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Spacelike hypersurfaces with constant rth mean curvature in steady state type spacetimes

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Abstract. We deal with spacelike hypersurfaces immersed with some constant rth mean curvature in a *steady state type spacetime*, that is, a generalized Robertson–Walker spacetime of the type $-\mathbb{R} \times_{e^t} M^n$. In this setting, supposing that the fiber M^n of the ambient space has nonnegative constant sectional curvature, we establish characterization results concerning domains of the spacelike slices $\{t\} \times M^n$. Afterwards, we apply such characterization results to study the uniqueness of complete spacelike hypersurfaces with one end in such a ambient space.

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1. Introduction

Let M^n be a connected, n-dimensional, oriented Riemannian manifold, $I \subset \mathbb{R}$ an open interval and $f : I \to \mathbb{R}$ a positive smooth function. Also, in the product manifold $\overline{M}^{n+1} = I \times M^n$ let π_I and π_M denote the projections onto the factors I and M^n, respectively.

The class of Lorentzian manifolds which will be of our concern here is the one obtained by furnishing $\overline{M}^{n+1}$ with the Lorentzian metric

$$\langle v, w\rangle_p = -\langle (\pi_I)_* v, (\pi_I)_* w\rangle_{\pi_I(p)} + (f \circ \pi_I)(p)^2 \langle (\pi_M)_* v, (\pi_M)_* w\rangle_{\pi_M(p)},$$

for all $p \in \overline{M}^{n+1}$ and $v, w \in T_p\overline{M}$. In such a case, we write

$$\overline{M}^{n+1} = -I \times_f M^n, \tag{1.1}$$

and say that $\overline{M}^{n+1}$ is a Lorentzian warped product space with warping function f.

According to the terminology due to Alías et al. [4], a warped product space (1.1) is called a *generalized Robertson–Walker (GRW) spacetime*. Note that, in this definition the fiber is not assumed to be of constant sectional curvature, in general. When this assumption holds and the dimension of the spacetime is 3, the GRW spacetime is a classical Robertson–Walker spacetime. Thus, GRW spacetimes widely extend Robertson–Walker spacetimes, and they include, for instance, the Einstein–de Sitter spacetime, Friedmann cosmological models, the static Einstein spacetime and the de Sitter spacetime. As it was already pointed out by Latorre and Romero [19], conformal changes of the metric of a GRW spacetime, with a conformal factor which only depends on universal time, produce new GRW spacetimes. Moreover, small deformations of the metric on the fiber of Robertson–Walker spacetimes also fit into the class of GRW spacetimes (for a thorough discussion about Robertson–Walker spacetimes, see for example Chapter 5 of [7]).

In this paper, we are interested in the study of spacelike hypersurfaces immersed with some constant rth mean curvature in a *steady state type spacetime*, that is, a GRW spacetime of the type $-\mathbb{R} \times_{e^t} M^n$. Such nomenclature, which was established by Albujer and Alías [1], is justified by the fact that, when its fiber M^n is the Euclidean space $\mathbb{R}^n$, such GRW spacetime is isometric to the half $\mathcal{H}^{n+1}$ of the de Sitter space $\mathbb{S}_1^{n+1}$, which models the so-called steady state space (for more details, see Sect. 3).

The importance of considering $\mathcal{H}^{n+1}$ comes from the fact that, in Cosmology, $\mathcal{H}^4$ is the steady state model of the universe proposed by Bondi and Gold [8], and Hoyle [18], when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous), but also at all times (cf. [25], Section 14.8, and [17], Section 5.2).

Before give details on our results, we present a brief outline of recent works which are directly related to our one.

In [1], Albujer and Alías used the well known generalized maximum principle of Omori–Yau [23, 26] to prove that, given a complete spacelike hypersurface Σ^n with constant mean curvature H in a steady state type spacetime $-\mathbb{R} \times_{e^t} M^n$, whose Riemannian fiber M^n is supposed to have nonnegative sectional curvature, if Σ^n lies between two slices of $-\mathbb{R} \times_{e^t} M^n$, then $H = 1$. Moreover, when $n = 2$, they concluded that the spacelike surface Σ^2 is necessarily a slice $\{t\} \times M^2$.

