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Isoptic characterization of spheres

Árpád Kurusa and Tibor Ódor

Abstract. If a convex body in $\mathcal{K} \in \mathbb{R}^n$ subtends constant visual angles over two concentric spheres exterior to $\mathcal{K}$, then it is a ball concentric to those spheres.

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1. Introduction

The *masking number*[1] $M_{\mathcal{K}}(P)$ of the convex body $\mathcal{K}$ at $P \notin \mathcal{K}$ as defined in [9, (7.1)] is the integral

$$M_{\mathcal{K}}(P) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \#(\partial\mathcal{K} \cap \ell(P, \mathbf{u}_\xi)) d\xi, \tag{1.1}$$

where # is the counting measure, $\partial\mathcal{K}$ denotes the boundary of $\mathcal{K}$, ξ is the spherical coordinate of the unit vector $\mathbf{u}_\xi \in \mathbb{S}^{n-1}$, and $\ell(P, \mathbf{u}_\xi)$ is the straight line through P having direction $\mathbf{u}_\xi$ (Fig. 1).

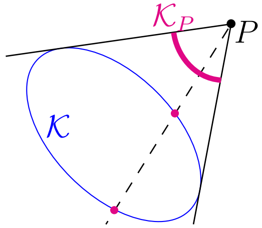

FIGURE 1 The masking number $M_{\mathcal{K}}(P)$ is twice the measure of the visual angle $\mathcal{K}_P$ of $\mathcal{K}$ at a point $P \notin \mathcal{K}$

[1] This is called the point projection in [1] or shadow picture in [3].

The set of points $P \in \mathbb{R}^n$, where a convex body $\mathcal{K} \subset \mathbb{R}^n$ has constant $\alpha \in (0, |\mathbb{S}^{n-1}|)$ masking number $M_{\mathcal{K}}(P)$ is called the *α-isomasker*[2] of the convex body $\mathcal{K}$. The α-isomasker of the convex body $\mathcal{K}$ in the plane is the set of the points where $\mathcal{K}$ subtends angles of constant $\frac{\alpha}{2} \in (0, \pi)$ measure, and it is called the *α-isoptic* of $\mathcal{K}$.

Following the conjecture of Klamkin [4] Nitsche proved in [13] that if two isoptics of $\mathcal{K}$ are concentric circles, then $\mathcal{K}$ is a disc. Nitsche also asked to consider the problem in higher dimensions.

We generalize Nitsche's result to higher dimensions in Theorem 5.1 as follows: if two isomaskers of a convex body are also isomaskers of a ball with the same masking numbers, then the body is that ball. We use an integral geometric method.

2. Preliminaries

We work in the Euclidean n-space $\mathbb{R}^n$ $(n \in \mathbb{N})$. Its unit ball is $\mathcal{B} = \mathcal{B}^n$ (in the plane the unit disc is $\mathcal{D}$), its unit sphere is $\mathbb{S}^{n-1}$ and the set of its hyperplanes is $\mathbb{H}$. The ball (resp. disc) of radius $\varrho > 0$ centered at the origin $\mathbf{0}$ is denoted by $\varrho\mathcal{B} = \varrho\mathcal{B}^n$ (resp. $\varrho\mathcal{D}$). The unit sphere centered at a point P is $\mathbb{S}^{n-1}_P$.

Using spherical coordinates $\xi = (\xi_1, \ldots, \xi_{n-1})$ every unit vector can be written in the form $\mathbf{u}_\xi = (\cos\xi_1, \sin\xi_1\cos\xi_2, \sin\xi_1\sin\xi_2\cos\xi_3, \ldots)$, the i-th coordinate of which is $u^i_\xi = (\prod_{j=1}^{i-1}\sin\xi_j)\cos\xi_i$ $(\xi_n := 0)$. In the plane we use $\mathbf{u}_\xi = (\cos\xi, \sin\xi)$ and $\mathbf{u}^\perp_\xi = \mathbf{u}_{\xi+\pi/2} = (-\sin\xi, \cos\xi)$. In analogy to this latter one, we introduce $\xi^\perp = (\xi_1, \ldots, \xi_{n-2}, \xi_{n-1} + \pi/2)$ for higher dimensions.

We introduce the notation $|\mathbb{S}^k| := 2\pi^{k/2}/\Gamma(k/2)$ for the standard surface measure of the k-dimensional sphere, where Γ is Euler's Gamma function.

The hyperplanes $\hbar \in \mathbb{H}$ are parametrized so that $\hbar(\mathbf{u}_\xi, r)$ is orthogonal to the unit vector $\mathbf{u}_\xi \in \mathbb{S}^{n-1}$ and contains the point $r\mathbf{u}_\xi$,[3] where $r \in \mathbb{R}$. For convenience we also use $\hbar(P, \mathbf{u}_\xi)$ to denote the hyperplane through the point $P \in \mathbb{R}^n$ with normal vector $\mathbf{u}_\xi \in \mathbb{S}^{n-1}$. For instance, $\hbar(P, \mathbf{u}_\xi) = \hbar(\mathbf{u}_\xi, \langle\overrightarrow{OP}, \mathbf{u}_\xi\rangle)$, where $O = \mathbf{0}$ is the origin and $\langle ., . \rangle$ is the usual inner product.

