J. Geom. 106 (2015), 313–320
© 2014 Springer Basel AG
0047-2468/15/020313-8
published online November 23, 2014
DOI 10.1007/s00022-014-0250-2

Journal of Geometry

The first eigenvalue of Laplace-type elliptic operators induced by conjugate connections

Udo Simon

Abstract. According to Tashiro–Obata, on a Riemannian manifold (M, g) with its Ricci curvature bounded positively from below, the first eigenvalue of the Laplacian on functions satisfies a simple inequality in terms of the scalar curvature, and equality characterizes the Riemannian sphere. We discuss a similar inequality for a certain elliptic operator on a manifold with conjugate connections. As application we characterize hyperellipsoids in Blaschke's unimodular-affine hypersurface theory.

Mathematics Subject Classification. 53C24, 53C40, 35P15, 53A15.

Keywords. Conjugate connections, Laplace-type operator, first eigenvalue, characterization of hyperellipsoids.

1. Introduction

We recall the notion of *a conjugate triple* (*conjugate connections*, resp.) (∇, g, ∇^*) from section 4.4 in [8]:

$$wg(u, v) = g(\nabla_w u, v) + g(u, \nabla^*_w v);$$

here (M, g) denotes a semi-Riemannian manifold, u, v, w denote vector fields and the connections ∇ and ∇^* are torsion free.
In the following we additionally assume that ∇^* is Ricci symmetric, that means its Ricci tensor Ric^* is symmetric. Again from section 4.4 in [8], for a given conjugate triple (∇, g, ∇^*) with torsion free connections, we have the following equivalences:

- ∇^* is Ricci symmetric,
- ∇ is Ricci symmetric,
- ∇^* admits a parallel volume form ω^*,
- ∇ admits a parallel volume form ω.

Each parallel volume form is unique modulo a non-zero constant factor.
M. Wiehe stated an extension of the so called Bochner–Lichnerowicz formula from Riemannian Geometry to the case of conjugate connections (see [10], Lemma 2.10); for the proof of this he used unpublished calculations of the present author (see Wiehe's remark [10], p. 15). As application of this formula he proved the following analogue of a theorem of Tashiro–Obata (for the theorem of Tashiro–Obata see e.g. [1], p. 179).

Theorem. *Let (M, g) be a closed Riemannian manifold with dimension $n \geq 2$ and let (∇, g, ∇^*) be a conjugate triple with torsion free and Ricci symmetric connections ∇ and ∇^*. Assume that there exists $0 < k \in \mathbb{R}$ such that the Ricci curvature $Ric\,(\nabla)$ of ∇ satisfies the inequality*

$$Ric\,(\nabla) \geq (n-1)k \cdot g.$$

Then the first eigenvalue λ_1^* of the operator $\Box^* := trace_g Hess^*$ satisfies the inequality

$$\lambda_1^* \geq nk.$$

Here $Hess^* f$ denotes the ∇^*-covariant Hessian of $f \in C^\infty(M)$.

In [5], Theorem 8.9, B. Opozda gave a new proof of Wiehe's Theorem.

In the theorem of Tashiro–Obata the discussion of the equality $\lambda_1 = nk$ leads to a characterization of the Riemannian sphere $\mathbb{S}^n(k)$. It is the aim of the present paper to investigate the case of equality $\lambda_1^* = nk$ in Wiehe's theorem. In our paper we use the notations from [8]. For local calculations we adopt the Einstein summation convention and raise and lower indices with respect to the Riemannian metric g.

The following Theorem is our main result.