Later on, by using an extension of the classical Hopf's theorem due to Yau [27] and imposing suitable conditions on both the rth mean curvatures and on the norm of the gradient of the height function, Camargo et al. [9] obtained another uniqueness results concerning complete spacelike hypersurfaces immersed in a steady state type spacetime.

Next, Aquino and de Lima [6] applied a generalized maximum principle developed in [11] in order to prove uniqueness theorems for the spacelike slices of a steady state type spacetime, under suitable conditions on both the rth mean curvatures and the angle between the Gauss map of the spacelike hypersurface and the unitary vector field ∂_t tangent to the $\mathbb{R}$-direction of the ambient space.

Meanwhile, as another suitable application of the generalized maximum principle of Omori–Yau, Colares and de Lima [13] obtained rigidity results concerning complete spacelike hypersurfaces into steady state type spacetimes. Moreover, they also study the uniqueness of entire vertical graphs in such ambient spacetimes.

Here, motivated by the works above described, first we extend a technique due to Colares and de Lima [12] in order to establish characterization results concerning domains entirely contained in a spacelike slice of a steady state type spacetime (cf. Theorem 1 and Corollary 1). Afterwards, we apply such characterization results in order to study the uniqueness of complete spacelike hypersurfaces with *one end* (that is, complete spacelike hypersurfaces which can be regarded as the union of a compact hypersurface whose boundary is contained into a slice of the ambient space, with a complete hypersurface diffeomorphic to a circular cylinder) immersed into a steady state type spacetime (cf. Theorem 2 and Corollary 2).

2. Spacelike hypersurfaces in GRW spacetimes

Proceeding with the context of the previous section, we recall that an n-dimensional connected manifold Σ^n immersed into a Lorentzian space is said to be a *spacelike hypersurface* if the metric on Σ^n induced from that of the ambient space is positive definite. It follows from the connectedness of Σ^n that one can uniquely choose a globally defined timelike unit vector field $N \in \mathfrak{X}(\Sigma)^{\perp}$, having the same time-orientation of ∂_t, that is, such that $\langle N, \partial_t \rangle < 0$. One then says that N is the *future-pointing Gauss map* of Σ^n. Here, ∂_t denotes the coordinate vector field induced by the universal time on the ambient GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$.

Associated to the shape operator $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ of Σ^n (with respect to N or $-N$) one has n algebraic invariants, namely, the elementary symmetric functions S_r of the principal curvatures $\kappa_1, \ldots, \kappa_n$ of A, given by

$$S_r = \sigma_r\left(\kappa_1, \ldots, \kappa_n\right) = \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \ldots \kappa_{i_r},$$

where, for $1 \leq r \leq n$, $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the rth elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$.

The *rth mean curvature* H_r of Σ^n is then defined by

$$\binom{n}{r} H_r = (-1)^r S_r.$$

In particular,

$$H_1 = -\frac{1}{n}\sum_{i=1}^{n}\kappa_i = -\frac{1}{n}\,\mathrm{tr}(A)$$

is the mean curvature H of Σ^n, which is its main extrinsic curvature.

It is also worth noting that H_2 defines a geometric quantity which is closely related to the (intrinsic) scalar curvature R of Σ^n. For instance, when $\overline{M}^{n+1}$ has constant sectional curvature $\overline{c}$, it easily follows from the Gauss' equation that

$$R = n(n-1)(\overline{c} - H_2)$$

Using the definiton of H, we can rewrite this expression as

$$R = n(n-1)\overline{c} + |A|^2 - n^2H^2,$$

where $|A|$ is the Hilbert–Schmidt norm of A (that is, $|A|^2 = \mathrm{tr}(A^*A)$, where A^* stands for the adjoint of A).

For what follows, we say that $p_0 \in \Sigma^n$ is an *elliptic point* of Σ^n if all principal curvatures $\kappa_i(p_0)$ are negative with respect to an appropriate choice of the Gauss map of Σ^n at p_0.

From the ideas of Montiel and Ros concerning Lemma 1 in [22] and their use of Garding's inequalities (cf. [15]), and taking into account our sign convention in the definition of the rth mean curvature, one easily derives the following result (see also Proposition 2.3 of [10]).

