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On the geometry of flat surfaces with a single singularity

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Abstract. If T is a flat torus with boundary and a conical singularity in its boundary then the isometry type of T is determined by the lengths of five closed geodesics. As a corollary the isometry type of a flat closed surface S with a single conical singularity is determined by the lengths of finitely many closed geodesics provided that S admits a special decomposition.

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1. Introduction: preliminaries

The study of the geometry of flat surfaces with conical singularities is an interesting subject with various open questions [10]. If S is a compact orientable surface of genus $g \geq 1$, it is well known [8] that S can be equipped with a flat structure with finitely many conical singularities s_i of angle $\theta(s_i)$, provided that the angles $\theta(s_i)$, $i = 1, .., n$ satisfy the Euler-type formula explained below. There is an interesting moduli space of such structures; in fact, there are several versions of such a moduli space, depending on whether we fix the singular points on the surface and their types or not.

Generally, it is difficult to find important properties which are valid for all flat surfaces with conical singularities. Special classes of such surfaces are easier to handle and such classes include the following: (1) Flat surfaces S with $\theta(s_i) > 2\pi$. Their universal covering $\widetilde{S}$ is a Hadamard space i.e. $\widetilde{S}$ satisfies the CAT(0) inequality, a fact that imposes special features on the behavior of geodesics of S [1]. (2) Translation surfaces [6]. In this case $\theta(s_i) = 2k\pi$, $k \in \mathbb{N}$, and the geometry of these surfaces has many similarities with the geometry of a flat torus. (3) Flat surfaces whose metric is defined by a quadratic differential [7]. This class is distinct from the previous one. Every singular flat metric defines uniquely a conformal structure on S and hence a hyperbolic structure on S. The space of flat metrics defined by quadratic differentials can be identified with a vector bundle over the Teichmü ller space of S, a fact that makes these metrics of particular interest.

In the present work we consider compact orientable flat surfaces S with a single conical singularity. Such surfaces are considered for example in [2], where some properties of their closed geodesics are studied. The main theorems of this paper are the following.

Theorem 1. *Let S be a flat torus of genus one with one boundary component ∂S and one singularity $s \in \partial S$. The geometry of S is determined by the lengths of five closed geodesics passing through s. Moreover, the area of T and the lengths of four simple closed geodesics of T determine two isometry types of T, see Theorem 13 below.*

As a corollary of this theorem we deduce the following.

Theorem 2. *Let S be a closed, orientable flat surface of genus $g \geq 3$ with one singularity s. Assume that there are g simple closed geodesics of S such that each one separates a torus of genus one from S. Then there are $5g + (g - 3)$ closed geodesics γ_i of S, all passing through s, whose lengths determine, up to isometry, the geometry of S. Furthermore, each γ_i can be chosen to be homotopic to a simple closed curve in S.*

If $g = 2$ we assume that there is a simple closed geodesic separating S into two tori. Then there are nine closed geodesics γ_i of S, all passing through s, whose lengths determine, up to isometry, the geometry of S. Furthermore, each γ_i is either simple or it is a union of two simple geodesic loops based at s, see Theorem 16 below.

We give below some basic definitions and results in order to put our problem in a general context.

The *standard flat cone* $C(v, \theta)$ is defined as the set $\{(r,t) : 0 \leq r,\ t \in \mathbb{R}/\theta\mathbb{Z}\}$ equipped with the metric $ds^2 = dr^2 + r^2dt^2$. We say that v (resp. θ) is the vertex (resp. the angle) of $C(v, \theta)$. By cutting $C(v, \theta)$ along a half-line starting from p we get a *flat sector*, say $S(v, \theta)$, of vertex v and of angle θ.

A *flat surface S with conical singularities* $s_1, ..., s_n$ is a compact, orientable surface with or without boundary ∂S which is equipped with a metric $d(\cdot, \cdot)$ such that:

- Every point $p \in Int(S) - \{s_1, ..., s_n\}$ has a neighborhood isometric to a disc in the Euclidean plane $\mathbb{E}^2$.
- Every point $p \in \partial S - \{s_1, ..., s_n\}$ has a neighborhood isometric to a neighborhood of a point on the boundary of the half-plane $\mathbb{E}^2_+ = \{(x, y) \in \mathbb{E}^2 : y \geq 0\}$.
- Every point $p \in Int(S) \cap \{s_1, ..., s_n\}$ has a neighborhood isometric to a neighborhood of the vertex v of a standard flat cone $C(v, \theta_i)$. The point s_i will be called a *conical singularity of angle θ_i*.
- Every point $p \in \partial S \cap \{s_1, ..., s_n\}$ has a neighborhood isometric to a neighborhood of the vertex v of the flat sector $S(v, \theta_i)$. The point s_i will be also called a *conical singularity* or a *boundary singularity of angle θ_i*. Note that if $\theta_i > \pi$ for each $s_i \in \partial S$, then the boundary ∂S is geodesic.

For brevity, the surface S will be denoted by $f.s.c.s.$ The angle θ_i is called the *angle of the (conical) singularity* s_i. The metric $d(\cdot,\cdot)$ is obviously a length metric and will be referred to as a *flat structure with conical singularities.*

Definition 3. *A geodesic segment is an isometry* $h : [t_1, t_2] \to S$.

A geodesic arc is a local isometry $h : [t_1, t_2] \to S$.

If $h(t_1) = a_1,\ h(t_2) = a_2$ *then we denote by* $[a_1, a_2]$ *the image of* h.

Let $I = [0, +\infty)$ *or* $I = (-\infty, +\infty)$. *A geodesic line (resp. geodesic ray) in* S *is a local isometry* $h : I \to S$ *where* $I = (-\infty, +\infty)$ *(resp.* $I = [0, +\infty)$*).*

A local geodesic $a : [t_1, t_2] \to T$ *such that* $a(t_1) = a(t_2) = s$ *will be called a geodesic loop based at* s.

A periodic geodesic line is called a closed geodesic of S.

As every locally compact, complete length space is geodesic (see Th. 1.10 in [3]) we immediately have that S is a geodesic space i.e. every two points of S can be joined by a geodesic segment.

The following Gauss–Bonnet formula is well known (see [4], p. 113).

Proposition 4. *Let* S *be a* $f.s.c.s.$ *with conical singularities* s_i, $i = 1, .., n$. *Let* $\theta_i = \theta(s_i)$ *be the angle of* s_i. *Let* g *be the genus of* S *and let* b *be the number of boundary components of* S. *Then we have*

$$\sum_{s_i \in Int(S)} (2\pi - \theta_i) + \sum_{s_j \in \partial S} (\pi - \theta_i) = (4 - 4g - 2b)\pi. \tag{1}$$

Now, let S be a compact orientable surface S with boundary and let $\theta_i \in (0, +\infty) - \{2\pi\}$, $1 \le i \le m$ and $\theta_j \in (0, +\infty) - \{\pi\}$, $m + 1 \le j \le n$. A well known result [8,9] asserts that if the above formula (1) is satisfied then there exist a flat structure with conical singularities s_i, $i = 1, .., n$ on S, such that:

(1) s_i is of angle θ_i;
(2) $s_i \in Int(S)$ for $1 \le i \le m$ and $s_j \in \partial S$ for $m + 1 \le j \le n$.

In the special case that S is a torus with a single singularity $s \in \partial S$ we deduce that the angle $\theta(s)$ at s must be equal to 3π. If S is closed surface of genus g then $\theta(s) = 4g\pi - 2\pi$.

Definition 5. *Let* a_1, a_2 *be two closed, piecewise differentiable curves of* S. *We will say that* a_1, a_2 *have a trivial intersection point* p *if there is a closed neighborhood* D *of* p *in* S *homeomorphic to a closed 2-disc such that:*

- $a_i \cap D = c_i$, $i = 1, 2$, *where* c_i *is a simple arc with* $\partial c_i \subset \partial D$;
- $c_1 \cap c_2 = \{p\}$;
- *each* c_i *can be isotoped in* D, *keeping its endpoints fixed, to an arc* c'_i *with* $c'_1 \cap c'_2 = \emptyset$.

Similarly, if a *is closed, piecewise differentiable curve, we define the notion of trivial self-intersection point of* a.

In the following proposition we denote by $\theta(s_i)$ the angle of a conical singularity $s_i \in S$.

Proposition 6. *Let S be a f.s.c.s. and let $\theta(s_i)$ be the angle of a conical singularity $s_i \in S$. We assume that if $s_i \in Int(S)$ then $\theta(s_i) > 2\pi$ and if $s_i \in \partial S$ then $\theta(s_i) > \pi$. Then we have,*

(i) In the homotopy class of each loop c based at a point x, there is a unique geodesic loop c_0 based at x. Furthermore, if c is a simple loop then c_0 can only have (finitely many) trivial self-intersection points which coincide with conical singularities of S.
(ii) In the free homotopy class of each closed curve c of S there is a closed geodesic c_0. Furthermore, if c is simple then c_0 can only have trivial self-intersection points which coincide with conical singularities of S.
(iii) Every two disjoint simple closed geodesics of S which are freely homotopic bound a Euclidean cylinder in S.

Remark that our hypothesis about the angles $\theta(s_i)$ implies that S satisfies locally the CAT(0) inequality. The proof of 6 follows from Proposition 1.2.4 and 1.2.6 of [5] and from the fact that if c is a simple loop (resp. simple closed curve) then the geodesic loop c_0 in the homotopy class of c (resp. the closed geodesic in the free homotopy class of c) cannot intersect itself transversely.