Theorem. *Let M be a closed and simply connected n-manifold and (M, g) be Riemannian with $n \geq 2$; further let (∇, g, ∇^*) be a conjugate triple with torsion free connections ∇, ∇^*, and assume that ∇^* is Ricci-symmetric; moreover, assume that ∇^* is projectively flat and that*

(i) $\exists\, k > 0$ *s.t. the Ricci-tensor $Ric\,(\nabla)$ of ∇ satisfies the inequality*

$$Ric\,(\nabla) \geq (n-1)k \cdot g,$$

(ii) *the first eigenvalue λ_1^* of the operator $\Box^*$ satisfies $\lambda_1^* = nk$.*

Then we have the following properties for (∇, g, ∇^*):

(a) *the eigenspace $E^*(\lambda_1^*)$ of $\Box^*$ has the dimension $\dim E^*(\lambda_1^*) = n + 1$;*
(b) *let f be a first eigenfunction of the operator $\Box^*$, i.e. $f \in E^*(\lambda_1^*)$; then there exists a positive function $\varrho : M \to \mathbb{R}$ s.t.*

(b.1) *the function* $f \cdot \varrho : M \to \mathbb{R}$ *satisfies the PDE-system*

$$Hess^*(f \cdot \varrho) + \tfrac{1}{n-1} \cdot Ric^* \cdot (f \cdot \varrho) = 0,$$

and its trace gives a Schrödinger type equation for the operator $\square^*$*;*

(b.2) *the function* ϱ *defines a new conjugate triple* $(\nabla^\sharp, g^\sharp, \nabla^{*\sharp})$ *as follows:* g *and* $g^\sharp$ *are conformally related:* $g^\sharp := \frac{1}{\varrho} \cdot g$*;*
∇^* *and* $\nabla^{*\sharp}$ *are projectively related:*

$$\nabla^{*\sharp}_u v - \nabla^*_u v = -d\ln\varrho(u)v - d\ln\varrho(v)u.$$

The (1.3)*-curvature tensors* $R^\sharp$ *and* $R^{*\sharp}$ *of* $\nabla^\sharp$ *and* $\nabla^{*\sharp}$*, resp., satisfy the relation*

$$R^\sharp(u,v)w = g^\sharp(v,w)u - g^\sharp(u,w)v = R^{*\sharp}(u,v)w;$$

from this last equation the connection $\nabla^\sharp$ *is also projectively flat;*

(b.3) *the eigenspaces* $E^*(\lambda_1 = nk)$ *of* $\square^*$ *and* $E^{*\sharp}(\lambda_1^\sharp = n)$ *of* $\square^{*\sharp}$ *coincide; in particular,* $\lambda_1^{*\sharp} = n$ *is the first eigenvalue of* $\square^{*\sharp}$ *; moreover, any first eigenfunction* $f \in E^{*\sharp}$ *satisfies the PDE-system*

$$Hess^{*\sharp} f + f \cdot g^\sharp = 0.$$

Finally, the connection $\nabla^\sharp$ *defines an operator* $\square^\sharp := trace_{g^\sharp} Hess^\sharp$*. This operator has the same first eigenvalue* $\lambda_1^\sharp = n = \lambda_1^{*\sharp}$*, and the corresponding eigenspaces coincide:* $E^\sharp = E^{*\sharp}$*.*

Proof of the Theorem Step 1 Following Theorem 2.11 and Remark 2.12 in [10], pp. 16–17, the equality $\lambda_1^* = nk$ implies:

$$Ric(grad_g f, grad_g f) = (n-1) \cdot k \cdot ||grad_g f||_g^2$$

and

$$Hess^* f + kfg = 0$$

for every first eigenfunction of the operator $\square^*$.

Step 2 The given assumptions, namely that (∇, g, ∇^*) is a conjugate triple with torsion free and Ricci-symmetric connections ∇, ∇^* and projectively flat ∇^*, imply that there exists a hypersurface immersion $x : M \to \mathbb{R}^{n+1}$ s.t. $x(M)$ is a hyperovaloid with a relative normalization (Y, y), where Y denotes a conormal field and y a relative normal, and where the given conjugate triple coincides with data induced from this relative normalization (see section 4.11 in [8]), namely:

- ∇ is the induced connection,
- g is the relative metric,
- ∇^* is the conormal connection.