Lemma 1. *Suppose that Σ^n has an elliptic point. If H_r is positive on Σ^n, then the same holds for H_k, $k = 1, \ldots, r-1$. Moreover,*

$$H_{k-1} \geq H_k^{(k-1)/k} \quad \text{and} \quad H \geq H_k^{1/k}$$

for $k = 1, \ldots, r$. Also, if $k \geq 2$, then equality happens in one of the above inequalities only at umbilical points.

Now, for $0 \leq r \leq n$, let $P_r : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ be the $r-$th *Newton transformation* of Σ^n, defined inductively by putting $P_0 = I$ (the identity of $\mathfrak{X}(\Sigma)$) and, for $1 \leq r \leq n$,

$$P_r = \binom{n}{r} H_r I + AP_{r-1}.$$

A standard fact concerning the Newton transformations is that

$$\mathrm{tr}\,(P_r) = (r+1)\binom{n}{r+1} H_r, \tag{2.1}$$

for $1 \leq r \leq n$, where $c_r = (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}$ (see, for instance, [2]).

If $\mathcal{D}(\Sigma)$ denotes the ring of the smooth real functions on Σ^n, then, associated to P_r, one has the second order linear differential operator $L_r : \mathcal{D}(\Sigma) \to \mathcal{D}(\Sigma)$, given by

$$L_r(\xi) = \mathrm{tr}(P_r\,\mathrm{Hess}\,\xi). \tag{2.2}$$

For a smooth function $g:\mathbb{R}\to\mathbb{R}$ and $\xi\in\mathcal{D}(\Sigma)$, it follows from the properties of the Hessian operator that

$$L_r(g\circ\xi)=g'(\xi)L_r(\xi)+g''(\xi)\langle P_r\nabla\xi,\nabla\xi\rangle. \tag{2.3}$$

Given a local coordinate frame $\left\{\frac{\partial}{\partial x_i}\right\}$ of Σ^n at a point p, by a direct computation, from (2.2) it is not difficult to obtain the following local expression of the linear operator L_r:

$$L_r(\xi)(p)=\sum_{i,j,k,l} g^{ik}t_{kl}g^{lj}\frac{\partial^2\xi}{\partial x_i\partial x_j}-\sum_{i,j,k,l,s} g^{ik}t_{kl}g^{lj}\Gamma^s_{ij}\frac{\partial\xi}{\partial x_s},$$

where

$$g_{ij}=\left\langle\frac{\partial\xi}{\partial x_i},\frac{\partial\xi}{\partial x_j}\right\rangle,\ G=(g_{ij}),\ G^{-1}=\left(g^{ij}\right),\ t_{ij}=P_r\left(\frac{\partial\xi}{\partial x_i},\frac{\partial\xi}{\partial x_j}\right),$$

and Γ^s_{ij} are the connection coefficients of ∇.

From the above local expression, we know that the linear operator L_r is elliptic if, and only if, P_r is positive definite. Clearly, $L_0=\Delta$, where Δ stands for the Laplacian–Beltrami operator on Σ^n. Thus, L_0 is always an elliptic operator. In the following, we quote two results giving sufficient conditions for this to happen for the operator L_r in general (see [3], Lemmas 3.2 and 3.3).

Lemma 2. *If $H_2>0$ on Σ^n, then P_1 is positive definite for a appropriate choice of the Gauss map N.*

Lemma 3. *Let Σ^n having an elliptic point with respect to an appropriate choice of Gauss map. If $H_{r+1}>0$ on Σ^n for some $2\le r\le n-1$, then P_k is positive definite for all $1\le k\le r$.*

In what follows, we consider two particular functions naturally attached to a spacelike hypersurface Σ^n immersed into a GRW spacetime $\overline{M}^{n+1}=-I\times_f M^n$, namely, the (vertical) *height function* $h=(\pi_I)|_\Sigma$ and the *support function* $\langle N,\partial_t\rangle$, where we recall that N denotes the future-pointing Gauss map of Σ^n.