2. The geometry of a flat torus with a boundary component and a single singularity on the boundary

Let T be a flat torus with one boundary component, say c, and one conical singularity $s \in c$. The goal of this section is to relate the geometry of T with the lengths of finitely many closed geodesics passing through s.

In the following, we will use the following notation. We denote by $R = (X_1, ..., X_n)$ a flat surface such that:

(1) it is homeomorphic to a closed 2-disc;
(2) it does not have conical singularities in its interior;
(3) it has boundary singularities at X_i, $i = 1, .., n$.

A flat surface R which satisfies (1)–(3) will be called a *flat n-gon* and X_i will be called *vertices* of R.

The flat n-gon R will be referred as a *Euclidean n-gon* if it can be isometrically embedded in $\mathbb{E}^2$.

If the diagonals X_iX_j and X_iX_k of R are contained in R, then we denote by $\measuredangle_R(X_i; X_j, X_k)$ the angle in R formed by X_iX_j and X_iX_k at X_i. The angle at the vertex X_i of R formed by the sides X_iX_{i-1} and X_iX_{i+1} will be denoted either by $\widehat{X_i}$ or by $\measuredangle_R(X_i; X_{i-1}, X_{i+1})$.

We have the following lemmata.

Lemma 7. *Let* P *be a flat n-gon with $n \leq 5$. We assume that each angle of* P *is smaller than 2π. Then P is isometric to a Euclidean pentagon.*

Proof. If $n = 3$ or $n = 4$ we may verify from formula (1) that each angle of P is smaller than 2π. If $n = 3$, it is easy to see that P isometric to a Euclidean triangle. If P has four vertices, say s_1, s_2, s_3, s_4, then there is a geodesic segment, say $[s_1, s_3] \subset P$, such that $[s_1, s_3] \cap P = \{s_1, s_3\}$. Cutting P along $[s_1, s_3]$ we obtain two Euclidean triangles and thus we may see that P is isometric to a Euclidean quadrilateral. Assume now that P has five vertices s_i, $i = 1, \dots, 5$. We may also prove that there is a geodesic segment, say $[s_1, s_3] \subset P$, such that $[s_1, s_3] \cap P = \{s_1, s_3\}$. So, $[s_1, s_3]$ separates P into a Euclidean triangle R and a Euclidean quadrilateral Q. Now, it is not hard to prove that even if Q is not convex, gluing R with Q along $[s_1, s_3]$ we take a Euclidean pentagon. Indeed, if the gluing surface is not a pentagon isometrically embedded in $\mathbb{E}^2$ then one of the angles of s_i should be greater or equal to 2π, a contradiction.

Finally, Fig. 1 shows that Lemma 7 is not valid if the number of vertices is greater than 5. □

Lemma 8. *Let S be a f.s.c.s. which is homeomorphic to $S^1 \times [0, 1]$. Assume that S has three boundary singularities s_i, $i = 1, 2, 3$ such that s_1 belongs to one boundary component of S and s_2, s_3 belong to the other boundary component. Then there is a geodesic arc, say $[s_1, s_2] \subset S$, such that $[s_1, s_2] \cap \partial S = \{s_1, s_2\}$ and if we cut and open S along $[s_1, s_2]$ we take a Euclidean polygon.*

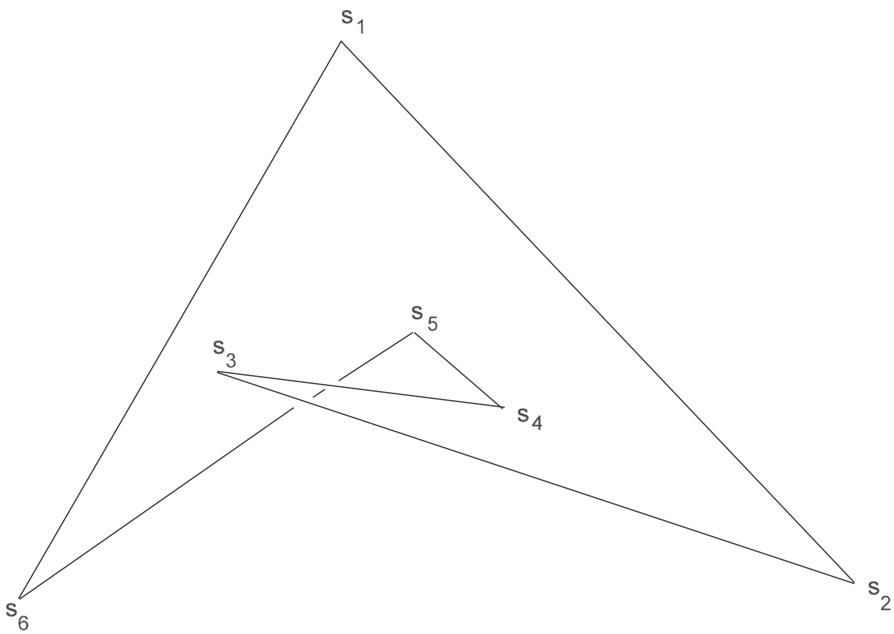

FIGURE 1 A flat, non-Euclidean hexagon

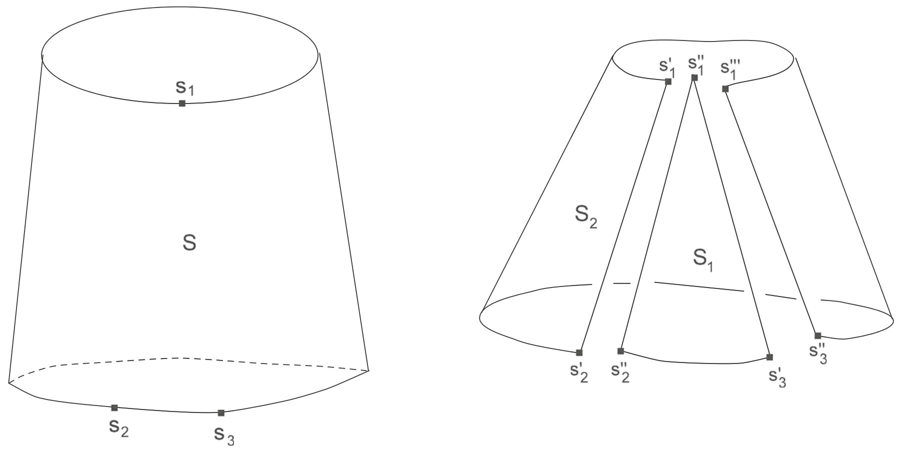

Figure 2 The annulus S decomposed in Euclidean polygons

Proof. Let us denote by $\theta(s_i)$ the angle of the conical singularity s_i. From the Gauss–Bonnet formula (1) we have $\theta(s_1)+\theta(s_2)+\theta(s_3)=3\pi$. Assume first that all angles $\theta(s_i)$ are smaller than 2π. Then it is easy to see that there is a geodesic arc $[s_1,s_2]$ in S such that $[s_1,s_2]\cap\partial S=\{s_1,s_2\}$. If we cut and open S along $[s_1,s_2]$ we take a surface which satisfies all the hypotheses of Lemma 7 and thus it is isometric to a Euclidean polygon with at most five vertices.

Now, let $\theta(s_1)\geq 2\pi$, see Fig. 2. Then there are geodesic arcs $[s_1,s_2]$ and $[s_1,s_3]$ in S such that $[s_1,s_2]\cap\partial S=\{s_1,s_2\}$ and $[s_1,s_3]\cap\partial S=\{s_1,s_3\}$. We cut S along $[s_1,s_2]$ and $[s_1,s_3]$ and we obtain two surfaces: a surface S_1 with three boundary singularities s_1'', s_2'', s_3' and a surface S_2 with four boundary singularities s_1', s_1''', s_3'', s_2', see Fig. 2. In the surface S_2 we have $\theta(s_2')<\pi$, $\theta(s_3'')<\pi$ but some of the angles $\theta(s_1')$, $\theta(s_1''')$ (at most one of them) can be $\geq 2\pi$. Without loss of generality assume that $\theta(s_1')<\pi$. Now, S_1 is a Euclidean triangle and S_2 is a Euclidean quadrilateral. Since $\theta(s_2')<\pi$ and $\theta(s_1')<\pi$, if we glue S_1 and S_2 by identifying $[s_1',s_2']$ and $[s_1'',s_2'']$ we take a Euclidean polygon (at most pentagon).

