.18. *Motions preserve the relation of betweenness.*

Proof. Let A, B, C be three different points of a line g with $(A.B.C)$. Since the group G is generated by the set S of line reflections, it is sufficient to prove that $(A^h.B^h.C^h)$ holds for every $h \in S$. Let a, b resp. c be the perpendiculars from A, B, C to h.

Since $c \mid h$ it is $T_g \cap T_c = \{1\}$ or $T_g \cap T_h = \{1\}$ (according to the Theorems 3.16 and 3.17). If $AB, BC \notin T_c$ (case 1) we can assume $AB, BC \in C_c^+$ (see Theorem 3.12). Hence $A^hB^h, B^hC^h \in C_c^+$ and $(A^h.B^h.C^h)$ (according to the Theorems 3.7, (b) and 3.12).

If $AB, BC \notin T_h$ (case 2) we can assume $AB, BC \in C_h^+$ (see Theorem 3.12). Hence $A^hB^h, B^hC^h \in C_h^-$ and $(A^h.B^h.C^h)$ (according to the Theorems 3.7, (b) and 3.12). □

3.1. Metric planes without double incidences

In this section we consider orderable metric planes which contain no points which have more than one joining line.

Let g and h be lines which admit partitions into sides $T = T_g \cup C_g^+ \cup C_g^-$ and $T = T_h \cup C_h^+ \cup C_h^-$. The partition into sides of g induces a partition into cones $T_a = (T_a \cap C_g^+) \cup (T_a \cap C_g^-)$ of the group of translations along a line a with $a \neq g$ and hence a linear order of the group T_a.

We call the partitions into sides of g and of h *compatible*, if the following condition holds:

(†) Let σ, τ be translations along a. If $\sigma, \tau \in C_g^+$ and $\sigma \in C_h^+$ then $\tau \in C_h^+$.

According to (†) the partitions into sides of g and h induce the same linear order on any line a which is distinct from g and h. If the metric plane has no points with more than one joining line then condition (†) is equivalent with (*) of Sect. 3. Hence the following theorem holds.

Theorem 3.19. *A metric plane without double incidences is orderable if and only if*

(1) *Every line admits a partition into sides.*
(2) *The partition into sides of any two lines g and h are compatible, i.e. they induce the same linear order on any line a with $a \neq g, h$.*

The relation $g \upharpoonright A, B$ has the following further properties (i) and (ii) if there are no points with more than one joining line. For a proof we can refer to [25].

(i) *Let* $A, B, C \mid k$ *and* $B \neq C$ *and* $A \mid a$ *and* $B \mid b$ *and* $C \mid c$ *and* $b, c \neq k$. *Then* $a \restriction B, C$ *if and only if either* $b \restriction A, C$ *or* $c \restriction A, B$.

(ii) *If* $A, B, C \mid k$ *and* $A, B \neq C$ *and* $g, h \mid C$ *and* $A, B \nmid g, h$ *then* $g \restriction A, B$ *if and only if* $h \restriction A, B$.

The properties (i) and (ii) were used as axioms in Karzel et al. [12, § 13]. The statement (ii) is the so-called *Geradenrelation* (see Sperner [22, § 3]).

We close this section with a theorem which is well known in orderable affine planes (a projection along lines through a point O preserves the betweenness relation on 'parallel' lines, i.e., on lines which have a common perpendicular through O).

Theorem 3.20. *Let* g *and* g' *be lines with a common perpendicular* k *and* O *a point on* k *which is not incident with* g *and* g'. *If* D, E, F *are points of* g *and* D', E', F' *points of* g' *and* d, e, f *lines through* O *with* $d \mid D, D'$ *and* $e \mid E, E'$ *and* $f \mid F, F'$ *then are equivalent* $(D.E.F)$ *and* $(D'.E'.F')$.

Proof. Let D, E, F, D', E', F', O and d, e, f, g, g', k be lines with the properties of the theorem and h the line with $h \mid O, k$. Then $h \restriction D, E, F$ and $h \restriction D', E', F'$ according to Theorem 3.8, (c). Since the reflection in O interchanges the sides of h (according to Theorem 3.7, (b)) and preserves the betweenness relation, we can assume that all points D, E, F, D', E', F' lie on the same side of h and that (O, D, D') and (O, E, E') and (O, F, F') hold.