On $\mathbb{H}$ we use the *kinematic density $d\hbar = drd\xi$* that is (up to a constant multiple) the only measure on $\mathbb{H}$ invariant with respect to the Euclidean motions [16].

By a convex body we mean a convex compact set $\mathcal{K} \subseteq \mathbb{R}^n$ with non-empty interior $\mathcal{K}^\circ$ and with piecewise C^1 boundary $\partial\mathcal{K}$. For a convex body $\mathcal{K}$ we let $p_{\mathcal{K}} \colon \mathbb{S}^{n-1} \to \mathbb{R}$ denote the support function of $\mathcal{K}$ defined by $p_{\mathcal{K}}(\mathbf{u}_\xi) = \sup_{\mathbf{x}\in\mathcal{K}}\langle\mathbf{u}_\xi, \mathbf{x}\rangle$. We also use notation $\hbar_{\mathcal{K}}(\mathbf{u}) = \hbar(\mathbf{u}, p_{\mathcal{K}}(\mathbf{u}))$.

[2] We reserve the word isoptic for the set of points where not only the measure, but also the shape of $\mathcal{K}_P$ is constant. A result toward this direction can be found in [12].

[3] Although $\hbar(\mathbf{u}_\xi, r) = \hbar(-\mathbf{u}_\xi, -r)$ this parametrization is locally bijective.

If the origin is in $\mathcal{K}^\circ$, then the *support function* of $\mathcal{K}$ is positive, otherwise the zero or even negative values appear in its image according to whether the origin is in $\partial\mathcal{K}$ or outside $\mathcal{K}$. If the origin is in $\mathcal{K}^\circ$, another useful function of a convex body $\mathcal{K}$ is its *radial function* $\varrho_{\mathcal{K}}\colon \mathbb{S}^{n-1} \to \mathbb{R}_+$ defined by $\varrho_{\mathcal{K}}(\mathbf{u}) = |\{r\mathbf{u} : r > 0\} \cap \partial K|$.

Assume that the origin $\mathbf{0}$ is an interior point of a convex body $\mathcal{K}$. Define $\mathbb{H}_0 := \{\hbar \in \mathbb{H} : \mathbf{0} \notin \hbar\}$, and let $\hat{\delta}\colon \mathbb{H}_0 \to \mathbb{R}^n$ and $\check{\delta}\colon \mathbb{R}^n \to \mathbb{H}_0$, the *dualizing maps*, be defined by

$$\hat{\delta}(\hbar(\mathbf{u}, r)) := -\frac{1}{r}\mathbf{u} \quad \text{and} \quad \check{\delta}(r\mathbf{u}) := \hbar\Big(-\mathbf{u}, \frac{1}{r}\Big), \tag{2.1}$$

respectively, where $\mathbf{u} \in \mathbb{S}^{n-1}$ is unit vector and $r > 0$. These functions are obviously inverses of each other, and it is an easy and well-known fact[4] that

$$\hat{\delta}(\{\hbar \in \mathbb{H} : \mathbf{v} \in \hbar\}) = \hbar\left(\frac{-\mathbf{v}}{|\mathbf{v}|}, \frac{1}{|\mathbf{v}|}\right) \quad \text{and} \quad \check{\delta}(\hbar(\mathbf{u}, r)) = \Big\{\hbar \in \mathbb{H} : \frac{-1}{r}\mathbf{u} \in \hbar\Big\}.$$

The dual body $\mathcal{K}^\star$ of $\mathcal{K}$ is bounded by $\partial\hat{\mathcal{K}} := \{\hat{\delta}(\hbar(\mathbf{u}, p_{\mathcal{K}}(\mathbf{u}))) : \mathbf{u} \in \mathbb{S}^{n-1}\}$. The dual body $\mathcal{K}^\star$, which is in fact the point reflection—to the origin $\mathbf{0}$—of the polar body $\mathcal{K}^*$ [17, Section 1.6], is convex, and its radial function is $\varrho_{\mathcal{K}^\star}(\mathbf{u}) = \frac{1}{p_{\mathcal{K}}(-\mathbf{u})}$ [17, Theorem 1.7.6]. Further, we have $(\mathcal{K}^\star)^\star = \mathcal{K}$ [17, Section 1.6].