Let now $\theta(s_2)\geq 2\pi$ (the case $\theta(s_3)\geq 2\pi$ is treated similarly). Then, we consider a geodesic arc $[s_1,s_2]$ in S which has the property $[s_1,s_2]\cap\partial S=\{s_1,s_2\}$, actually any such arc in S has this property. By cutting S along $[s_1,s_2]$ we take a surface S_1 with five conical singularities s_1', s_1'', s_2'', s_3, s_2' in the boundary, see Fig. 3a. If the angles at these points are smaller than 2π then by Lemma 7 S_1 is a Euclidean pentagon. If not, we have either $\theta(s_2')\geq 2\pi$ or $\theta(s_2'')\geq 2\pi$. Without loss of generality assume that $\theta(s_2'')\geq 2\pi$. Then at the other vertices the angles are smaller than π. This implies that the geodesic segment $[s_2'',s_1']$ of S_1 has the property $[s_2'',s_1']\cap\partial S_1=\{s_2'',s_1'\}$, see Fig. 3a. We cut S_1 along $[s_2'',s_1']$ and we take two new surfaces S_2 and S_3, see Fig. 3b. Assume that S_3 is the surface which contains s_3 in its boundary as a boundary singularity. S_3 has s_2' as a vertex and let us denote by s_{12}', s_{22}'' the

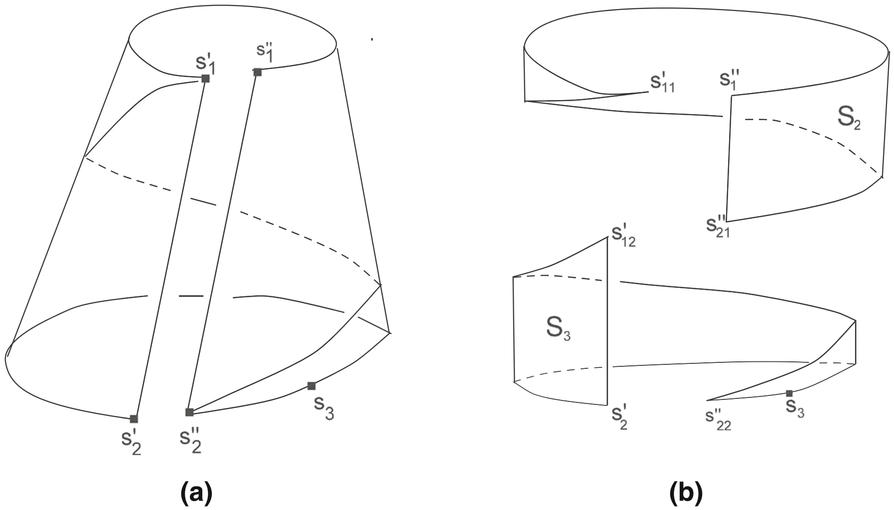

FIGURE 3 The annulus S decomposed in Euclidean polygons

other vertices of S_3. Also S_2 has s_1'' as a vertex and let us denote by s_{11}', s_{21}'' the other vertices of S_2. In our notation we assume that s_{21}'', s_{22}'' correspond to the vertex s_2'' of S_1, while s_{11}', s_{12}' correspond to the vertex s_1' of S_1. Remark that S_2 is a Euclidean triangle and S_3 is a Euclidean quadrilateral. Additionally in S_3, either $\theta(s_2') < \pi$ or $\theta(s_{22}'') < \pi$.

If $\theta(s_2') < \pi$ (resp. $\theta(s_{22}'') < \pi$) and given that $\theta(s_{12}') < \pi$, we glue S_2 to S_3 by identifying $[s_2', s_{12}']$ with $[s_{21}'', s_1'']$ (resp. $[s_{22}'', s_{12}']$ with $[s_{21}'', s_{11}']$) and we obtain a Euclidean pentagon. □

Lemma 9. *There is a pair of simple geodesic loops, say a, b, passing through s such that a intersects b transversely and if we cut and open T along a and b we take a Euclidean pentagon.*

Proof. Let γ be a simple closed curve in T which is not freely homotopic to c and let γ_0 be the closed geodesic in the homotopy class of γ. By taking a parallel translation of γ_0, if it is necessary, we may assume that γ_0 passes through s. From 6, γ_0 can only have trivial self-intersection points at s. Therefore γ_0 consists of simple geodesic loops based at s. Pick one of them and label it as a.

Considering the geodesic loop a we cut and open T along a. Then we take a cylinder C which satisfies all hypothesis of Lemma 8. Therefore there exists a geodesic arc $[s_1, s_2]$ in C as in Lemma 8. Obviously this arc defines a geodesic loop b in T which with a define the required loops. □

Lemma 10. *Let $\mathrm{Q} = (A, B, C, D)$ be a Euclidean quadrilateral. Then Q is uniquely determined from the lengths of its sides, its area and the fact that $\measuredangle_{\mathrm{Q}}(A; D, B) + \measuredangle_{\mathrm{Q}}(C; D, B)$ is greater or not than π.*

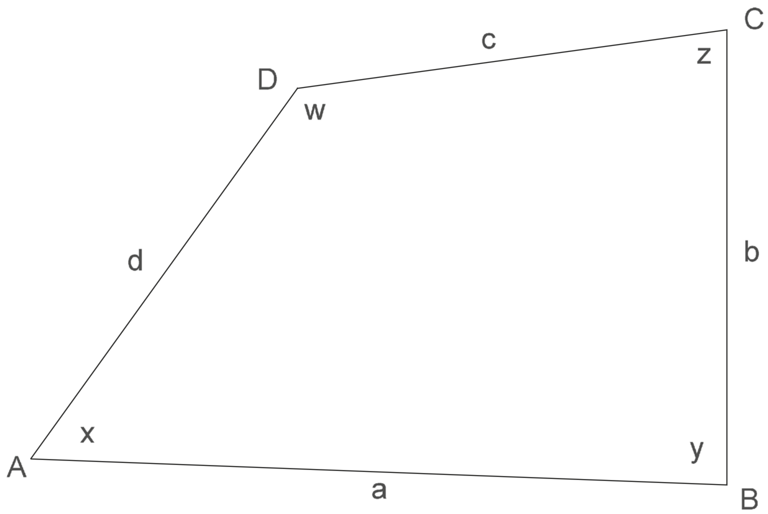

Figure 4 A Euclidean quadrilateral

Proof. Let a, b, c, d be the lengths of sides AB, BC, CD, DA respectively and let $x = \measuredangle_Q(A; D, B)$, $y = \measuredangle_Q(B; A, C)$, $z = \measuredangle_Q(C; D, B)$, $w = \measuredangle_Q(D; A, C)$, see Fig. 4. We set $\phi = (x + z)/2$, $s = (a + b + c + d)/2$ and let E be the area of $\mathcal{Q}$. Then we have the Bretschneider's formula:

$$E^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \phi \tag{2}$$

Actually this formula is valid for convex quadrilateral but it is not difficult to show that it is true for any quadrilateral.

Now E is a function of ϕ, i.e. $E = E(\phi)$ and $2E(\phi)E'(\phi) = abcd \sin(2\phi)$. Since $0 < 2\phi < 2\pi$, the function $E(\phi)$ is increasing when $0 < \phi \leq \pi/2$ and decreasing when $\pi/2 \leq \phi < \pi$. Therefore for a given value of E, there are at most two angles ϕ_1, ϕ_2 such that $E = E(\phi_1) = E(\phi_2)$. On the other hand, it is either $x + z \leq \pi$ or $x + z > \pi$. So it suffices to show that Q is uniquely determined from the lengths of its sides, its area and from ϕ with $2\phi = x + z \leq \pi$. To prove this, we have

$$a^2 + d^2 - 2ad \cos x = b^2 + c^2 - 2bc \cos z.$$

This formula follows from the fact that since $x + z \leq \pi$ the diagonal BD is lying inside Q.

We set

$$F(x, \phi) = a^2 + d^2 - 2ad \cos x - (b^2 + c^2 - 2bc \cos(2\phi - x)),\ 0 < x < \pi,\ 0 < \phi \leq \pi/2.$$

So, we have $F(x, \phi) = 0$ and

$$\frac{\partial F}{\partial x} = 2ad \sin x + 2bc \sin(2\phi - x) = 2ad \sin x + 2bc \sin(2\phi - x).$$

Therefore it is $\partial F/\partial x > 0$, which implies that $x = x(\phi)$ is a function of ϕ. Now, from the formula (2), for a given value of area E, $\cos^2 \phi$ is determined.

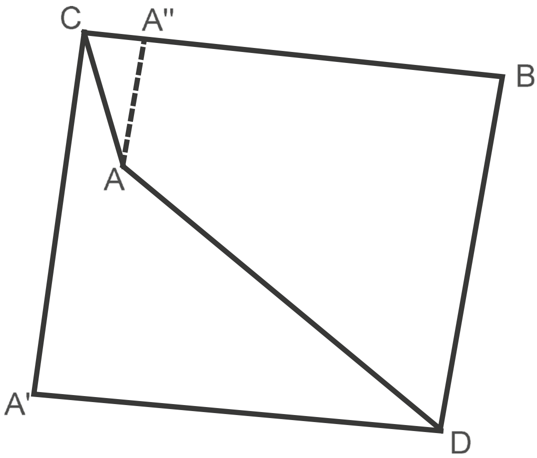

Figure 5 A Euclidean quadrilateral with $\widehat{B} + \widehat{D} < \pi$ and $|CB| = |AD|$

Since $0 < \phi \leq \pi/2$ the angle ϕ is determined and also x is determined as a function of ϕ. Therefore Q is determined. $\square$

Lemma 11. *Let* Q $= (A, D, B, C)$ *be a quadrilateral such that* $|CB| = |DA|$ *and* $\widehat{B} + \widehat{D} < \pi$. *Then* $|AC| < |BD|$. *If furthermore,* $\widehat{D} < \widehat{B}$ *then* $\widehat{A} + \widehat{D} > \pi$.

Proof. In order to prove $|AC| < |BD|$ we assume first that $\widehat{D} < \widehat{B}$ (if $\widehat{B} < \widehat{D}$ we work similarly). Then if we consider the parallelogram $P' = (A', C, B, D)$ the vertex A belongs in the interior of P', see Fig. 5. Therefore, $|DA| + |AC| < |DA'| + |A'C| = |BC| + |BD|$ and hence $|AC| < |BD|$.

Now, since $\widehat{D} < \widehat{B}$, there exists A'' in the interior of CB such that AA'' is parallel to DB. Hence $\widehat{A} + \widehat{D} > \pi$. $\square$

Now, we have the following basic proposition.