For a smooth function ϕ on $\overline{M}^{n+1}$, let $\overline{\nabla}\phi$ and $\nabla\phi$ respectively denote the gradient of ϕ on $\overline{M}^{n+1}$ and that of its restriction to Σ^n. A simple computation shows that

$$\overline{\nabla}\pi_I=-\langle\overline{\nabla}\pi_I,\partial_t\rangle\partial_t=-\partial_t,$$

so that

$$\nabla h=(\overline{\nabla}\pi_I)^\top=-\partial_t^\top=-\partial_t-\langle N,\partial_t\rangle N. \tag{2.4}$$

Therefore,

$$|\nabla h|^2=\langle N,\partial_t\rangle^2-1, \tag{2.5}$$

where $|\cdot|$ stands for the norm of a vector field on Σ^n.

From Lemma 4.1 of [3], we have that

$$L_rh=-(\log f)'(h)\left(c_rH_r+\langle P_r\nabla h,\nabla h\rangle\right)-c_rH_{r+1}\langle N,\partial_t\rangle. \tag{2.6}$$

On the other hand, from Corollary 8.2 of [3] we have that

$$\begin{aligned}\Delta(f(h)\langle N,\partial_t\rangle) = {} & nf(h)\langle \nabla H,\partial_t\rangle + nHf'(h) + f(h)\langle N,\partial_t\rangle |A|^2\\ & -(n-1)f(h)\langle N,\partial_t\rangle(\log f)''(h)|\nabla h|^2\\ & + f(h)\langle N,\partial_t\rangle \mathrm{Ric}_M(N^*,N^*),\end{aligned} \tag{2.7}$$

where Ric_M stands for the Ricci tensor of the Riemannian fiber M^n, and $N^* = (\pi_M)_*(N)$. Moreover, if $-I\times_f M^n$ is a RW spacetime, from Corollary 8.4 of [3], we also have that

$$\begin{aligned}L_r(f(h)\langle N,\partial_t\rangle) = {} & \binom{n}{r+1} f(h)\langle \nabla H_{r+1},\partial_t\rangle + c_r H_{r+1} f'(h)\\ & + \binom{n}{r+1} f(h)\langle N,\partial_t\rangle \left(nHH_{r+1} - (n-r-1)H_{r+2}\right)\\ & + f(h)\langle N,\partial_t\rangle \left(\frac{\kappa_M}{f^2(h)} - (\ln f)''(h)\right)\\ & \times \left(c_r H_r |\nabla h|^2 - \langle P_r \nabla h, \nabla h\rangle\right),\end{aligned} \tag{2.8}$$

where κ_M stands for the sectional curvature of the Riemannian fiber M^n.

3. Uniqueness results in steady state type spacetimes

Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz–Minkowski space ($n\geq 2$), that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric defined by

$$\langle v,w\rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2}w_{n+2},$$

for all $v,w\in\mathbb{R}^{n+2}$. We define the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$

$$\mathbb{S}_1^{n+1} = \left\{p\in L^{n+2}; \langle p,p\rangle = 1\right\}.$$

The induced metric from $\langle\,,\rangle$ makes $\mathbb{S}_1^{n+1}$ into a Lorentz manifold with constant sectional curvature one. Let $a\in\mathbb{L}^{n+2}$ be a past-pointing null vector, that is, $\langle a,a\rangle = 0$ and $\langle a, e_{n+2}\rangle > 0$, where $e_{n+2} = (0,\ldots,0,1)$. Then the open region of the de Sitter space $\mathbb{S}_1^{n+1}$, given by

$$\mathcal{H}^{n+1} = \left\{x\in\mathbb{S}_1^{n+1}; \langle x,a\rangle > 0\right\}$$

is the so-called *steady state space* (cf. [20], Example 4.2). Observe that $\mathcal{H}^{n+1}$ is a noncomplete manifold, being only half of the de Sitter space. Its boundary, as a subset of $\mathbb{S}_1^{n+1}$, is the null hypersurface

$$\mathcal{L}_0 = \left\{x\in\mathbb{S}_1^{n+1}; \langle x,a\rangle = 0\right\},$$

whose topology is that of $\mathbb{R}\times\mathbb{S}^{n-1}$ (see [21], Section 2).