A strictly positive integrable function $\omega\colon \mathbb{R}^n \setminus \mathcal{B} \to \mathbb{R}_+$ is called *weight* and the integral

$$V_\omega(f) := \int\limits_{\mathbb{R}^n \setminus \mathcal{B}} f(x)\omega(x)dx$$

of an integrable function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called the *volume of f with respect to the weight ω* or simply the *ω-volume of f*. For the volume of the indicator function $\chi_{\mathcal{S}}$ of a set $\mathcal{S} \subseteq \mathbb{R}^n$ we use the notation $V_\omega(\mathcal{S}) := V_\omega(\chi_{\mathcal{S}})$ as a shorthand. If several weights are indexed by $i \in \mathbb{N}$, then we use the even shorter notation $V_i(\mathcal{S}) := V_{\omega_i}(\mathcal{S}) = V_i(\chi_{\mathcal{S}}) := V_{\omega_i}(\chi_{\mathcal{S}})$.

Finally we introduce a utility function χ that takes relations as argument and gives 1 if its argument is fulfilled. For example $\chi(1 > 0) = 1$, but $\chi(1 \le 0) = 0$ and $\chi(x > y)$ is 1 if $x > y$ and it is zero if $x \le y$. However we still use χ also as the indicator function of the set given in its subscript.

3. Dualizing the masking function

For any point $P \in \mathbb{R}^n$ define the sets $\bar{\mathcal{K}}_P$ and $\mathcal{K}_P$ in the unit sphere $\mathbb{S}^{n-1}_P$ centered at P that contains exactly those points $X \in \mathbb{S}^{n-1}_P$ for which the

[4]Embed the space $\mathbb{R}^n$ of $\mathcal{K}$ into $\mathbb{R}^{n+1}$ in such a way that the $(n+1)$th coordinate of every point is 1 and the $(n+1)$th coordinate axis intersects $\mathcal{K}$ in its inner point $\mathbf{0} \in \mathbb{R}^n$.

hyperplane $\hbar(P, \overrightarrow{PX})$ and the straight line $\ell(P, \overrightarrow{PX})$, respectively, intersects $\mathcal{K}$. Then, by (1.1) and some easy observations we have

$$M_{\mathcal{K}}(P) = \frac{1}{2}\int\limits_{\mathbb{S}^{n-1}} \#(\partial\mathcal{K} \cap \ell(P, \mathbf{u}_\xi))d\xi = \int\limits_{\mathcal{K}_P} 1\, d\xi = \frac{1}{|\mathbb{S}^{n-2}|}\int\limits_{\bar{\mathcal{K}}_P} 1\, d\xi$$
$$= \frac{1}{|\mathbb{S}^{n-2}|}\int\limits_{\mathbb{S}^{n-1}} \chi(\hbar(P, \mathbf{u}_\xi) \cap \mathcal{K} \neq \emptyset)d\xi.$$

From this we obtain

$$\begin{aligned} |\mathbb{S}^{n-2}| M_{\mathcal{K}}(P) &= \int\limits_{\mathbb{S}^{n-1}} \chi(\langle \mathbf{u}_\xi, P\rangle \le p_K(\mathbf{u}_\xi))\, d\xi \\ &= |\mathbb{S}^{n-1}| - \int\limits_{\mathbb{S}^{n-1}} \chi(\langle \mathbf{u}_\xi, P\rangle \ge p_K(\mathbf{u}_\xi))\, d\xi \\ &=: |\mathbb{S}^{n-1}| - M^\star_{\mathcal{K}}(\check{\delta}(P)). \end{aligned} \tag{3.1}$$

Assuming $\mathbf{0} \in \mathcal{K}^\circ$ one can reformulate the last integral to obtain

$$\begin{aligned} M^\star_{\mathcal{K}}(\check{\delta}(P)) &= \int\limits_{\mathbb{S}^{n-1}} \chi\left(\langle -\mathbf{u}_\xi \varrho_{K^\star}(-\mathbf{u}_\xi), -\mathbf{u}\rangle \ge \frac{1}{r}\right) d\xi \\ &= \int\limits_{\mathbb{S}^{n-1}} \chi\left(\varrho_{K^\star}(-\mathbf{u}_\xi) \ge \frac{\frac{1}{r}}{\langle -\mathbf{u}_\xi, -\mathbf{u}\rangle}\right) d\xi \\ &= \int\limits_{\check{\delta}(P)} \chi\left(\mathbf{x} \in \mathcal{K}^\star\right) \left|\frac{d\xi}{d\mathbf{x}}\right| d\mathbf{x}, \end{aligned}$$

where $P = r\mathbf{u}$, $r > 0$, $\mathbf{u} \in \mathbb{S}^{n-1}$, and $|\frac{d\xi}{d\mathbf{x}}|$ is the Jacobian of the map $\mathbf{x} \mapsto \xi$ given by $\mathbf{x} = -|\mathbf{x}|\mathbf{u}_\xi$. Let $\mathbf{x} = \frac{-1}{r}\mathbf{u} + \varrho\mathbf{u}_\psi$, where $\mathbf{u} \perp \mathbf{u}_\psi \in \mathbb{S}^{n-1}$ and ψ is a spherical coordinate on $\mathbb{S}^{n-2}$ such that $\xi = (\xi, \psi)$. Then by rotational invariance we obtain immediately that $\left|\frac{d\xi}{d\mathbf{x}}\right| = |\mathbf{x}|^{2-n}\left|\frac{d\xi}{d\varrho}\right|$, where $\tan\xi = \frac{\varrho}{1/r}$ and so