Let $T = T_e \cup C_e^+ \cup C_e^-$ be the partition into sides of e. Since $(D.E.F)$ we can assume $ED \in C_e^+$ and $EF \in C_e^-$. For the proof of $(D'.E'.F')$ it is sufficient to show $E'D' \in C_e^+$ and $E'F' \in C_e^-$.

The points D', E', F' of g' lie on the same side of g (since k is a common perpendicular of g and g'). If $T = T_g \cup C_g^+ \cup C_g^-$ is the partition into sides of g then we can assume $DD', EE', FF' \in C_g^+$.

(O, D, D') and $D \mid g$ imply that O and D' lie on different sides of g, i.e., $DO, EO, FO \in C_g^-$ and hence $OD, OE, OF \in C_g^+$ and $OD', OE', OF' \in C_g^+$ (since $OD' = OD \cdot DD'$).

Since $ED \in C_e^+$ and $E, O \mid e$ it is $OD \in C_e^+$ and since the partitions into sides of e and g are compatible it is $OD, DD', OD' \in C_e^+$ according to (†). By Theorem 3.5 the statement $OD' \in C_e^+$ implies $E'D' \in C_e^+$.

An analogical argument shows $E'F' \in C_e^-$. This proves the theorem. □

4. The Pasch axiom in ordered metric geometry

The Pasch axiom was formulated by Hilbert [8] under the assumption that any two distinct points are incident with a unique line. This assumption does not necessarily hold in metric planes. Hence we have to formulate the Pasch axiom in a more detailed way (which is equivalent with Hilbert's formulation if any two points have a unique joining line).

Pasch axiom *Let A, B, C be three non-collinear points and g a line which is not incident with A, B or C. If g is incident with a point X on a joining line of A, B, which lies between A and B, then g is incident either with a point Y on a joining line of B, C which lies between B and C or with a point Z on a joining line of A, C which lies between A and C.*

We want to characterize the ordered metric planes which satisfy the Pasch axiom. Since the weak Pasch axiom (see Sect. 3) holds in every ordered metric plane we ask—in other words—for the missing link between the weak and the full Pasch axiom. To answer this question we introduce the following axioms.

Axiom A1. For A, B there exists c with $A, B \mid c$.*

Definition 4.1. We say that a line a lies on a side of a line g (which we denote by $g \restriction a$) if all points of a lie on the same side of g.

Axiom L. Lines a, g have no point of intersection if and only if $a \restriction g$.

According to *Axiom A1** any two points have a joining line. According to *Axiom L* two points A and B lie on different sides of a line g if and only if A and B have a joining line which intersects g.

Theorem 4.2. *The Pasch axiom holds in an orderable metric plane if and only if Axiom A1* and Axiom L hold.*

Proof. See H. Struve and R. Struve [25, Theorem 5.3]. □

The Pasch axiom implies the existence but not the uniqueness of a joining line (the real Galilean plane contains points with more than one joining line and the order of the field of real numbers induces an order of the metric plane which satisfies the Pasch axiom; see [25, Section 4]).

5. Coordinatization of ordered metric planes

In the preceding section we defined the concept of an ordered metric plane. We now show that the order of the geometrical structure corresponds to an order of the associated algebraic structure (the field of coordinates).

Bachmann planes. Let (G, S) be a (non-elliptic) Bachmann plane, i.e., a metric plane with the property that any two points have a unique joining line.

According to the main theorem of [1, Sections 6 and 11] a Bachmann plane can be extended to a pappian projective plane (the *projective ideal plane*) by introducing ideal points and ideal lines. The *ideal points* are the pencils of lines $S(ab) = \{c : abc \in S\}$ with $a \neq b$. The set of lines through a point E is called a *proper pencil* (or a proper ideal point). The proper pencils correspond in a one-to-one way to the points of the Bachmann plane.

Let O be a fixed point. Ideal lines are defined by means of contractions with center O (see [1, p. 307]). A contraction is a mapping from S into S which is induced by the product of two semi-rotations χ_{uv} and χ_{vu} about O (with lines

$u, v \mid O$ and $u \nmid v$) which map a line a on the axis of the glide reflections auv resp. avu.