If necessary we translate $x(M)$ s.t. the origin of $\mathbb{R}^{n+1}$ lies inside the hyperovaloid, thus, for an appropriate orientation of the normalization, the support function $\varrho := - < Y, x >$ satisfies $\varrho > 0$. We fix a determinant form over the vector space $\mathbb{R}^{n+1}$; for another choice of a determinant form there would appear a constant non-zero factor in the calculations below.

Step 3 Let f be a first eigenfunction of the operator $\Box^*$. Define the vector valued mapping $a : M \to \mathbb{R}^{n+1}$ by

$$a := -g^{ir}\partial_i f\, \partial_r x + kf \cdot x.$$

Denote the Levi-Civita connection of g by $\nabla(g)$. Then

$$\nabla(g)_i a = P_i^k \partial_k x$$

where the operator P satisfies

$$P_i^k = \nabla(g)_i f^k - C_i^{kr}\, \nabla(g)_r f + kf \cdot \delta_i^k = \nabla_i^* f^k + kf \cdot \delta_i^k;$$

the components C_i^{kr} come from the cubic form tensor C of the relative hypersurface (x, Y, y), see section 4.4.3 in [8]. As f is a first eigenfunction of $\Box^*$, we have trace $P = 0$.

Step 4 Following an idea of Blaschke (p. 215 in [2]), below we derive an integral formula for the elementary symmetric functions $\mathcal{P}_k$ of P. We use the determinant tensor ε of Ricci. While we set $\mathcal{P}_o := 1$, for $k = 1, \dots, n$ the functions $\mathcal{P}_k$ are defined as follows:

$$n!\, \mathcal{P}_k := \varepsilon^{i_1 i_2 \dots i_k i_{k+1} \dots i_n} \varepsilon_{j_1 j_2 \dots j_k i_{k+1} \dots i_n} P_{i_1}^{j_1} \dots P_{i_k}^{j_k}.$$

Define the vector field Ω in terms of local coordinates:

$$\Omega^i := \varepsilon^{i i_2 \dots i_n}\, \det(x, a, a_{i_2}, x_{i_3}, \dots, x_{i_n}).$$

Then its divergence in terms of $\nabla(g)$ reads:

$$\begin{aligned}
\nabla(g)_i \Omega^i &= \varepsilon^{i i_2 \dots i_n}\, \left(\det(x_i, a, a_{i_2}, x_{i_3}, \dots, x_{i_n}) + \det(x, a_i, a_{i_2}, x_{i_3}, \dots, x_{i_n})\right) \\
&= \varepsilon^{i i_2 \dots i_n} \cdot (-kfP_{i_2}^s\, \det(x, x_i, x_s, x_{i_3}, \dots, x_{i_n})) \\
&\quad + \ \varepsilon^{i i_2 \dots i_n} \cdot P_i^p P_{i_2}^s \det(x, x_p, x_s, x_{i_3}, \dots, x_{i_n}) \\
&= \varepsilon^{i i_2 \dots i_n} \left(-kfP_{i_2}^s \cdot \varepsilon_{i s i_3 \dots i_n} + P_i^p P_{i_2}^s \cdot \varepsilon_{p s i_3 \dots i_n}\right) \cdot < Y, x > \\
&= n! \cdot < Y, x > \mathcal{P}_2.
\end{aligned}$$

In the foregoing calculation we obey the skew-symmetry of the ε-tensor, the symmetry of second covariant derivatives, the higher dimensional cross product construction for the conormal

$$\varepsilon_{i_1 i_2 \dots i_n} \cdot Y = [x_{i_1}, \dots, x_{i_n}]$$

and the relation $\mathcal{P}_1 = 0$. Thus we arrive at the integral formula

$$\int \varrho \cdot \mathcal{P}_2\, \omega(g) = 0,$$

where $\omega(g)$ denotes the Riemannian volume form.