We observe that the steady state space $\mathcal{H}^{n+1}$ can also be expressed in an isometrically equivalent way as the RW spacetime

$$-\mathbb{R}\times_{e^t}\mathbb{R}^n.$$

To see it, take $b\in\mathbb{L}^{n+2}$ another null vector such that $\langle a,b\rangle=1$ and consider the map $\Phi:\mathcal{H}^{n+1}\to-\mathbb{R}\times_{e^t}\mathbb{R}^n$ defined by

$$\Phi(x)=\left(\ln(\langle x,a\rangle),\frac{x-\langle x,a\rangle b-\langle x,b\rangle a}{\langle x,a\rangle}\right).$$

Then it can easily be checked that Φ is an isometry between both spaces which conserves time orientation (see [1], Section 4).

Following the ideas of Albujer and Alías [1], we now consider a natural extension of the steady state space $\mathcal{H}^{n+1}=-\mathbb{R}\times_{e^t}\mathbb{R}^n$. Let M^n be a connected n-dimensional Riemannian manifold and consider the GRW spacetime

$$-\mathbb{R}\times_{e^t}M^n.$$

We will refer to such wider family of GRW spacetimes as *steady state type spacetimes*. For instance, when M^n is the flat n-torus we get the de Sitter cusp as defined in [14]. In this setting, we will state and prove our first result.

Theorem 1. *Let Σ^n be a compact immersed spacelike hypersurface of a steady state type spacetime $-\mathbb{R}\times_{e^t}M^n$, whose Riemannian fiber M^n has nonnegative constant sectional curvature κ_M. Suppose that Σ^n lies over a slice $M_t=\{t\}\times M^n$, for some $t\in\mathbb{R}$, with its boundary $\partial\Sigma$ contained in M_t. If one of the following conditions is satisfied:*

- *(i) H_2 is a constant with $1\le H\le H_2$, or*
- *(ii) H_{r+1} is a constant (with $r\ge 2$) with $1\le H_r\le H_{r+1}$ and there exists an elliptic point in Σ^n,*

then Σ^n is a domain of M_t.

Proof. We can assume, without lost of generality, that $t=0$. Consequently, since we are assuming that Σ^n is over M_0, we have that h is nonnegative. In this setting, we define on Σ^n the function ξ given by

$$\xi=c\,e^h-\langle N,K\rangle, \tag{3.1}$$

where h is the vertical height function of Σ^n, $K=e^h\partial_t$, N is the future-pointing Gauss map of Σ^n and c is an arbitrary positive constant. From equations (2.3), (2.6) and (2.8) we have

$$\begin{aligned}L_r(\xi)&=-c\,c_r\,e^h(H_r+\langle N,\partial_t\rangle H_{r+1})-e^hc_r\,H_{r+1}\\&\quad-\binom{n}{r+1}e^h\langle N,\partial_t\rangle(nHH_{r+1}-(n-r-1)H_{r+2})\\&\quad-e^{-h}\langle N,\partial_t\rangle\kappa_M\left(c_rH_r|\nabla h|^2-\langle P_r\nabla h,\nabla h\rangle\right),\end{aligned} \tag{3.2}$$

where $c_r=(r+1)\binom{n}{r+1}$.

We also note that we are in position to apply either Lemma 2 or Lemma 3 in order to conclude that L_k is elliptic, for any $k\in\{0,\ldots,r\}$. Then, since the fact that L_r is elliptic is equivalent to P_r be positive definite, from (2.1) we get

$$\langle P_r\nabla h, \nabla h\rangle \le \operatorname{tr}(P_r)|\nabla h|^2 = c_r H_r|\nabla h|^2. \tag{3.3}$$

Thus, taking into account that $\langle N, \partial_t\rangle \le -1$ and $\kappa_M \ge 0$, from (3.2) and (3.3) we obtain

$$\begin{aligned} L_r(\xi) \ge & -c\, c_r\, e^h(H_r + \langle N, \partial_t\rangle H_{r+1}) - e^h c_r\, H_{r+1} \\ & -\binom{n}{r+1} e^h \langle N, \partial_t\rangle (nHH_{r+1} - (n-r-1)H_{r+2}). \end{aligned} \tag{3.4}$$