For contractions the following holds (see [1, §6,2]):

(†) A contraction maps a proper pencil into a proper pencil.
(‡) For any improper pencil which is not a pencil of perpendiculars of a line through O there exists a contraction with center O which takes the improper pencil into a proper one.

Ideal lines are sets of ideal points. A set of pencils that can be transformed by a contraction with center O into the set of pencils which have a common line g is called an *ideal line*. An ideal line whose pencils have a common line a is a *proper (ideal) line*.

The set of pencils of perpendiculars for a line through O is called the *line at infinity* of the projective ideal plane. We denote the affine specialization with respect to this line at infinity by $\mathcal{A}$. A non-elliptic Bachmann plane can be represented as a subplane of $\mathcal{A}$ which contains with every point all lines of $\mathcal{A}$ which are incident with this point.

A contraction with center O of the Bachmann plane induces a dilatation with center O of $\mathcal{A}$, i.e., a collineation with fixed point O which maps every line of $\mathcal{A}$ onto a parallel line (see Bachmann [1, p. 307] and [3, p. 79]). We denote the group of dilatations of $\mathcal{A}$ which is generated by the set of contractions with center O of the Bachmann plane by $\mathcal{D}(O)$. The group $\mathcal{D}(O)$ is a subgroup of the full group of dilatations of $\mathcal{A}$ with center O. According to (†) and (‡) any finite set of collinear points of $\mathcal{A}$ can be mapped by an element of $\mathcal{D}(O)$ onto a set of collinear points of the Bachmann plane.

We now show that the ideal affine plane of an orderable Bachmann plane is orderable and this implies—as is well-known—that the field of coordinates is orderable.

We start with an extension of the notion of beetweenness of an ordered Bachmann plane to the affine ideal plane $\mathcal{A}$ and define that a point B of $\mathcal{A}$ lies between points A and C of $\mathcal{A}$ (which we denote by $[A.B.C]$) if A, B, C are collinear and if there exists a dilatation of $\mathcal{D}(O)$ which maps A, B, C into points A', B', C' of the Bachmann plane with $(A'.B'.C')$.

We show that this definition does not depend on the choice of the dilatation of $\mathcal{D}(O)$: Let A, B, C be points of $\mathcal{A}$ and $\delta, \kappa \in \mathcal{D}(O)$ dilatations which map A, B, C onto collinear points $A\delta, B\delta, C\delta$ resp. $A\kappa, B\kappa, C\kappa$ of an orderable Bachmann plane. The points $A\delta, B\delta, C\delta$ are mapped by the dilatation $\delta^{-1}\kappa$ of $\mathcal{A}$ onto the points $A\kappa, B\kappa$ resp. $C\kappa$. Hence $(A\delta.B\delta.C\delta)$ is equivalent with $(A\kappa.B\kappa.C\kappa)$ (according to Theorem 3.20). This shows that the betweenness relation on $\mathcal{A}$ is well defined.

Since dilatations κ and κ^{-1} of $\mathcal{A}$ with center O are collineations of $\mathcal{A}$ there are equivalent $[A.B.C]$ and $[A\kappa.B\kappa.C\kappa]$ and $(A\kappa.B\kappa.C\kappa)$ (if $A\kappa, B\kappa$ and $C\kappa$ are points of the Bachmann plane).

Next, we extend the notion of sidedness of an orderable Bachmann plane and say that two distinct points A, B of $\mathcal{A}$, which are not incident with a line g of $\mathcal{A}$, are *on the same side of* g (which we denote by $g \upharpoonleft\!\downharpoonright A, B$) if g and the joining line h of A and B are parallel lines or if g and h have a common point Q with $[Q.A.B]$ or $[Q.B.A]$. If A, B are not on the same side of g (which is equivalent with $[A.Q.B]$) we say that A, B are *on different sides of* g.

Since dilatations κ and κ^{-1} of $\mathcal{A}$ with center O are collineations and preserve the betweenness relation the statements $g \upharpoonleft\!\downharpoonright A, B$ and $g\kappa \upharpoonleft\!\downharpoonright A\kappa, B\kappa$ and $g\kappa \upharpoonright A\kappa, B\kappa$ (if $A\kappa$ and $B\kappa$ are points of the Bachmann plane) are equivalent.