$$\frac{d\xi}{d\varrho} = \frac{r}{1 + r^2\varrho^2}.$$

Thus, we obtain

$$\begin{aligned} M^\star_{\mathcal{K}}(\check{\delta}(P)) &= \int\limits_{\check{\delta}(P)} \chi\big(\mathbf{x} \in \mathcal{K}^\star\big)|\mathbf{x}|^{2-n}\frac{|P|}{1 + |P|^2(|\mathbf{x}|^2 - |P|^{-2})}\, d\mathbf{x} \\ &= \int\limits_{\check{\delta}(P)} \chi\big(\mathbf{x} \in \mathcal{K}^\star\big)\frac{1/|P|}{|\mathbf{x}|^n}\, d\mathbf{x}, \end{aligned} \tag{3.2}$$

where $d\mathbf{x}$ is the standard surface measure on the hyperplane $\check{\delta}(P)$.

4. Measures of convex bodies

In view of (3.2) it is natural to consider the following transforms.

Let $\mathcal{M}$ and $\mathcal{K}$ be convex bodies such that $\mathbf{0} \in \mathcal{M} \subseteq \mathcal{K}^\circ$. Let $\nu\colon \mathbb{H} \to C^1(\mathbb{R}^n)$ be a *function of weights*, that is, $\nu_\hbar$ is a weight for every $\hbar \in \mathbb{H}$. Then the *weighted section function* of $\mathcal{K}$ with respect to $\mathcal{M}$, the so called *kernel*, is defined by

$$\mathrm{S}^\nu_{\mathcal{M};\mathcal{K}}(\mathbf{u}) = \int\limits_{\langle \mathbf{x},\mathbf{u}\rangle = p_{\mathcal{M}}(\mathbf{u})} \chi(\mathbf{x}\in\mathcal{K})\nu_{\hbar_{\mathcal{M}}(\mathbf{u})}(\mathbf{x})\, d\mathbf{x}_{\hbar_{\mathcal{M}}(\mathbf{u})}, \tag{4.1}$$

where $d\mathbf{x}_{\hbar_{\mathcal{M}}(\mathbf{u})}$ is the usual surface measure on $\hbar_{\mathcal{M}}(\mathbf{u})$ (Fig. 2).

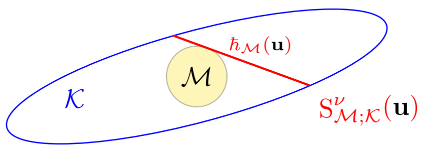

FIGURE 2 Section of $\mathcal{K}$ with respect to the kernel $\mathcal{M}$

The function $\nu\colon \mathbb{H} \to C^1(\mathbb{R}^n)$ of weights is called *rotationally symmetric* if for every $\hbar \in \mathbb{H}$, $\mathbf{x} \in \hbar$ and $D \in SO(n)$ one has $\nu_{D\hbar}(D\mathbf{x}) = \nu_\hbar(\mathbf{x})$, where $D \in SO(n)$ acts naturally on $\mathbb{H}$. Assume that $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{u},\mathbf{v} \in \mathbb{S}^{n-1}$. If $|\mathbf{x}| = |\mathbf{y}|$ and $\langle \mathbf{x},\mathbf{u}\rangle = \langle \mathbf{y},\mathbf{v}\rangle$, then there is a $D \in SO(n)$ such that $D\mathbf{x} = \mathbf{y}$ and $D\mathbf{u} = \mathbf{v}$. Thus we have the following lemma immediately.

Lemma 4.1. *The function ν of weights is rotationally symmetric if and only if there is a function $\bar\nu\colon \mathbb{R}^3 \to \mathbb{R}$ such that $\nu_{\hbar(\mathbf{u},r)}(\mathbf{x}) = \bar\nu(r, \langle \mathbf{x},\mathbf{u}\rangle, |\mathbf{x}|)$.*

If the kernel body is a ball, i.e. $\varrho\mathcal{B}$, we use the notation $\mathrm{S}^\nu_{\varrho;\mathcal{K}} := \mathrm{S}^\nu_{\varrho\mathcal{B};\mathcal{K}}$ as a shorthand.