Step 5 We recall that the support function ϱ is nowhere zero on M; moreover, the equation $\mathcal{P}_1 = 0$ implies $\mathcal{P}_2 \le 0$. Thus the integral formula gives $\mathcal{P}_2 = 0$ and we finally get $P = 0$. From Step 3 we get $a = const$ and

$$-kf \cdot \varrho \ = < Y, a > .$$

Step 6 According to Satz 3.1 and its proof in [6] any function of the type $< Y, a >$ satisfies the system (b.1) in the assertion of our Theorem. As $a \in \mathbb{R}^{n+1}$ is arbitrary, the space of such functions has dimension $(n+1)$. From the positivity of ϱ then also the eigenspace $E^*(\lambda_1^*)$ has the same dimension; see [3].

Step 7 We rewrite the equation in Step 5:

$$f = < Y^\sharp, a^\sharp >$$

where $Y^\sharp := \frac{1}{\varrho} \cdot Y$ and $a^\sharp := -\frac{1}{k} \cdot a \in \mathbb{R}^{n+1}$. One easily verifies that the conormal field $Y^\sharp$ defines another relative hypersurface geometry on $x(M)$, namely the so called *centroaffine geometry* (see sections 3.4.3 and 6.6.4.iii in [9] and section 6.3 in [8]). From 6.3.3 in [8] the asserted relations on the curvature tensors in (b.2) of our Theorem follow. Finally one verifies the system $Hess^{*\sharp} f + f \cdot g^\sharp = 0$; this proves (b.3) above.

2. Observations for conjugate connections

- Let (∇, g, ∇^*) be a conjugate triple with torsion free connections ∇, ∇^*; moreover assume that ∇^* is Ricci-symmetric and that the symmetric $(1,2)$-difference tensor $K := \nabla^* - \nabla$ is trace free (so called *apolarity*). If both Ricci tensors Ric and Ric^* satisfy the same inequality

$$Ric \geq (n-1)k \cdot g \quad \text{and} \quad Ric^* \geq (n-1)k \cdot g$$

for some constant $k \in \mathbb{R}$ then the Ricci tensor $Ric\,(g)$ of the Levi-Civita connection $\nabla(g)$ satisfies the same inequality $Ric\,(g) \geq (n-1)k \cdot g$.

Proof. Calculate the following relation for the Ricci tensors in local terms (cf. formula 4.4.10.f on p. 60 in [8]):

$$2R(g)_{ij} = R_{ij} + R^*_{ij} + 2K_{irs}K_j^{rs}.$$

The symmetric tensor field with components $K_{irs}K_j^{rs}$ is semi-positive definite, thus the inequalities for Ric and Ric^* imply the asserted inequality for $Ric\,(g)$.

- In the foregoing assume that (M, g) is complete Riemannian and that $k > 0$; then both inequalities $Ric \geq (n-1)k \cdot g$ and $Ric^* \geq (n-1)k \cdot g$ together imply

$$Ric\,(g) \geq (n-1)k \cdot g,$$

and now Myers' theorem [4] implies that (M, g) is compact. Following results of Lichnerowicz–Obata (see pp. 179–180 in [1]) the inequality $Ric\,(g) \geq (n-1)k \cdot g$ implies an inequality for the first eigenvalue of the Laplacian:

$$\lambda_1(\Delta) \geq n\ k;$$

here equality holds if and only if the manifold (M, g) is isometrically diffeomorphic to a sphere.

- We call a conjugate triple a *Blaschke structure* if the difference tensor K satisfies an apolarity condition:

$$K_{ij}^{j} = 0.$$

In this special case the elliptic operators $\Box$ and $\Box^*$ coincide with the Laplacian Δ_g on functions.

