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The calculus of reflections and the order relation in hyperbolic geometry

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Abstract. It is well known that the calculus of reflections (developed by Hjelmslev, Bachmann et al.) enables the derivation of a large part of Euclidean and non-Euclidean geometry without using assumptions about order and continuity. We show in this article that the calculus of reflections can conversely be used to *introduce* a relation of order in hyperbolic geometry. Our investigations are based on the famous 'Endenrechnung' of Hilbert which was formulated purely in terms of the calculus of reflections by F. Bachmann. We then discuss some implications of these results and show that the calculus of reflections enables (1) the introduction of an order relation in a Pappian projective line and (2) to define an axiom system for hyperbolic planes which seems to be as simple as the famous axiom system of Menger who only used the notion of point-line incidence to axiomatize plane hyperbolic geometry.

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1. Introduction

While it is well known, that the calculus of reflections enables the derivation of a large part of Euclidean and non-Euclidean geometry without using assumptions about order and continuity (see Hjelmslev [7,8] and Bachmann [2]), it seems to be an open question whether the calculus can be used to *introduce* concepts of order and separation.

In the foundations of geometry the relations of betweenness and order are introduced in hyperbolic geometry either as undefined notions which satisfy certain axioms (see Hilbert [5,6] or in a modern setting Hartshorne [4]) or they are defined in an algebraic way (see Bachmann [2, §15] who uses an algebraic representation of the group of motions of a hyperbolic plane to show that the field of coordinates has a subset of positive elements which induces a relation of order on the set of points of the hyperbolic plane).

We show in Sect. 2 that in hyperbolic geometry a relation of order can be formulated purely in terms of the calculus of reflections without assuming axioms of order and without making reference to a special algebraic representation of the underlying geometrical structure. We proceed in two steps and show that on the set of ends a relation of separation can be introduced which induces a relation of order on the set of points of a hyperbolic plane. To elucidate this approach we prove some theorems about separation and order which show that our definitions satisfy well-known axioms.

In the subsequent sections we discuss some implications of our results. In Sect. 3 we answer a question of Pambuccian who gives in [19] an extensive overview of the axiomatics of ordered geometry. Starting with the one-dimensional case he considers projective lines over fields of characteristic $\neq 2$. Their groups $PGL_2(K)$ of projective collineations were axiomatized by Bachmann [1] and it is natural to ask, whether Bachmann's axiom system—in which individuals are Möbius transformations and the only primitive notion is the binary operation of composition of Möbius transformations—allows us, to bring a projective order relation into the picture [19, Subsection 2.7]. We answer this question in the affirmative.

In Sect. 4 we use the results of Sect. 2 to define an axiom system for hyperbolic planes which seems to be as simple as the famous axiom system of Menger [13] who only used the notion of point-line incidence to axiomatize plane hyperbolic geometry (for axiomatizations of hyperbolic geometry see Pambuccian [20]).

2. The relation of order in hyperbolic geometry

We show in this section that in hyperbolic geometry a relation of order can be formulated purely in terms of the calculus of reflections. To this end we proceed in two steps and show that on the set of ends a relation of separation can be introduced which induces a relation of order on the set of points of the hyperbolic plane.

As starting point of our investigations we choose the group-theoretic axiom system of Bachmann [2, §14] (which was proposed by Klingenberg [10] and Bergau [3]) for plane hyperbolic geometry which does not contain any axioms of order (in contrast to the axiomatizations of Hilbert [6] and Hartshorne [4]).

Basic assumption. Let G be a group which is generated by an invariant set S of involutory elements.

Notation: The elements of S will be denoted by lowercase Latin letters. The set of involutory elements of S^2 will be denoted by P and their elements by uppercase letters $A, B, \ldots$ The 'stroke relation' $\alpha \mid \beta$ is an abbreviation for the statement that α, β and $\alpha\beta$ are involutory elements. The statement $\alpha, \beta \mid \delta$ is an abbreviation of $\alpha \mid \delta$ and $\beta \mid \delta$.