Proposition 12. *Let* a, b *be two simple geodesic loops which satisfy the assumptions of Lemma* 9. *Then there are simple closed geodesics* a_0 *and* b_0 *in* T, *freely homotopic to* a, b *respectively, such that* $a_0 \cap b_0 = \{s\}$.

Proof. Let x, y, z be the angles at s, formed by the geodesic loops a and b, as they are shown in Fig. 6i. Cutting and opening T along a, b we take a pentagon $P = (A, D, B, C, E)$ see Fig. 6ii. It is $\measuredangle_P(A; D, E) = x$, $\measuredangle_P(D; A, B) = y$, $\measuredangle_P(B; C, D) = z$. We distinguish the following cases:

(I) there is some angle $\phi \in \{x, y, z\}$ such that $\phi \geq \pi$.

In this case we distinguish the following subcases:

(I_1) $y \geq \pi$;
(I_2) $y < \pi$, $x \geq \pi$, $z \geq \pi$;
(I_3) $y < \pi$, $x \geq \pi$, $z < \pi$ (similarly, $y < \pi$, $z \geq \pi$, $x < \pi$).

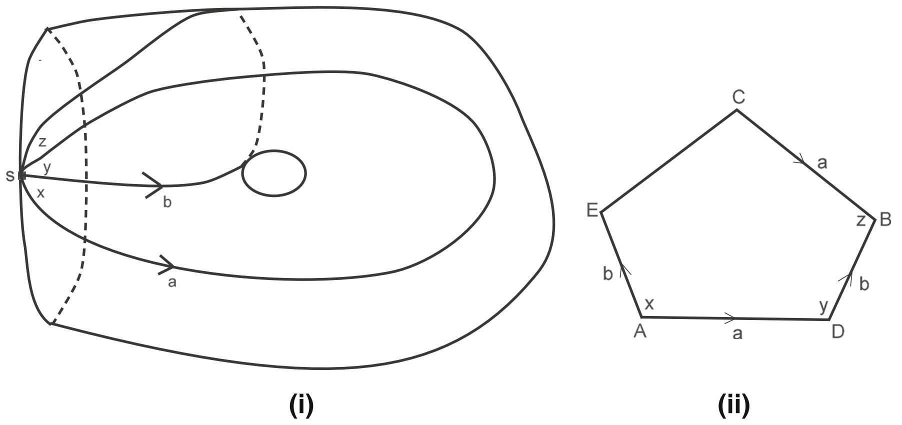

FIGURE 6 Two simple closed geodesics containing the conical singularity

(II) each angle x, y, z is smaller than π.

In this case we distinguish the subcases:

(II_1) P is convex;
(II_2) P is non-convex, $y + x \geq \pi$ and $y + z < \pi$ (similarly, P is non-convex, $y + z \geq \pi$ and $y + x < \pi$);
(II_3) P is non-convex, $y + x < \pi$ and $y + z < \pi$;
(II_4) P is non-convex, $y + x \geq \pi$ and $y + z \geq \pi$.

We will examine each subcase separately and we will prove the existence of geodesics a_0, b_0.

(I_1) : Obviously a and b are geodesics of T. Hence, we set $a_0 = a$ and $b_0 = b$.
(I_2) : Also a and b are geodesics of T. Hence we set again $a_0 = a$ and $b_0 = b$.
(I_3) : Obviously a is a geodesic of T and we set $a_0 = a$. Now, if $y + z \geq \pi$ then b is a geodesic of T and hence we set $b_0 = b$. If $y + z < \pi$ then we consider the diagonal AC which is always lying inside P, see Fig. 7. Let $Q = (A, D, B, C)$ and $n = \measuredangle_Q(C; A, B)$, $m = \measuredangle_Q(A; C, D)$. Since $y + z < \pi$ we have $m + n \geq \pi$. Therefore AC is a geodesic of T which is additionally freely homotopic to b. So we set $b_0 = AC$.

We come now to the case (II).

(II_1) : Set, $u = \measuredangle_P(A; B, E)$, $u' = \measuredangle_P(A; B, D)$, $t = \measuredangle_P(B; A, C)$, $t' = \measuredangle_P(B; A, D)$, $r = \measuredangle_P(E; A, B)$, $w = \measuredangle_P(B; A, E)$, see Fig. 8i.

We remark that $t > u'$ or $u > t'$. Indeed, if $t \leq u'$ and $u \leq t'$ then we will take a contradiction by showing that:

$$|AD| + |DB| > |AE| + |EC| + |CB| \iff a + b > b + |EC| + a \qquad (3)$$

For it, we remark $AB \subset P$ since P is convex. So we consider a parallelogram $P' = (A, D, B, D')$ such that: $\measuredangle_{P'}(A; B, D') = \measuredangle_{P'}(B; A, D) = \measuredangle_P(B; A, D)$,

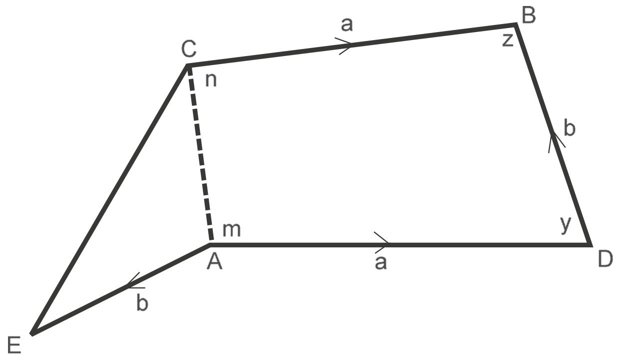

FIGURE 7 A pentagon with $y + z < \pi$

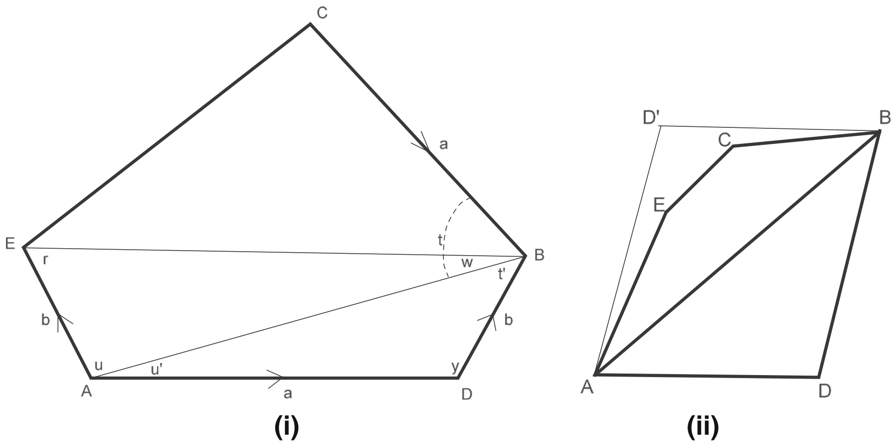

FIGURE 8 Existence of simple closed geodesics containing the conical singularity in the case of a convex pentagon

$\measuredangle_{P'}(B; A, D') = \measuredangle_{P'}(A; B, D) = \measuredangle_P(A; B, D)$. Therefore $(AE \cup EC \cup CB) \subset (A, B, D')$ which implies 3, see Fig. 8ii.

- If $t > u'$ and $u \geq t'$ then $y + x \geq \pi$ and $y + z > \pi$. Therefore the geodesic loops a and b are both geodesics of T. So we set $a_0 = a$ and $b_0 = b$.
- If $t > u'$ and $u < t'$ then $y + z > \pi$ and so b is a geodesic but a is not a geodesic. In this case we consider the diagonal EB which is freely homotopic to a. In order EB to be a geodesic of T it must be $r+w+t' \geq \pi$. From the triangle (E, A, B) we have $r + u + w = \pi$ and since $u < t'$ we deduce that $r + w + t' > \pi$. Therefore we set $b_0 = b$ and $a_0 = EB$.
- If $t \leq u'$ and $u > t'$ we work similarly.

Therefore the proposition is proven in the case (II_1).

Assuming now that P is non-convex we claim that one of the diagonal AC or BE of P is lying inside P. Indeed, assuming that AC and BE are not contained

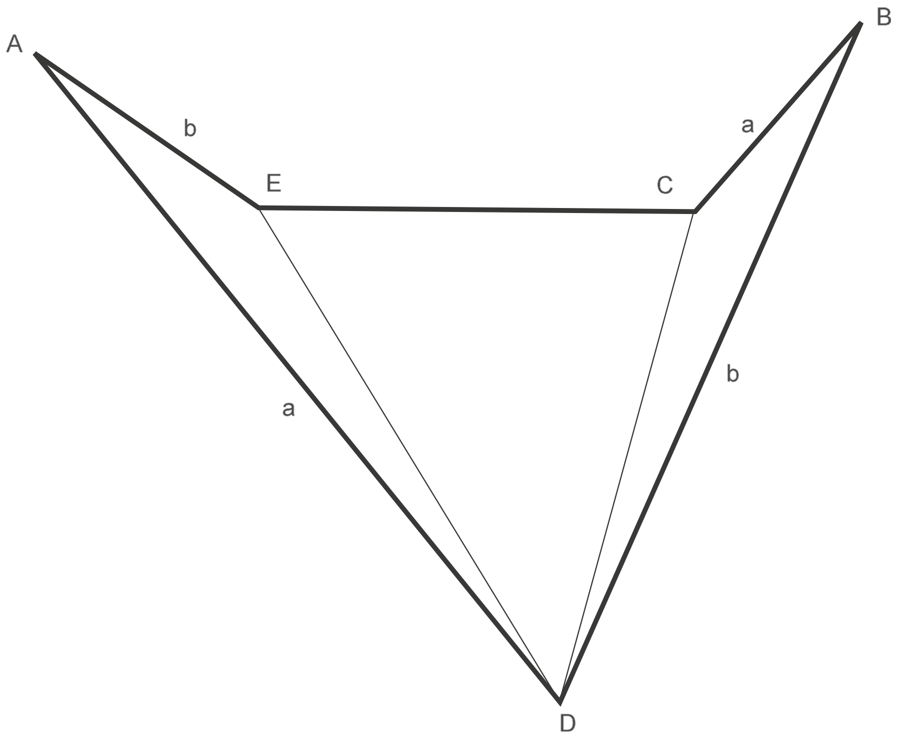

Figure 9 A Euclidean pentagon not containing its diagonals AC and BE

in P we have the configuration of Fig. 9. Then it is easy to prove that $|AD|$ + $|BD| > |AE|$ + $|EC|$ + $|CB|$ which gives a contradiction since $AD = a$, $BD = b$, $BC = a$, $AE = b$. Therefore our claim is proven.