Now, we claim that

$$nHH_{r+1} - (n-r-1)H_{r+2} \;\ge\; (r+1)H_{r+1}^{(r+2)/(r+1)}. \tag{3.5}$$

In fact, taking into account our restrictions on H_r and H_{r+1}, from Lemma 1 we have that

$$H_r \;\ge\; H_{r+1}^{r/(r+1)} \;>\; 0 \quad \text{and} \quad H \;\ge\; H_r^{1/r}.$$

Moreover, from Newton inequalities (cf. [16], Theorem 144; see also [10], Proposition 2.3), we have that

$$H_{r+2} \le \frac{H_{r+1}^2}{H_r}.$$

Next, from these above inequalities, we obtain that

$$\begin{aligned} HH_{r+1} - H_{r+2} &\ge \frac{H_{r+1}}{H_r}\,(HH_r - H_{r+1}) \;\ge\; \frac{H_{r+1}}{H_r}\left(HH_r - H_r^{(r+1)/r}\right) \\ &= H_{r+1}\left(H - H_r^{1/r}\right) \;\ge\; 0. \end{aligned} \tag{3.6}$$

Thus, from (3.6) we have that

$$\begin{aligned} nHH_{r+1} - (n-r-1)H_{r+2} = {} & nHH_{r+1} - nH_{r+2} + (r+1)H_{r+2} \\ & + (r+1)HH_{r+1} - (r+1)HH_{r+1}. \end{aligned}$$

After a simple algebraic computation and using the inequality (3.6), we conclude from above expression that

$$\begin{aligned} nHH_{r+1} - (n-r-1)H_{r+2} &= (n-r-1)(HH_{r+1} - H_{r+2}) + (r+1)HH_{r+1} \\ &\ge (r+1)H_{r+1}^{(r+2)/(r+1)}, \end{aligned}$$

and our affirmation is shown.

Hence, taking into account once more our hypothesis on H_r and H_{r+1}, from (3.4) and (3.5) we obtain

$$\begin{aligned} \frac{1}{c_r}\, L_r(\xi) &\ge e^h\left\{-c\, H_r - c\,\langle N, \partial_t\rangle H_{r+1} - H_{r+1} - \langle N, \partial_t\rangle H_{r+1}^{(r+2)/(r+1)}\right\} \\ &\ge e^h\left\{-c\, H_r + c\, H_{r+1} - H_{r+1} + H_{r+1}^{(r+2)/(r+1)}\right\} \\ &= e^h\left\{c\,(H_{r+1} - H_r) + H_{r+1}(H_{r+1}^{1/(r+1)} - 1)\right\}. \end{aligned}$$

Thus, from above expression we conclude that $L_r(\xi) \geq 0$ on Σ^n. Now, the maximum principle ensures that

$$\xi \leq \xi\big|_{\partial\Sigma}. \tag{3.7}$$

Consequently, from (3.1) and (3.7), it follows that

$$c\,e^h \;\leq\; c\,e^h - \langle N, e^h\partial_t\rangle \;=\; \xi \;\leq\; \xi\big|_{\partial\Sigma} \;\leq\; c+\alpha,$$

where $\alpha = \max_{\partial\Sigma}|\langle N, \partial_t\rangle|$. Thus, we have that

$$e^h \leq 1 + \frac{\alpha}{c}. \tag{3.8}$$

Therefore, since the positive constant c is arbitrary and $h \geq 0$, from inequality (3.8) we get that the height function h vanishes identically on Σ^n and, hence, we conclude that $\psi(\Sigma^n) \subset M_0$ and this finishes the proof. □

From equation (2.7), we can reason as in the proof of Theorem 1 in order to obtain the following:

Corollary 1. *Let Σ^n be a compact immersed spacelike hypersurface of a steady state type spacetime $-\mathbb{R}\times_{e^t} M^n$, whose Riemannian fiber M^n has nonnegative Ricci curvature. Suppose that Σ^n lies over a slice $M_t = \{t\}\times M^n$, for some $t \in \mathbb{R}$, with its boundary $\partial\Sigma$ contained in M_t. If Σ^n has constant mean curvature $H \geq 1$, then Σ^n is a domain of M_t.*

Remark 1. We observe that the restriction on the sectional curvature of the fiber of the ambient space in Theorem 1 corresponds, in the context of steady state type spacetimes, to the strong null convergence condition, which was established by Alías and Colares [3]. Moreover, the restriction on the Ricci curvature of the fiber in Corollary 1 is exactly the so-called null convergence condition (see, for instance, [5] and [20]).