Hence the properties of the relation $g \upharpoonright A, B$ of an ordered Bachmann plane which we proved in Sect. 3 (i.e., Theorem 3.8, (a), (b), the weak Pasch axiom and the properties (i) and (ii) in Sect. 3.1) are also satisfied by the points, lines and the relation $g \upharpoonleft\!\downharpoonright A, B$ of the affine plane $\mathcal{A}$. This shows that the field of coordinates of $\mathcal{A}$ is orderable (see Sperner [22] and Ellers and Karzel [3, Chapter 6] or Karzel et al. [12]).

Remark. Our proof method provides a substitute for the consideration of germs of orderings (see Hessenberg and Diller [7, §61]) and for the construction of a singular pseudo-metric of a Bachmann plane (see Kunze [13]). For an alternative introduction of an order relation in hyperbolic geometry based on the calculus of reflections see R. Struve [26].

We close this section with a remark on how the partition into sides of a line g of a Bachmann plane can be obtained if the associated field K of coordinates is orderable. An order of K induces in a well-known way a linear order on the group T_g of translations along a line g of the Bachmann plane. It remains to show how a partition $T = T_g \cup C_g^+ \cup C_g^-$ of the partial group T of translations with cones C_g^+ and C_g^- of T can be obtained (see Definition 3.4).

Let O be a point on g and $h = Og$ and $T_g = T_g^+ \cup T_g^-$ and $T_h = T_h^+ \cup T_h^-$ the associated partition into cones of the groups T_g and T_h of translations along g resp. h. Let A, B be points with $AB \notin T_g$. We denote the feet of the perpendiculars from A and B to g resp. to h by C and D resp. by E and F. We can assume $OC \in T_g^+$ and $OE \in T_h^+$.

The cone C_g^+ can be defined as the set of translations AB with the property $OE, EF \in T_h^+$ if $E \neq F$ and $CD \in T_g^+$ if $E = F$. If C_g^- is the set of translations AB with the property $OE \in T_h^+$ and $EF \in T_h^-$ if $E \neq F$ and $CD \in T_g^-$ if $E = F$ then $T = T_g \cup C_g^+ \cup C_g^-$.

Pasch-free ordered Bachmann planes. In this section we give an example of an ordered Euclidean plane (i.e. the axioms of an orderable Bachmann plane hold and the field of coordinates is orderable) with the property that the Pasch axiom is not satisfied.

In an orderable Bachmann plane the Pasch axiom holds if and only if the associated field of coordinates is orderable and the set of points of the Bachmann plane is a convex subset of the set of points of the associated affine ideal plane (see Pejas [21]).

Let $\mathrm{E}(\mathbb{Q}, 1)$ be the Euclidean plane over the field $\mathbb{Q}$ of rational numbers with the orthogonality constant $k = 1$. Points can be represented by pairs (x, y) of elements of $\mathbb{Q}$ and lines by triples $[u, v, w]$ with $u \neq 0$ or $v \neq 0$ (proportional triples represent the same line). A point (x, y) and a line $[u, v, w]$ are *incident* if $ux + vy + w = 0$. Lines $[u, v, w]$ and $[u', v', w']$ are *orthogonal* if $vv' + uu' = 0$ (see Bachmann [1]).

We consider the smallest subplane M of $\mathrm{E}(\mathbb{Q}, 1)$ which is a singular Bachmann plane and which contains the points $(0, 0)$ and $(1, 0)$. The points of M have as coordinates rational numbers which satisfy additional number-theoretic conditions which imply that M does not contain the point $(\frac{1}{3}, 0)$ (see Bachmann [1, §19,2]). Hence the set of points of M is not a convex subset of the set of points of the associated affine ideal plane which shows that the axiom of Pasch does not hold.

M is the smallest subplane of the Euclidean plane over the field of real numbers which is a Bachmann plane (see [1, §19,2]).

Metric planes with more than one or no joining line at all. Minkowskian and Galilean planes over fields of characteristic $\neq 2$ and Euclidean planes over commutative local rings (which are not fields) are metric planes which contain points with more than one or no joining line at all. According to H. Struve and R. Struve [25] these metric planes are orderable if and only if the associated field of coordinates is orderable.