In the following we assume, without loss of generality, that $AC \subset P$.

(II_2) : Since $x + y \geq \pi$ we have that a is geodesic of T. On the other hand, we have $AC \subset P$ and AC is freely homotopic to b in T. Now, from the quadrilateral (A, C, B, D) we deduce that $\measuredangle_P(A; C, D) + \measuredangle_P(C; A, B) > \pi$ since $y + z < \pi$. Therefore AC is a geodesic of T and we set $a_0 = a$ and $b_0 = AC$, see Fig. 10.

(II_3) : As is the case (II_2) we may prove that $AC = b'$ is a geodesic of T and AC is freely homotopic to b, see Fig. 11. Now we cut the triangle (A, C, E) and we glue it to P along $DB = AE = b$. Then we take a new pentagon $Q = (A, D, Z, B, C)$ as in Fig. 11.

Considering the quadrilateral $(A, C, B, D) \subset P$ we have $z + y < \pi$ and from Lemma 11 it follows that $|b'| = |AC| < |b| = |DB|$. Therefore, from triangle (E, A, C) we deduce that $\measuredangle_P(E; C, A) < \pi/2$. If $z \geq \pi/2$ then $y < z$ and from Lemma 11 we take again that $\measuredangle_P(A; C, D) + y > \pi$. This gives a contradiction

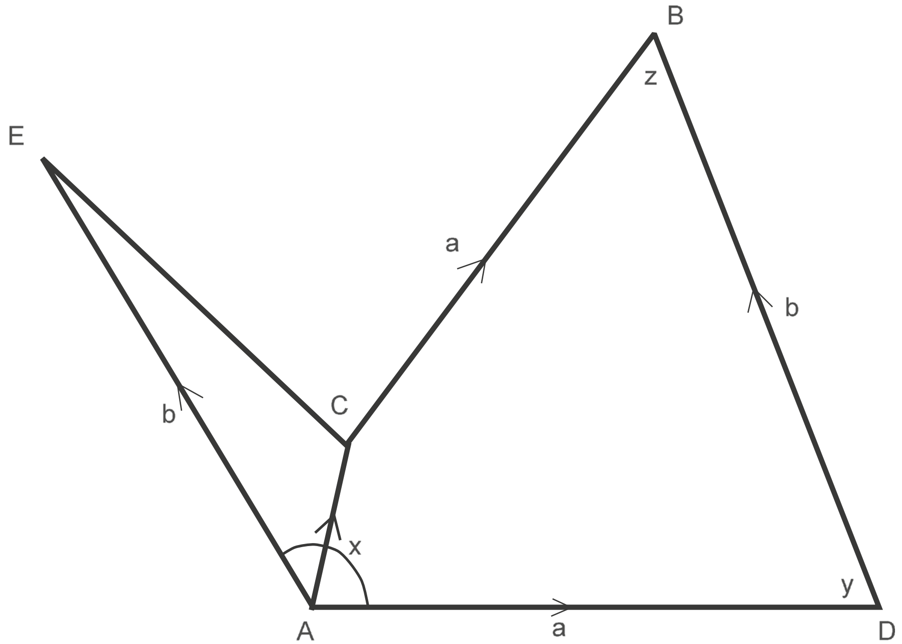

FIGURE 10 Existence of simple closed geodesics containing the conical singularity in the case of a non-convex pentagon with $y + x \geq \pi$ and $y + z < \pi$

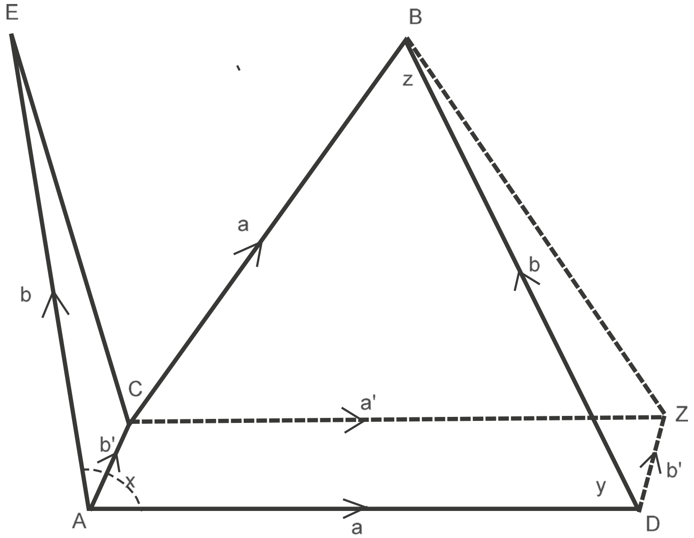

FIGURE 11 Existence of simple closed geodesics containing the conical singularity in the case of a non-convex pentagon with $y + x < \pi$ and $y + z < \pi$

since by hypothesis we have $x+y<\pi$. Therefore $z<\pi/2$ and $\measuredangle_Q(B;C,Z) = \measuredangle_P(B;C,D)+\measuredangle_P(E;A,C)<\pi$. This implies that CZ is contained in Q.

Now in Q the diagonal CZ is a geodesic of T. Indeed, we have that $\measuredangle_Q(C;A,Z)+\measuredangle_Q(Z;C,D)>\pi$ since $\measuredangle_Q(A;C,D)+\measuredangle_Q(D;A,Z)=x+y<\pi$. Furthermore, we have CZ is freely homotopic to a. Therefore we set $a_0=CZ$ and $b_0=AC$.

(II_4) : Then $a_0=a$ and $b_0=b$.

□

Remark. In the proof of Proposition 12 we have used the fact that the polygon P is Euclidean in order to prove an inequality between the lengths of sides of P with the help of Fig. 9. All the other proofs work equally well if P is just a flat polygon.

Now we are able to prove the following theorem.

Theorem 13. *Let T be a flat torus with one boundary component c and one conical singularity $s\in c$. Then,*

- (i) *There are five closed geodesics whose lengths determine T under isometry; four of them are simple and one consists of the union of two simple geodesic loops based at s.*
- (ii) *The area of T and the lengths of four simple closed geodesics of T determine two isometry types of T.*

Proof. ¿From Proposition 12, there are simple closed geodesics a and b passing through s, see Fig. 6. We cut and open T along a,b and we take a flat pentagon $P=(A,D,B,C,E)$. Without loss of generality, we may assume that $a= AD=CB$, $b=AE=DB$; this notation means that the oriented segment AD is identified with the oriented segment CB (similarly for AE and DB) in order to take T.

Recall that $x=\measuredangle_P(A;D,E)$, $y=\measuredangle_P(D;A,B)$, $z=\measuredangle_P(B;C,D)$. Since a and b are geodesics we have

$$x+y\geq\pi,\ \ y+z\geq\pi \tag{4}$$

We distinguish the following cases.

(1) $x<\pi$, $y<\pi$, $z<\pi$;
(2) $x\geq\pi$, $y<\pi$, $z<\pi$ (similarly, $z\geq\pi$, $y<\pi$, $x<\pi$);
(3) $y<\pi$, $x\geq\pi$, $z\geq\pi$;
(4) $y\geq\pi$, $x<\pi$, $z<\pi$;
(5) $x\geq\pi$, $y\geq\pi$, $z<\pi$ (similarly, $z\geq\pi$, $y\geq\pi$, $x<\pi$).

In the following we will study each case separately. Our goal is to find additional closed geodesics in T whose lengths, with the lengths of a and b, determine T up to isometry. In most cases the additional closed geodesics appear as diagonals of pentagons.

Case (1): Let P be the flat pentagon obtained by cutting T along a and b. Because of the condition (4) each angle of P is $< 2\pi$ and thus P is a Euclidean pentagon, from Proposition 7. Since $x < \pi$, $y < \pi$, $z < \pi$ the diagonal AB is contained in P and it is always a geodesic of T. Indeed, it is sufficient to prove that

$$\measuredangle_P(A; B, E) + \measuredangle_P(B; A, C) + y \geq \pi \tag{5}$$

¿From the relation (4) we have $\measuredangle_P(A; B, E) \geq \measuredangle_P(B; A, D)$ and $\measuredangle_P(B; A, C) \geq \measuredangle_P(A; B, D)$ and since $y + \measuredangle_P(B; A, D) + \measuredangle_P(A; B, D) = \pi$ we obtain the relation (5).