According to Section 5 of [12], we say that a complete spacelike hypersurface Σ^n immersed in a steady state type spacetime $-\mathbb{R}\times_{e^t} M^n$ has *one end* $\mathcal{C}^n$ if, for each $t \in \mathbb{R}$ such that $M_t \cap \Sigma^n \neq \emptyset$, Σ^n can be regarded as

$$\Sigma^n = \Sigma_t \cup \mathcal{C}^n,$$

where Σ_t is a connected compact hypersurface whose boundary is contained into the slice $M_t = \{t\}\times M^n$ and $\mathcal{C}^n$ is a manifold diffeomorphic to the circular cylinder $[t, +\infty)\times\mathbb{S}^{n-1}$ which lies in a region of $-\mathbb{R}\times_{e^t} M^n$ of the form $[t, +\infty)\times M^n$ or $(-\infty, t]\times M^n$.

Now, let Σ^n be a complete hypersurface with one end of a steady state type spacetime $-\mathbb{R}\times_{e^t} M^n$. We say that Σ^n is *tangent from above at the infinity* to a slice M_t, if either Σ^n is a slice $M_{\widetilde{t}}$, for some $\widetilde{t} \geq t$, or, for all $\widetilde{t} \geq t$, one of the following conditions is satisfied

- (*i*) $M_{\widetilde{t}} \cap \Sigma^n = \emptyset$;
- (*ii*) $M_{\widetilde{t}} \cap \Sigma^n \neq \emptyset$ and the compact part $\Sigma_{\widetilde{t}}$ of Σ^n lies over $M_{\widetilde{t}}$.

From Theorem 1, we obtain the following uniqueness result:

Theorem 2. *Let Σ^n be a complete immersed spacelike hypersurface with one end of a steady state type spacetime $-\mathbb{R}\times_{e^t} M^n$, whose Riemannian fiber M^n is complete with nonnegative constant sectional curvature. Suppose that Σ^n is tangent from above at the infinity to a slice $M_t = \{t\}\times M^n$, for some $t\in\mathbb{R}$. If one of the following conditions is satisfied:*

(i) H_2 is a constant with $1\leq H\leq H_2$, or
(ii) H_{r+1} is a constant (with $r\geq 2$) with $1\leq H_r\leq H_{r+1}$ and there exists an elliptic point in Σ^n,

then Σ^n is a slice $M_{\widetilde{t}}$, for some $\widetilde{t}\geq t$.

Proof. Suppose by contradiction that Σ^n is not a slice of $-\mathbb{R}\times_{e^t} M^n$. Then, there are constants $t_2 > t_1$ such that the $M_{t_1}\cap\Sigma^n\neq\emptyset$ and $M_{t_2}\cap\Sigma^n\neq\emptyset$. Consequently, from Theorem 1, we get that $\Sigma^n_{t_1}\subset M_{t_1}$ and $\Sigma^n_{t_2}\subset M_{t_2}$. Hence, since $\Sigma^n_{t_2}\subset\Sigma^n_{t_1}$, we arrive at a contradiction. $\square$

From Corollary 1, we can reason as in the proof of Theorem 2 to get the following:

Corollary 2. *Let Σ^n be a complete immersed spacelike hypersurface with one end of a steady state type spacetime $-\mathbb{R}\times_{e^t} M^n$, whose Riemannian fiber M^n is complete with nonnegative Ricci curvature. Suppose that Σ^n is tangent from above at the infinity to a slice $M_t = \{t\}\times M^n$, for some $t\in\mathbb{R}$. If Σ^n has constant mean curvature $H\geq 1$, then Σ^n is a slice M_t, for some $t\in\mathbb{R}$.*

Remark 2. Colares jointly with the second author obtained uniqueness and nonexistence results concerning complete constant mean curvature spacelike hypersurfaces with one end and over a spacelike hyperplane of the steady state space $\mathcal{H}^{n+1}$ (cf. [12], Theorem 5.3 and Corollary 5.4).

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