Lemma 4.2. *Let the convex body $\mathcal{K}$ contain the ball $\varrho\mathcal{B}$. Then for any rotationally symmetric function ν of weights we have*

$$\int\limits_{\mathbb{S}^{n-1}} \mathrm{S}^\nu_{\varrho;\mathcal{K}}(\mathbf{u}_\xi)d\xi = |\mathbb{S}^{n-2}| \int\limits_{\mathcal{K}\setminus\varrho\mathcal{B}} \bar\nu(\varrho,\varrho,|\mathbf{x}|)\frac{(|\mathbf{x}|^2-\varrho^2)^{\frac{n-3}{2}}}{|\mathbf{x}|^{n-2}}\, d\mathbf{x}, \tag{4.2}$$

Proof. Define the function μ^ε of weights by

$$\mu^\varepsilon_{\hbar(\mathbf{u},r)}(\mathbf{x}) := \nu_{\hbar(\mathbf{u},r)}(\mathbf{x} + (r - \langle \mathbf{x},\mathbf{u}\rangle)\mathbf{u})\chi(0 \le \langle \mathbf{x},\mathbf{u}\rangle - r \le \varepsilon),$$

where $\varepsilon > 0$. Now we can write[5]

$$\int\limits_{\mathbb{S}^{n-1}} \mathrm{S}^\nu_{\varrho;\mathcal{K}}(\mathbf{u}_\zeta)d\zeta = \int\limits_{\mathbb{S}^{n-1}} \int\limits_{\langle \mathbf{x},\mathbf{u}_\zeta\rangle = \varrho} \nu_{\hbar(\mathbf{u}_\zeta,\varrho)}(\mathbf{x})\chi(\mathbf{x}\in\mathcal{K})\, d\mathbf{x}_\hbar\, d\zeta$$

[5]Similar calculation is given in [11]. It is given here for the sake of completeness.

$$= \int_{\mathbb{S}^{n-1}} \lim_{\varepsilon\to 0} \left(\frac{1}{\varepsilon} \int_{\langle \mathbf{x},\mathbf{u}_\zeta\rangle \geq \varrho} \mu^{\varepsilon}_{\hbar(\mathbf{u}_\zeta,\varrho)}(\mathbf{x})\chi(\mathbf{x}\in\mathcal{K})\,d\mathbf{x}\right) d\zeta$$

$$= \lim_{\varepsilon\to 0} \left(\frac{1}{\varepsilon} \int_{\mathbb{S}^{n-1}} \int_{\langle \mathbf{x},\mathbf{u}_\zeta\rangle \geq \varrho} \mu^{\varepsilon}_{\hbar(\mathbf{u}_\zeta,\varrho)}(\mathbf{x})\chi(\mathbf{x}\in\mathcal{K})\,d\mathbf{x}\,d\zeta\right)$$

$$= \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \lim_{\varepsilon\to 0} \left(\frac{1}{\varepsilon} \int_{\langle \mathbf{x},\mathbf{u}_\zeta\rangle \geq \varrho} \mu^{\varepsilon}_{\hbar(\mathbf{u}_\zeta,\varrho)}(\mathbf{x})\,d\zeta\right) d\mathbf{x}.$$

As ν is rotationally symmetric, $\nu_{\hbar(\mathbf{u},\langle\mathbf{x},\mathbf{u}\rangle)}(\mathbf{x}) = \bar{\nu}(\langle\mathbf{x},\mathbf{u}\rangle, \langle\mathbf{x},\mathbf{u}\rangle, |\mathbf{x}|)$, and this implies $\mu^{\varepsilon}_{\hbar(\mathbf{u}_\zeta,\varrho)}(\mathbf{x}) = \bar{\nu}(\varrho,\varrho,|\mathbf{x}|)\chi(0 \leq \langle\mathbf{x},\mathbf{u}_\zeta\rangle - \varrho \leq \varepsilon)$. Therefore, letting $|\mathbf{x}|\mathbf{u}_\xi = \mathbf{x}$, where $\mathbf{u}_\xi \in \mathbb{S}^{n-1}$, the calculation above continues as

$$\int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\varrho;\mathcal{K}}(\mathbf{u}_\zeta)d\zeta$$

$$= \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \bar{\nu}(\varrho,\varrho,|\mathbf{x}|) \lim_{\varepsilon\to 0} \left(\frac{1}{\varepsilon} \int_{\langle \mathbf{x},\mathbf{u}_\zeta\rangle \geq \varrho} \chi(0 \leq \langle\mathbf{x},\mathbf{u}_\zeta\rangle - \varrho \leq \varepsilon)\,d\zeta\right) d\mathbf{x}.$$

As

$$\lim_{\varepsilon\to 0} \left(\frac{1}{\varepsilon} \int_{\langle \mathbf{x},\mathbf{u}_\zeta\rangle \geq \varrho} \chi(0 \leq \langle\mathbf{x},\mathbf{u}_\zeta\rangle - \varrho \leq \varepsilon)\,d\zeta\right)$$

$$= \lim_{\varepsilon\to 0} \left(\frac{|\mathbb{S}^{n-2}|/|\mathbf{x}|}{\varepsilon/|\mathbf{x}|} \int_{\frac{(\varrho+\varepsilon)}{|\mathbf{x}|}}^{\frac{\varrho}{|\mathbf{x}|}} \sqrt{1-\lambda^2}^{\,n-3}\,d\lambda \right) = \frac{|\mathbb{S}^{n-2}|}{|\mathbf{x}|}\sqrt{1-\left(\frac{\varrho}{|\mathbf{x}|}\right)^2}^{\,n-3},$$

the lemma is proved. □

Although the following lemma was already proved as Lemma 5.3 in [11], we present it here for the sake of completeness with its short proof.