Next we assert that if DE and DC are contained in P then either DE or DC is a geodesic in T. Indeed, DC is a geodesic if and only if

$$\measuredangle_P(C; B, D) + \measuredangle_P(D; B, C) + x \geq \pi \tag{6}$$

Also DE is a geodesic if and only if

$$\measuredangle_P(E; A, D) + \measuredangle_P(D; A, E) + z \geq \pi \tag{7}$$

Obviously, the sum of angles in relations (6) and (7) is equal to 2π. Therefore, at most one of the relations (6) or (7) is valid.

Finally, without loss of generality, we assume that DC is contained in P but DE is not contained in P, see Fig. 12. Since DE is not contained in P we have $\measuredangle_P(C; B, E) \geq \pi$ and therefore $\measuredangle_P(E; A, C) \leq \pi$. This implies that $AC \subset P$. We have also that $DC \subset P$.

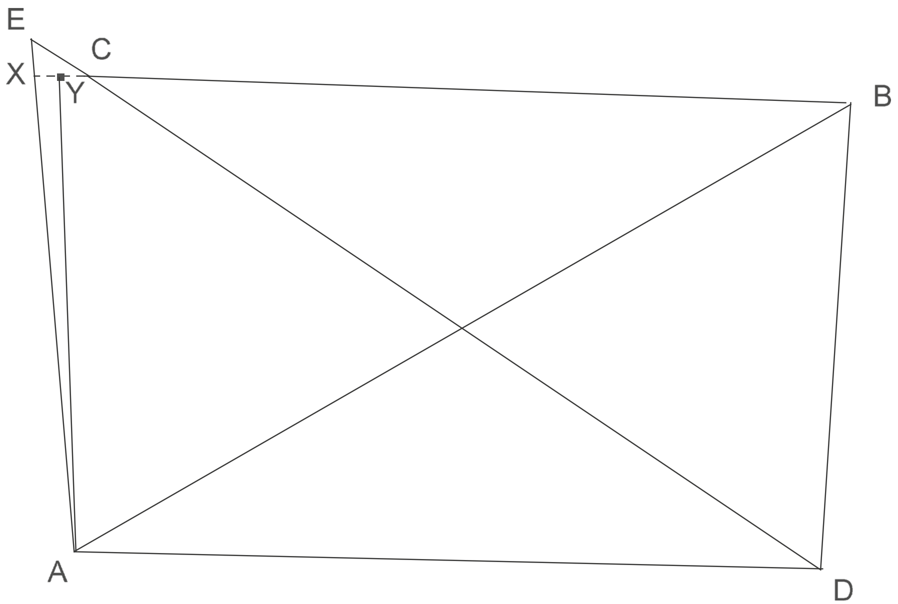

FIGURE 12 A Euclidean pentagon with $x < \pi$, $y < \pi$, $z < \pi$

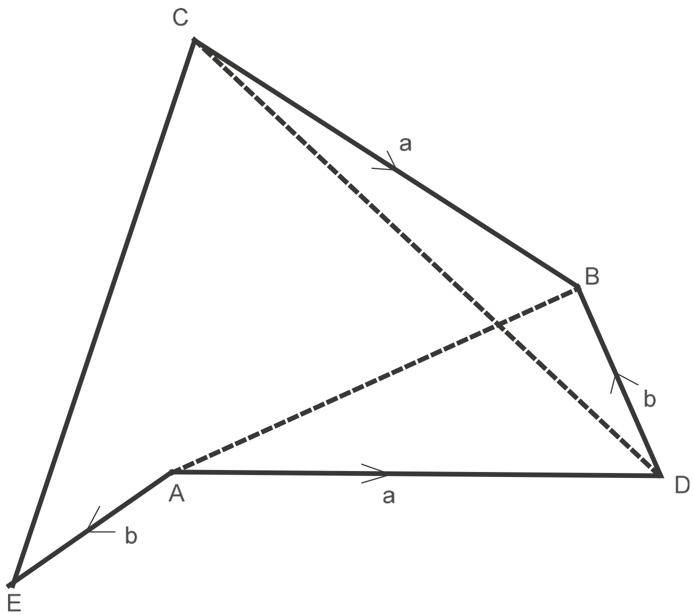

FIGURE 13 A Euclidean pentagon with $x \geq \pi$, $y < \pi$, $z < \pi$

Now, $|CB| = |AD| = |a|$ and $y + z \geq \pi$, therefore $|AC| \geq |b|$. The extension of DC intersects AE to a point X. We have that $|AX| \leq |b|$, $|AC| \geq |b|$, therefore there is a point $Y \in CX$ such that $|AY| = |b|$. It is to see now that $\measuredangle_P(A; Y, D) \geq \measuredangle_P(B; C, D) = z$. Therefore, $x \geq z$. This implies that $\measuredangle_P(C; B, D) + x + \measuredangle_P(D; B, C) \geq \pi$. This last condition implies that DC is a geodesic in T.

Therefore, in each case P is determined from the lengths of geodesics AB and DE or DC and the lengths of edges of P,which correspond to the geodesics a and b of T.

Case (2): We may easily prove that $AB \subset P$ and $CD \subset P$ and we will show that AB and CD are geodesics of T, see Fig. 13. Indeed, AB is a geodesic if and only if

$$\measuredangle_P(A; B, E) + y + \measuredangle_P(B; C, A) \geq \pi \tag{8}$$

Now, BD is a geodesic and hence $\measuredangle_P(D; B, A) + \measuredangle_P(B; D, A) + \measuredangle_P(B; C, A) \geq \pi$. On the other hand, $\measuredangle_P(A; B, D) + \measuredangle_P(D; B, A) + \measuredangle_P(B; D, A) = \pi$. Comparing these two relations we have $\measuredangle_P(B; C, A) \geq \measuredangle_P(A; B, D)$.

Finally, AD is a geodesic therefore $\measuredangle_P(A; B, E) + \measuredangle_P(A; D, B) + \measuredangle_P(D; B, A) \geq \pi$ and since $\measuredangle_P(B; C, A) \geq \measuredangle_P(A; B, D)$ we obtain the relation (8).

We may prove also that CD is a geodesic. Therefore P is determined from the lengths of geodesics AB, CD and the lengths of edges of P.

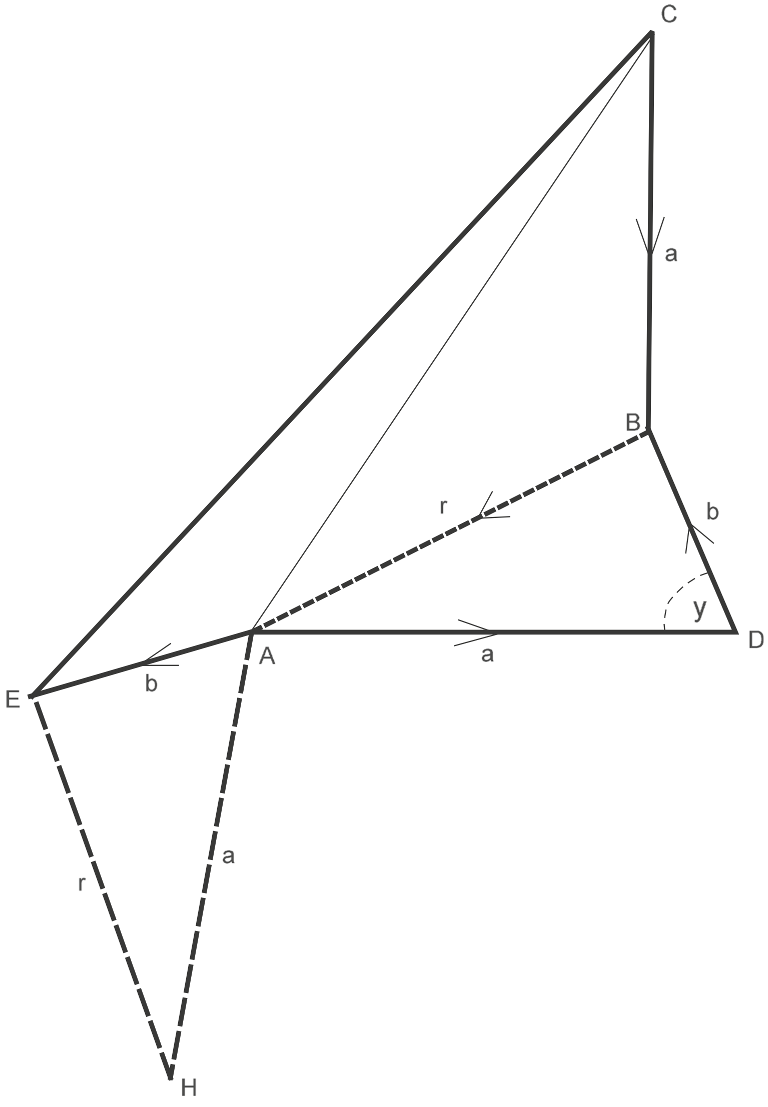

FIGURE 14 A Euclidean pentagon with $x \geq \pi$, $y < \pi$, $z \geq \pi$

Case (3): The diagonal $r = BA$ is contained in P and it is always a geodesic of T, see Fig. 14. Indeed, it is sufficient to prove that

$$\begin{aligned}
&\measuredangle_P(A;B,E) + \measuredangle_P(D;B,A) + \measuredangle_P(B;C,A) \geq \pi \iff \\
&\measuredangle_P(A;B,E) + \pi - \measuredangle_P(A;B,D) - \measuredangle_P(B;A,D) + \measuredangle_P(B;C,A) \geq \pi \iff \\
&\measuredangle_P(A;B,E) + \measuredangle_P(B;C,A) \geq \measuredangle_P(A;B,D) + \measuredangle_P(B;A,D) \iff \\
&\measuredangle_P(A;B,E) + \measuredangle_P(B;C,A) + \measuredangle_P(A;B,D) + \measuredangle_P(B;A,D) \geq \\
&\geq 2\measuredangle_P(A;B,D) + 2\measuredangle_P(B;A,D) \iff \\
&\measuredangle_P(A;D,E) + \measuredangle_P(B;C,A) \geq 2\measuredangle_P(A;B,D) + 2\measuredangle_P(B;A,D)
\end{aligned}$$

But $\measuredangle_P(A; D, E) + \measuredangle_P(B; C, A) \geq 2\pi$, while $2\measuredangle_P(A; B, D) + 2\measuredangle_P(B; A, D) < 2\pi$.Therefore our assertion is proven.