Appendix

In the preceding sections we showed that in a metric plane the calculus of reflections allows the introduction of the notions of betweenness and order.

In this appendix we axiomatize ordered metric planes in a first-order language. By this we do not claim that our axiom system is simple or preferable to its competitors, but simply that the theory can be expressed in a first-order language.

For reasons of simplicity we consider planes with the property that any pair of distinct points has a unique joining line (Bachmann planes). The basis for our axiomatization is the axiom system for Wolff planes given in Pambuccian and R. Struve [18, Section 4].

The axiom system can be expressed with two sorts of individual variables (elements $a, b, c, \ldots$ of a set S and elements $U, V, W, \ldots$ of a set $\mathcal{C}$), a binary operation ϱ on S and a relation ϑ on $S \times \mathcal{C}$.

The elements of S are to be interpreted as 'lines' and the elements of $\mathcal{C}$ as 'cones of the partial group of translations' and the operation $\varrho(a, b)$ as 'the reflection of line b in line a' and the relation ϑ as 'a relation which assigns to each line an associated cone of the partial group of translations'.

To improve the readability of the axioms, we introduce the following abbreviations:

$$a_1 \ldots a_n = 1 \quad \Leftrightarrow \quad (\forall x)\varrho(a_1, \ldots \varrho(a_n, x) \ldots) = x$$

$$g_1 \ldots g_n = h_1 \ldots h_m \quad \Leftrightarrow \quad (\forall x)\varrho(g_1, \ldots \varrho(g_n, x) \ldots) = \varrho(h_1, \ldots \varrho(h_m, x) \ldots)$$

$$a \,|\, b \quad \Leftrightarrow \quad a \neq b \wedge (ab)^2 = 1$$

$$J(abc) \quad \Leftrightarrow \quad abc \neq 1 \wedge (abc)^2 = 1$$

$$pq \,|\, a \quad \Leftrightarrow \quad p \,|\, q \wedge J(pqa)$$

We think of the pair (p, q) with $p \,|\, q$ as a 'point', namely the intersection point of p and q. If $p \,|\, q$ then $pq \,|\, a$ may be read as 'the point pq lies on a'.

We present the axioms in informal language (their formalization being straightforward) and define two sets P and T (which correspond in a first-order language to unary predicates) with an intended interpretation as the set of points resp. translations of the Bachmann plane (cp. Sect. 2).

(1) $pq \in P \quad \Leftrightarrow \quad p \,|\, q$; the elements of P are denoted by $A, B, \ldots$
(2) $ab \in T \quad \Leftrightarrow \quad (\exists e)\, a, b \,|\, e$; the elements of T are denoted by $\sigma, \tau, \ldots$

If $\tau \in T$ and $\tau = ab$ then we denote the element $ba \in T$ by τ^{-1} and the element $\tau^{-1} g \tau$ by g^τ.

The axioms are:

Axiom H1. $a^2 = 1$

Axiom H2. $\varrho(a, b) = aba$

Axiom H3. For A, B there exists c with $A, B \mid c$.

Axiom H4. If $A, B \,|\, c, d$ then $A = B$ or $c = d$.

Axiom H5. If $a, b, c \,|\, E$ then there exists d with $abc = d$.

Axiom H6. If $a, b, c \,|\, e$ then there exists d with $abc = d$.

Axiom H7. There exist g, h, j with $g \,|\, h$ and neither $j \,|\, h$ nor $j \,|\, g$ nor $gh \,|\, j$.

Axiom H8. $U \subseteq T$

Axiom H9. $1 \in U$

Axiom H10. If $\alpha, \beta \in U$ and $\alpha \cdot \beta \in T$ then $\alpha \cdot \beta \in U$.

Axiom H11. If $\alpha, \beta \in U$ and $\alpha \cdot \beta = 1$ then $\alpha = \beta = 1$.

Axiom H12. For $U \in \mathcal{C}$ there exists $a \in S$ with $\vartheta(a, U)$.

Axiom H13. If $a \in S$ then there are $U, V \in \mathcal{C}$ with $U \neq V$ and $\vartheta(a, U)$, $\vartheta(a, V)$.