Lemma 4.3. *Let ω_i $(i = 1, 2)$ be weights, let $\mathcal{K}$ and $\mathcal{L}$ be convex bodies containing the unit ball $\mathcal{B}$, and let $c \geq 1$.*

1. *If $cV_1(\mathcal{K}) \leq V_1(\mathcal{L})$ and there is a constant $c_{\mathcal{K}}$ such that*

$$\omega_2(X) \geq c_{\mathcal{K}}\omega_1(X), \quad \text{if } X \notin \mathcal{K},$$
$$\omega_2(X) = c_{\mathcal{K}}\omega_1(X), \quad \text{if } X \in \partial\mathcal{K},$$
$$\omega_2(X) \leq c_{\mathcal{K}}\omega_1(X), \quad \text{if } X \in \mathcal{K},$$

where equality may occur only in a set of measure zero, then $cV_2(\mathcal{K}) \leq V_2(\mathcal{L})$.

2. *If* $V_1(\mathcal{K}) \le cV_1(\mathcal{L})$ *and there is a constant* $c_{\mathcal{L}}$ *such that*

$$\begin{aligned} \omega_2(X) &\le c_{\mathcal{L}}\omega_1(X), \quad \textit{if } X \notin \mathcal{L},\\ \omega_2(X) &= c_{\mathcal{L}}\omega_1(X), \quad \textit{if } X \in \partial\mathcal{L},\\ \omega_2(X) &\ge c_{\mathcal{L}}\omega_1(X), \quad \textit{if } X \in \mathcal{L}, \end{aligned}$$

where equality may occur only in a set of measure zero, then $V_2(\mathcal{K}) \le cV_2(\mathcal{L})$.

In both cases equality in the resulted inequality implies $\mathcal{K} = \mathcal{L}$ *and* $c = 1$.

Proof. In both statements $\mathcal{K}\triangle\mathcal{L} = \emptyset$ implies $V_1(\mathcal{K}) = V_1(\mathcal{L})$, hence $c = 1$ and $V_1(\mathcal{K}) = V_1(\mathcal{L})$.

Assume from now on that $\mathcal{K}\triangle\mathcal{L} \neq \emptyset$.

We prove here only (1) since the verification of (2) is similar.

Having (1) we proceed as

$$\begin{aligned} &V_2(\mathcal{L}) - cV_2(\mathcal{K})\\ &\quad = V_2(\mathcal{L}) - V_2(\mathcal{K}) + (1-c)V_2(\mathcal{K}) = V_2(\mathcal{L}\setminus\mathcal{K}) - V_2(\mathcal{K}\setminus\mathcal{L}) + (1-c)V_2(\mathcal{K})\\ &\quad = \int\limits_{\mathcal{L}\setminus\mathcal{K}} \frac{\omega_2(x)}{\omega_1(x)}\omega_1(x)dx - \int\limits_{\mathcal{K}\setminus\mathcal{L}} \frac{\omega_2(x)}{\omega_1(x)}\omega_1(x)dx + (1-c)V_2(\mathcal{K})\\ &\quad > c_{\mathcal{K}}(V_1(\mathcal{L}\setminus\mathcal{K}) - V_1(\mathcal{K}\setminus\mathcal{L})) + (1-c)V_2(\mathcal{K})\\ &\quad = c_{\mathcal{K}}(V_1(\mathcal{L}) - V_1(\mathcal{K})) + (1-c)V_2(\mathcal{K})\\ &\quad \ge (c-1)(c_{\mathcal{K}}V_1(\mathcal{K}) - V_2(\mathcal{K})) = (c-1)\left(\int\limits_{\mathcal{K}} \left(c_{\mathcal{K}} - \frac{\omega_2(x)}{\omega_1(x)}\right)\omega_1(x)dx\right) \ge 0. \end{aligned}$$

This implies $V_2(\mathcal{L}) - cV_2(\mathcal{K}) > 0$.