Now, we examine two cases:

$$\measuredangle_P(A; B, E) + y < \pi \tag{9}$$

or

$$\measuredangle_P(B; C, A) + y < \pi \tag{10}$$

First, we study the case (9). By cutting P along AB we take a triangle ABD that we glue back along the side AE. Then we obtain a new pentagon, say $Q = (E, C, B, A, H)$, see Fig. 14. We have that Q has its angles at the vertices H, A smaller than π. Therefore, we are reduced to the case (1) or (2), depending of the angle $\measuredangle_P(B; C, A)$ respectively. Therefore, the isometry type of Q is determined by the lengths of simple closed geodesics and so the same is true for P.

The case (10) is treated similarly.

Therefore, in order to complete our study, it suffices to study the case where,

$$\measuredangle_P(A; B, E) + y \geq \pi \text{ and } \measuredangle_P(B; C, A) + y \geq \pi \tag{11}$$

In this case, the knowledge of the area of P implies that the area of quadrilateral $P' = (A, B, C, E)$ is known. We set $\theta = \measuredangle_P(A; B, E) + \measuredangle_P(C; E, B)$. For P' we know the lengths of its edges and its area. Now, from Lemma 10 we have P' is determined from the length of its sides, its area and from the fact that $\theta \geq \pi$ or $\theta < \pi$. Therefore the lengths of four simple closed geodesics of T and its area determine two isometry types of T.

Finally, we will prove that there is an additional closed geodesic whose length with the lengths of the previous four simple closed geodesics determine T up to isometry. Indeed, in the quadrilateral (A, B, C, E) at least one of the diagonals AC, BE lies inside it. Without loss of generality, we assume that AC has this property. Then, it is easy to prove that $d = AC \cup DB$ is a geodesic; notice here that the orientation from A to C and then from D to B is important in order to define d. Obviously, this geodesic has a (non trivial) self-intersection point. Now, if we know the length of d, and since we know the length of $b = DB$, we know the length of the geodesic loop AC. Therefore the isometry type of T is determined by the lengths of geodesics a, b, r, c, d, see Fig. 14.

Case (4): It is easy to verify that the diagonals CD and ED are contained in P and furthermore that exactly one of them is a geodesic of T, see Fig. 15. Indeed, CD is a geodesic if and only if

$$\measuredangle_P(C; B, D) + x + \measuredangle_P(D; B, C) \geq \pi \tag{12}$$

and ED is a geodesic if and only if

$$\measuredangle_P(E; A, D) + z + \measuredangle_P(D; A, E) \geq \pi \tag{13}$$

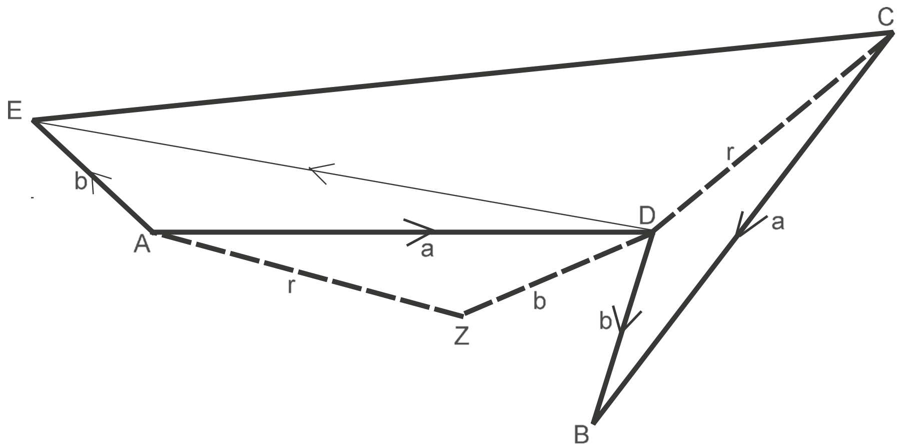

FIGURE 15 A Euclidean pentagon with $x < \pi$, $y \geq \pi$, $z < \pi$

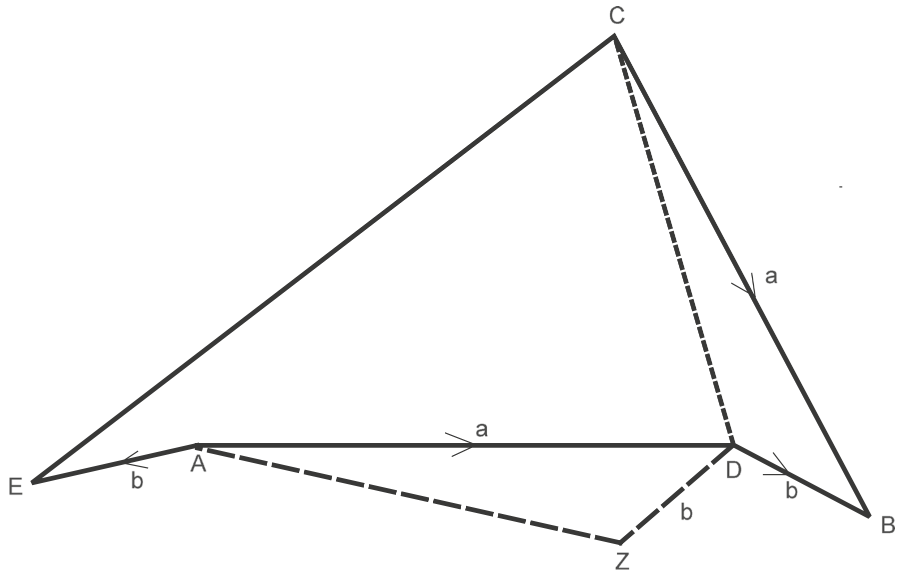

FIGURE 16 A Euclidean pentagon with $x \geq \pi$, $y \geq \pi$, $z < \pi$

However, the sum of all angles in the relations (12) and (13) is equal to the sum of all angles of P minus the angles of triangle (C, D, E) and hence this sum is equal to 2π. Therefore our assertion is proven.

In the following we assume, without loss of generality that CD is a geodesic of T. We cut the triangle (B, C, D) from P. Then we glue it to (A, D, C, E) along the side $AD = CB = a$. Thus, we take a new pentagon $Q = (A, Z, D, C, E)$, see Fig. 15. Obviously, we have $\measuredangle_Q(Z; A, D) < \pi$ and hence we fall in one of the previous cases (1) or (2) or (3).

Case (5): The diagonal CD is a geodesic of P. For this, it is sufficient to prove that $\measuredangle_P(D;B,C) + \measuredangle_P(A;E,D) + \measuredangle_P(C;B,D) \geq \pi$, see Fig. 16. However $\measuredangle_P(A;E,D) = x \geq \pi$, hence our assertion follows.

Now we cut the triangle (D,C,B) from P and we glue it to (A,D,C,E) along $AD = CB = a$. Then we take a new pentagon $Q = (A,Z,D,C,E)$, see Fig. 16. Then $\measuredangle_Q(Z;A,D) < \pi$ and $\measuredangle_Q(A;Z,E) \geq \pi$. We look at the angle $\measuredangle_Q(D;Z,C)$. Then, if $\measuredangle_Q(D;Z,C) \geq \pi$ we fall in the case (3) and if $\measuredangle_Q(D;Z,C) < \pi$ we fall in the case (2).

Therefore in each case we can find closed geodesics of T whose lengths determine the geometry of T. □

Finally, we will prove the following lemma which shows that in the case (3) and hence probably in the cases (4) and (5), there do not exist finitely many simple closed geodesics whose lengths determine T.

Lemma 14. *In the case* (3) *and under the assumption* (11)*, there do not exist closed geodesics freely homotopic to simple closed curves, whose lengths determine the isometry type of* T.

Proof. Let $a = AD = CB$, $b = DB = AE$ be the simple closed geodesics of Theorem 13 and $r = BA$ the simple closed geodesic of case (3) in 13. Notice here that in what follows, a, b, r are oriented as it is shown in Figs. 14 or 17. Let γ be a simple closed curve of T, equipped with an orientation. Then either γ is freely homotopic to the boundary c or c^{-1} (choosing an orientation for c) or γ is freely homotopic to a curve of the form $a^m b^n$, where $m, n \in \mathbb{Z}$ and $(|m|, |n|) = 1$. Indeed, this follows from the following simple topological fact: if S is a torus and $S_0 \subset S$ is a torus with one hole, then two simple closed curves γ, γ' are isotopic in S if and only if they are isotopic in S_0.