Axiom H14. If $\vartheta(a, U)$, $\vartheta(a, V)$ and $\vartheta(a, W)$ then $U = V$ or $U = W$ or $V = W$.

Axiom H15. If $\tau \in T \backslash \{1\}$ and $\vartheta(a, U)$ then either $\tau \in U$ or $\tau^{-1} \in U$ or $a^\tau = a$.

Axiom H16. If $\vartheta(a, U)$ and $\vartheta(b, V)$ and $g \neq a, b$ and $\sigma, \tau \in U$ with $g^\sigma = g^\tau = g$ then $\sigma, \tau \in V$ or $\sigma, \tau \notin V$.

Axiom H1, corresponding to the fundamental assumption (or *Grundannahme*) of [1], states that reflections in lines are involutions. *Axiom H2* states that, for all line-reflections a and b, aba is a line-reflection as well, namely $\varrho(a, b)$. The axioms *H3–H7* correspond to the axioms for Bachmann planes given in [1, §3,2]. *Axiom H3*, corresponding to Axiom 1 of [1], states that any two points have a joining line. *Axiom H4*, corresponding to Axiom 2 of [1], states that the joining line of two distinct points is unique. *Axiom H5*, corresponding to Axiom 3 of [1], states that the composition of three reflections in lines with a common point is a reflection in a line. *Axiom H6*, corresponding to Axiom 4 of [1], states that the composition of three reflections in lines with a common perpendicular is a reflection in a line. *Axiom H7*, corresponding to Axiom D of [1], states that there exist three lines in a general position.

Axiom H8 states that the elements of an element of $\mathcal{C}$ are translations. *Axiom H9* states that an element of $\mathcal{C}$ contains at least the trivial translation (the identity). *Axiom H10* states that elements of $\mathcal{C}$ contain with two elements their product (if the product is a translation). *Axiom H11* states that elements of $\mathcal{C}$ do not contain a (non-trivial) translation and the inverse one. *Axiom H12* states that each element of $\mathcal{C}$ is associated to a line. *Axiom H13* states that to any line there are associated at least two distinct elements of $\mathcal{C}$. *Axiom H14* states that to any line there are associated at most two elements of $\mathcal{C}$. *Axiom H15* states that an element of $\mathcal{C}$, which is associated to a line a, contains one element of every pair (τ, τ^{-1}) of translations which do not have a as a fixed line. *Axiom H16* states that if two elements U and V of $\mathcal{C}$ which are associated to lines a and b contain a translation with a fixed line $g \neq a, b$ then every translation with fixed line g which is an element of U is an element of V, and conversely.

Theorem 5.1. *The axioms H1–H16 axiomatize ordered Bachmann planes.*

Proof. Let $\mathcal{A}$ be the theory axiomatized by *H1-H16*. Let $\mathcal{M} =< \mathrm{S}, \mathrm{C}, \varrho, \vartheta >$ be a model of $\mathcal{A}$. For each $g \in S$, we can define a bijective map $\sigma_g : \mathrm{S} \to \mathrm{S}$ by $\sigma_g(h) = \varrho(g, h)$. Let G denote the subgroup of $Sym(S)$ generated by the σ_g. If $S := \{\sigma_g : g \in \mathrm{S}\}$ then (G, S) is a group with a distinguished set S of generators, satisfying the fundamental assumption and the axioms *H3–H7* which correspond to the axioms for Bachmann planes given in [1, §3,2].

In a Bachmann plane a translation can be represented as the product of reflections in lines a and b which have a common perpendicular. The set T of translations forms a partial group. According to the *Axioms H8–H11* the elements of $\mathcal{C}$ are cones of T. According to the axioms *H12–H15* the relation ϑ describes the partition into sides of lines, i.e., they ensure that to a line a there exist two elements U and V of $\mathcal{C}$ with $\vartheta(a, U)$ and $\vartheta(a, V)$ and $T = T_a \cup U \cup V$ and $T_a \cap U = T_a \cap V = U \cap V = \{1\}$. By *Axiom H16* the partition into sides of any two lines a and b are compatible, i.e. they induce the same linear order on any line g with $g \neq a, b$.

This shows that the Bachmann plane is orderable (according to Theorem 3.19) and that the axiom system characterizes orderable Bachmann planes. □

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