The lemma is proved. □

5. Spherical isomaskers

First we calculate the integral of the masking function $M_{\mathcal{K}}$ of the convex body $\mathcal{K} \subset \bar{r}\mathcal{B}^n$ over the sphere $\bar{r}\mathbb{S}^{n-1}$ ($\bar{r} > 0$). Starting with Eq. (3.1) we get

$$\begin{aligned} \int\limits_{\mathbb{S}^{n-1}} M_{\mathcal{K}}(\bar{r}\mathbf{u}_\xi)\, d\xi &= \frac{1}{|\mathbb{S}^{n-2}|}\int\limits_{\mathbb{S}^{n-1}} |\mathbb{S}^{n-1}| - M^\star_{\mathcal{K}}(\check{\delta}(\bar{r}\mathbf{u}_\xi))\, d\xi\\ &= \frac{|\mathbb{S}^{n-1}|^2}{|\mathbb{S}^{n-2}|} - \frac{1}{|\mathbb{S}^{n-2}|}\int\limits_{\mathbb{S}^{n-1}} M^\star_{\mathcal{K}}(\check{\delta}(\bar{r}\mathbf{u}_\xi))\, d\xi. \end{aligned}$$

Assuming $\mathbf{0} \in \mathcal{K}^\circ$ we can continue by using (3.2) and (4.1) and obtain

$$\int_{\mathbb{S}^{n-1}} M_{\mathcal{K}}(r\mathbf{u}_\xi)\, d\xi = \frac{|\mathbb{S}^{n-1}|^2}{|\mathbb{S}^{n-2}|} - \frac{1}{|\mathbb{S}^{n-2}|} \int_{\mathbb{S}^{n-1}} \int_{\hbar(-\mathbf{u}_\xi, \frac{1}{r})} \chi\big(\mathbf{x} \in \mathcal{K}^\star\big) \frac{1/\bar{r}}{|\mathbf{x}|^n}\, d\mathbf{x}\, d\xi.$$

This means

$$\int_{\mathbb{S}^{n-1}} M_{\mathcal{K}}(r\xi)\, d\xi = \frac{|\mathbb{S}^{n-1}|^2}{|\mathbb{S}^{n-2}|} - \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\bar{r}};\mathcal{K}^\star}(\mathbf{u}_\xi)\, d\xi, \tag{5.1}$$

where $\nu_{\hbar(\mathbf{u},r)}(\mathbf{x}) = r|\mathbf{x}|^{-n}$. Having this we are ready to prove the following generalization of Nitsche's result [13].

Theorem 5.1. *Let $\varrho_2 > \varrho_1 > \bar{r} > 0$ and let $\mathcal{K}$ be a convex body contained in the interior of $\varrho_1\mathcal{B}^n$. If the sphere $\varrho_1\mathbb{S}^{n-1}$ is the common α-isomasker and $\varrho_2\mathbb{S}^{n-1}$ is the common β-isomasker of the convex body $\mathcal{K}$ and $\bar{r}\mathcal{B}$, then $\mathcal{K} = \bar{r}\mathcal{B}$.*

Proof. By the conditions we have $M_{\mathcal{K}}(\varrho_1\mathbf{u}) = \alpha = M_{\bar{r}\mathcal{B}^n}(\varrho_1\mathbf{u})$ and $M_{\mathcal{K}}(\varrho_2\mathbf{u}) = \beta = M_{\bar{r}\mathcal{B}^n}(\varrho_2\mathbf{u})$ for every $\mathbf{u} \in \mathbb{S}^{n-1}$.

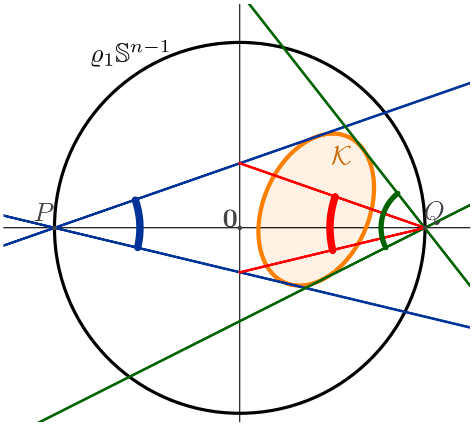

FIGURE 3 $M_{\mathcal{K}}(P)$ is clearly smaller than $M_{\mathcal{K}}(Q)$

Some elementary observations and reasoning illustrated in Fig. 3 implies that $\mathcal{K}^\circ$ contains the common center $\mathbf{0}$ of the balls $\bar{r}\mathcal{B}$, $\varrho_1\mathcal{B}^n$ and $\varrho_2\mathcal{B}^n$.

Now Eq. (5.1) implies

$$\int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_1};\mathcal{K}^\star}(\mathbf{u}_\xi)\, d\xi = \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_1};(\bar{r}\mathcal{B}^n)^\star}(\mathbf{u}_\xi)\, d\xi = \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_1};\frac{1}{\bar{r}}\mathcal{B}^n}(\mathbf{u}_\xi)\, d\xi,$$

$$\int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_2};\mathcal{K}^\star}(\mathbf{u}_\xi)\, d\xi = \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_2};(\bar{r}\mathcal{B}^n)^\star}(\mathbf{u}_\xi)\, d\xi = \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_2};\frac{1}{\bar{r}}\mathcal{B}^n}(\mathbf{u}_\xi)\, d\xi.$$