The condition (11) in case (3) of Theorem 13, implies that the curves ar^{-1}, $r^{-1}b$ are geodesics. Also, $a^{-1}b$ and ab^{-1} are geodesics since the angles x and z are greater or equal to π, see Fig. 17.

Take now an oriented simple closed curve γ of T which is not freely homotopic to c or to c^{-1}. Therefore, γ is freely homotopic to one of the curves $a^m b^n$, $a^{-m} b^n$, $a^m b^{-n}$, $a^{-m} b^{-n}$, where $m,\, n > 0$ and $(m, n) = 1$. Then the geodesic γ_0 in the free homotopy class of γ must be one of the following curves, $a^{m-1} r^{-1} b^{n-1}$ or $a^{-m} b^n$ or $a^m b^{-n}$ or $a^{1-m} r b^{1-n}$. Therefore the image of γ_0 runs the images of geodesics a, b, r. So the length of γ does not give any more information about the geometry of T. This proves our lemma. □

Remark 15. If the singularity s is in the interior of T then T is determined, up to isometry, from the lengths of five closed geodesics and the area of T. Indeed, if $c = \partial T$ we may translate c_0 into a parallel geodesic c_0 passing through s. Let $T_0 \subset T$ be the torus bounded by c_0. The geometry of T_0, and hence the area of T_0, is determined by the lengths of five closed geodesics, see Theorem 13. Therefore, the area of annulus $A \subset T$ with $\partial A = c \cup c_0$ is known. This area with the length of c determine A up to isometry.

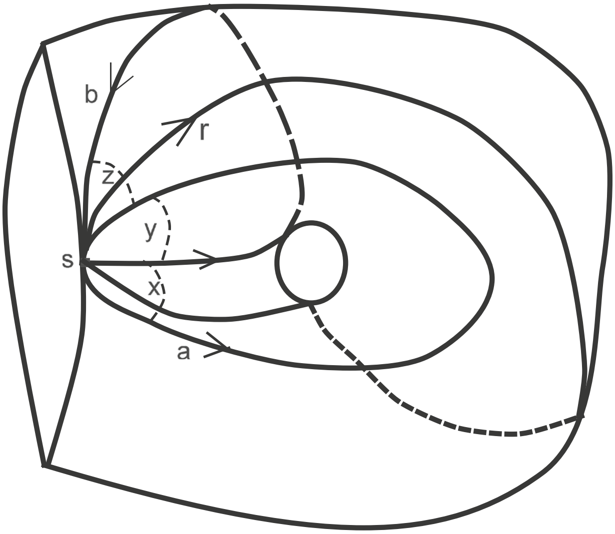

FIGURE 17 Three simple closed geodesics containing the conical singularity

3. Closed flat surfaces with one singularity

Theorem 16. *Let S be a closed, orientable flat surface of genus $g \geq 3$ with one singularity s. Assume that there are g simple closed geodesics of S such that each one separates a torus of genus one from S. Then there are $5g + (g - 3)$ closed geodesics γ_i of S, all passing through s, whose lengths determine, up to isometry, the geometry of S. Furthermore, each γ_i can be chosen to be homotopic to a simple closed curve in S.*

If $g = 2$ we assume that there is a simple closed geodesic separating S into two tori. Then there are nine closed geodesics γ_i of S, all passing through s, whose lengths determine, up to isometry, the geometry of S. Furthermore, seven of the γ_i are simple and each one of the two remaining is a union of two simple geodesic loops based at s.

Proof. If $g = 2$, then S is decomposed, via a simple closed geodesic c passing through s, into two tori T_1, T_2, see Fig. 18. From the demonstration of Theorem 13, from each $i = 1, 2$, the isometry type of T_i is determined by five simple closed curve a_i^k, $k = 1, 2, 3, 4, 5$, all passing through s, such that:

(1) $a_1^1 = a_2^1 = c$;
(2) a_i^k are simple closed geodesics for $k = 2, 3, 4$;
(3) a_i^5 is a simple geodesic loop, not necessarily geodesic.

Consider now a geodesic γ whose image is $a_1^5 \cup a_2^2$ and oriented as in Fig. 18. Obviously γ is a geodesic which has a self intersection point. Therefore, if we know the length of γ we deduce the length of a_1^5 since a_2^2 is a geodesic of known

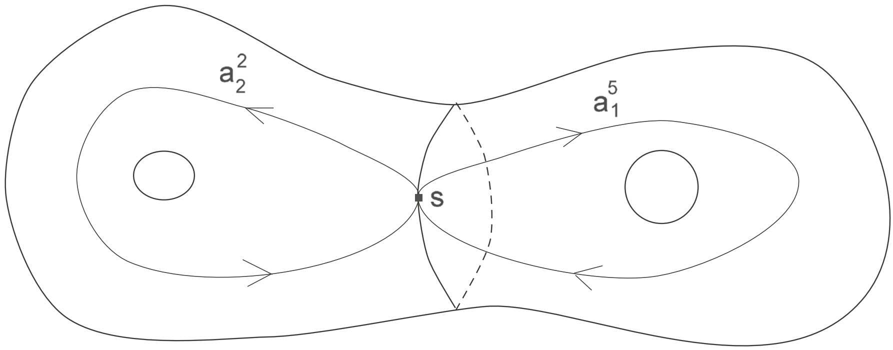

Figure 18 A decomposition of S into two Euclidean tori

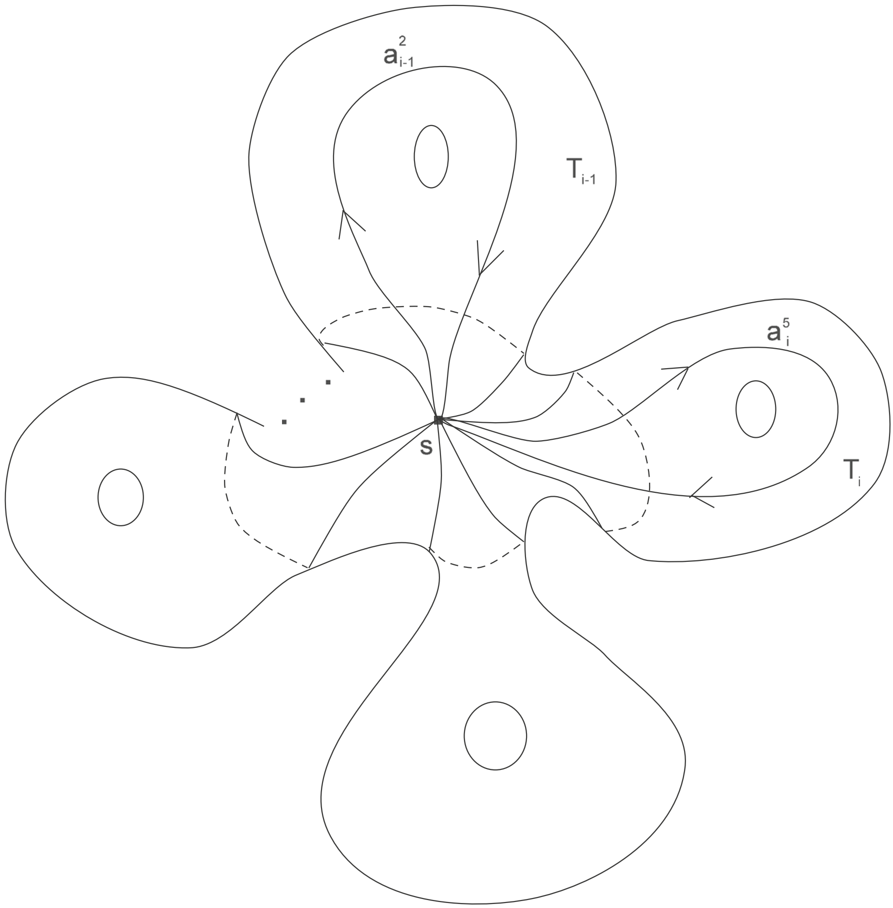

Figure 19 A decomposition of S into Euclidean tori

length. This proves our theorem for $g = 2$. Actually, in order to define γ we could combine a_1^5 with any geodesic in $\{a_2^2, a_2^3, a_2^4\}$.

If $g \geq 3$, then S can be decomposed into g tori T_i, $i = 1, 2, .., g$ and a flat polygon say F. As before, for each i there are inside T_i five simple closed geodesic loops a_i^k, $k = 1, .., 5$, all based at s, such that:

(1) $a_i^1 = c_i$;
(2) a_i^k are simple closed geodesic, $\forall\, k = 2, 3, 4$;
(3) a_i^5 is a simple geodesic loop, not necessarily geodesic;
(4) the lengths of a_i^k determine the isometry type of T_i for each i.

Consider now an arbitrary a_i^5 and without loss of generality we may assume that the angle w at s, between c_i and a_i^5 is $\geq \pi$, see Fig. 19. Then, considering for example the geodesic a_{i-1}^2 the union $a_i^5 \cup a_{i-1}^2 = \gamma_i$, where a_{i-1}^2, a_i^5 are oriented as in Fig. 19, forms a closed geodesic with a trivial self intersection point. Therefore γ_i is homotopic to simple closed curve of S. Now, from the length of γ_i we deduce the length of a_i^5, since the length of simple closed geodesic a_{i-1}^2 is known.

Furthermore, we may prove that the flat polygon F can be triangulated by diagonals of F and it is easy to check that each such diagonal is a geodesic of S. This proves our theorem for $g \geq 3$. □

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