3. Observations for Blaschke hypersurfaces

Consider a connected, oriented $C^\infty-$manifold of dimension $n \geq 2$ and an affine immersion $x : M \rightarrow \mathbb{R}^{n+1}$ into the affine space $\mathbb{R}^{n+1}$ equipped with a unimodular structure; assume that x is locally strongly convex and that the induced structure is *equiaffine* (see section 6.2 in [8]), that means x is equipped with an appropriately oriented *affine normal* y s.t. the Blaschke metric g is positive definite; x together with the normalization y is called a *Blaschke hypersurface* with conjugate triple (∇, g, ∇^*), where ∇ is the *induced connection* and ∇^* the *conormal connection*; the triple satisfies an apolarity condition. If the *affine shape operator* S is positive definite then the symmetric *Weingarten form* $S^\flat$ with $S^\flat(u, v) := g(Su, v)$ can be interpreted as *affine spherical metric* of the *affine spherical indicatrix* (or *affine Gauß map*, resp.) $y : M \rightarrow \mathbb{R}^{n+1}$; see section 4.6 in [8].
$S^\flat$ and Ric^* satisfy the relation $Ric^* = (n-1)S^\flat$, moreover we have

$$(n-1)Ric + Ric^* = nH \cdot g,$$

where $nH := trace\, S$ is the *affine mean curvature*, and the Ricci tensor $Ric\,(g)$ of the Blaschke metric g satisfies

$$Ric\,(g)_{ij} = K_{irs}K_j^{rs} + \tfrac{n-2}{2} S_{ij}^\flat + \tfrac{n}{2} \cdot Hg_{ij}.$$

The foregoing facts admit the following observations:

1. If the spherical metric $S^\flat$ is bounded below, namely $S^\flat \geq k \cdot g$ for some $k \in \mathbb{R}$, then $Ric^* \geq (n-1)kg$ and also $Ric \geq (n-1)kg$, and thus finally also $Ric\,(g) \geq (n-1)kg$. Thus all three Ricci curvatures Ric, Ric^* and $Ric(g)$ satisfy the same inequality.
2. If $n \geq 2$, $S^\flat \geq 0$ and $H \geq \frac{2(n-1)}{n} \cdot k$ then $Ric\,(g) \geq (n-1)kg$.
3. If $n = 2$ and $H \geq k$ then $Ric\,(g) \geq kg$, that means the Gauß curvature is bounded below by k.

Remark. The assumption that $Ric\,(g)$ is bounded below plays a role in the maximum principle of Omori–Yau; for applications in Blaschke's affine hypersurface theory see e.g. [7].

The foregoing statements can be applied to metrically complete (so called *affine complete*) Blaschke hypersurfaces. We consider the case $k > 0$. We give an example:

Theorem. *Let $x : M \to \mathbb{R}^{n+1}$ be a locally strongly convex Blaschke hypersurface s.t.*

(i) *the Blaschke metric is complete,*
(ii) $\exists\ 0 < k \in \mathbb{R}$ *s.t.* $Ric^* \geq (n-1) \cdot kg$,
(iii) $\exists\ f \in C^\infty$ *s.t.* $\Box^* f + nkf = 0$.

Then $x(M)$ is a hyperellipsoid.

Proof. It follows from (ii) above that the Weingarten form satisfies $S^\flat \geq kg$, and thus the affine mean curvature $H \geq k$. Now from the observations above we also know that $Ric\,(g) \geq (n-1)kg$ and therefore M is compact. As a consequence of the apolarity condition, on a Blaschke hypersurface the operators Δ and $\Box^*$ coincide. Thus from the assumption (iii) the Laplacian has its first eigenvalue $\lambda_1 = nk$. Then Obata's result (see [1], p. 180) implies that (M, g) is isometrically diffeomorphic to the canonical Euclidean sphere, that means $x(M)$ must be a hyperellipsoid.

Acknowledgment

I thank B. Opozda for discussions on the topic.

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Udo Simon
Institut Mathematik,
MA 8-3, TU Berlin
Straße 17. Juni 136
10623 Berlin, Germany
e-mail: simon@math.tu-berlin.de

Received: September 9, 2014.