As the function ν of weights having $\bar{\nu}(\varrho_2, \varrho_2, r) = \varrho_2 r^{-n}$ is obviously rotational invariant, (4.2) implies

$$\int\limits_{\mathcal{K}^\star \setminus \frac{1}{\varrho_2}\mathcal{B}^n} \frac{\left(|\mathbf{x}|^2 - \varrho_2^{-2}\right)^{\frac{n-3}{2}}}{|\mathbf{x}|^{2n-2}}\, d\mathbf{x} = \int\limits_{\frac{1}{\bar{r}}\mathcal{B}^n \setminus \frac{1}{\varrho_2}\mathcal{B}^n} \frac{\left(|\mathbf{x}|^2 - \varrho_2^{-2}\right)^{\frac{n-3}{2}}}{|\mathbf{x}|^{2n-2}}\, d\mathbf{x},$$

and

$$\int\limits_{\mathcal{K}^\star \setminus \frac{1}{\varrho_1}\mathcal{B}^n} \frac{\left(|\mathbf{x}|^2 - \varrho_1^{-2}\right)^{\frac{n-3}{2}}}{|\mathbf{x}|^{2n-2}}\, d\mathbf{x} = \int\limits_{\frac{1}{\bar{r}}\mathcal{B}^n \setminus \frac{1}{\varrho_1}\mathcal{B}^n} \frac{\left(|\mathbf{x}|^2 - \varrho_1^{-2}\right)^{\frac{n-3}{2}}}{|\mathbf{x}|^{2n-2}}\, d\mathbf{x}.$$

Let $\bar{\omega}_1(r) := r^{2-2n}(r^2 - \varrho_1^{-2})^{\frac{n-3}{2}}$, $\bar{\omega}_2(r) := r^{2-2n}(r^2 - \varrho_2^{-2})^{\frac{n-3}{2}}$, and let $\omega_1(\mathbf{x}) := \bar{\omega}_1(|\mathbf{x}|)$, $\omega_2(\mathbf{x}) := \bar{\omega}_2(|\mathbf{x}|)$. Then $\frac{\omega_1}{\omega_2}$ is clearly a constant, say $c_{\mathcal{L}}$, on $\frac{1}{\bar{r}}\mathcal{B}^n$, and

$$\frac{\bar{\omega}_1(r)}{\bar{\omega}_2(r)} = \frac{\left(r^2 - \varrho_1^{-2}\right)^{\frac{n-3}{2}}}{\left(r^2 - \varrho_2^{-2}\right)^{\frac{n-3}{2}}} = \left(1 - \frac{\varrho_1^{-2} - \varrho_2^{-2}}{r^2 - \varrho_1^{-2}}\right)^{\frac{n-3}{2}}$$

shows that $\frac{\bar{\omega}_1}{\bar{\omega}_2}$ is strictly monotone increasing.

The above observations show that the conditions in (2) of Lemma 4.3 are satisfied for $\mathcal{K}^\star$, $\mathcal{L} := \frac{1}{\bar{r}}\mathcal{B}^n$ and $c = 1$, hence $V_2(\mathcal{K}^\star) \leq V_2(\mathcal{L})$, and equality implies $\mathcal{K}^\star = \mathcal{L}$ and $c = 1$.

As $\mathcal{K} = (\mathcal{K}^\star)^\star = (\mathcal{L})^\star = \bar{r}\mathcal{B}^n$, the theorem is proved. □

6. Discussion

To have a complete generalization of Nitsche's result [13] from the point of view of Theorem 5.1, one should prove that if a convex body $\mathcal{K}$ has two spherical isomaskers of values $\alpha_1 \neq \alpha_2$, then there is a ball $\bar{r}\mathcal{B}^n$ with the same α_1- and α_2-isomaskers of radius $\varrho_1 \neq \varrho_2$. Although Nitsche proved this in the plane, the authors conjecture that this is no longer valid in higher dimensions.

Conjecture 6.1. *There are positive values $\alpha_1 \neq \alpha_2$ and $\varrho_1 \neq \varrho_2$ such that there is a non-spherical convex body $\mathcal{K} \subset \mathbb{R}^n$ the α_1- and α_2-isomaskers of which are spheres of radius $\varrho_1 \neq \varrho_2$, respectively.*

However note that it is proved in [7] that if two convex bodies in the plane have rotational symmetry of angle $2(\pi - \nu)$ and have common ν-isoptic, then that ν-isoptic is a circle.

In higher dimensions the only positive result the authors know about is the surprisingly easy [5, Theorem 2]. It states that if a convex body $\mathcal{K} \subset \mathbb{R}^n$ has an isoptic $\mathcal{I}$ in the sense of a k-dimensional angles for any $1 < k < n-1$, then $\mathcal{K}$ is reconstructible from $\mathcal{I}$.

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Árpád Kurusa and Tibor Ódor
Bolyai Institute
University of Szeged
Aradi vértanúk tere 1.
Szeged 6720
Hungary
e-mail: kurusa@math.u-szeged.hu;
odor@math.u-szeged.hu

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