Axiom A1. For A, B there exists c with $A, B \mid c$.
Axiom A2. If $A, B \mid c, d$ then $A = B$ or $c = d$.
Axiom A3. If $a, b, c \mid e$ then $abc \in S$.
Axiom A4. If $a, b, c \mid E$ then $abc \in S$.
Axiom D. There exist a, b, c with $a \mid b$ and $a \nmid c$ and $b \nmid c$ and $ab \nmid c$.
Axiom $\neg V^*$. There exist a, b with $\alpha \nmid a, b$ for all $\alpha \in S \cup P$.
Axiom H. If $a, b, c \mid D$ and if there is no $\alpha \in S \cup P$ with $\alpha \mid a, g$ or $\alpha \mid b, g$ or $\alpha \mid c, g$ then $a = b$ or $a = c$ or $b = c$.

The axiom system has a twofold interpretation. It characterizes the group of motions of a hyperbolic plane by axioms which the reflections $A, B, C, \dots$ in points and the reflections $a, b, c, \dots$ in lines satisfy. So according to *Axiom A3* and *Axiom A4* the *theorem of three reflections* holds: If three lines have a common point or a common perpendicular, then the product of the reflections in these lines is a line reflection.

The group of motions of a hyperbolic plane allows the recovery of the geometry of the underlying hyperbolic plane. The points and lines of a hyperbolic plane are in one-to-one correspondence with the reflections in points and lines and geometric relations such as incidence or orthogonality correspond to group-theoretical equations among reflections. Hence we can assign to (G, S) a geometrical structure, called the *group plane* of (G, S) (cp. [2, §20,2]). Elements of S are called *lines* and elements of P *points*. Lines $a, b \in S$ are called *orthogonal* if $ab \in P$ (written $a \mid b$). A point A and a line b are called *incident* if Ab is involutory (written $A \mid b$). Two lines a, b are *connected* if they have a common point or a common perpendicular. The mapping $a \to a^\alpha, A \to A^\alpha$ of S onto S and P onto P is called the *motion* induced by $\alpha \in G$ (we write β^α instead of $\alpha^{-1}\beta\alpha$). A set $S(a, b) = \{c : abc \in S\}$ of lines is called a *pencil*. A pencil $S(a, b)$ is called *ordinary* if a, b are connected and *singular* otherwise. Following Hilbert [5] a singular pencil of lines is called an *end*. We denote ends by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$.

Bachmann's axioms for plane hyperbolic geometry can be formulated in the geometrical language of the group plane (the second interpretation of the axiom system): *Axiom A1* and *Axiom A2* can be interpreted as the existence and uniqueness of a joining line of two points, *Axiom A3* and *Axiom A4* as three lines which have a common point or a common perpendicular lie in a pencil, *Axiom* $\neg V^*$ as the existence of an end and *Axiom H* as the hyperbolic parallel axiom (if A, g are not incident then there are at most two lines through A which have neither a common point nor a common perpendicular with g).

In this way the calculus of reflections allows the formulation of geometric theorems as theorems about elements of the group of motions which can be proved by group-theoretic calculations.

According to the main theorem of Bachmann [2, §6 and §11] the following holds: a hyperbolic plane can be extended to a Pappian projective plane (the *projective ideal plane*) which can be coordinatized over a commutative field of characteristic $\neq 2$. The orthogonality of lines induces a hyperbolic polarity in the ideal plane (with the set of ends as the set of points of the absolute conic) which can be described by a symmetric bilinear form *f*. The group G can be represented as the orthogonal group $O_3(K, f)$.

We are interested in the properties of ends. The following propositions hold according to Bachmann [2, §14 and §10,4]):

(1) Two lines of an end have neither a common point nor a common line.
(2) A line is an element of at most two ends.
(3) A line, which is element of an end, is the element of a second end.
(4) Any two ends have a common line.
(5) If $\mathcal{A}, \mathcal{B}$ are ends and $g \in \mathcal{A}, \mathcal{B}$ then $\mathcal{A}^E = \mathcal{B}$ (resp. $\mathcal{A}^h = \mathcal{B}$) if and only if $E \mid g$ (resp. $h \mid g$).

Following the famous 'Endenrechnung' of Hilbert [7] the following theorem is proved by Bachmann (see [2, §15,1]):

Theorem 2.1. *Let $\mathcal{I}$ ('infinity') be an arbitrary fixed end. On the set K of ends $\neq \mathcal{I}$ of (G, S) an addition $+$ and a multiplication $\cdot$ can be introduced with the following properties.*

(1) $(K, +, \cdot)$ *is a field of characteristic* $\neq 2$.
(2) *The groups G and $PGL_2(K)$ are isomorphic.*

We now show that Bachmann's axiom system allows us to bring the notion of separation into the picture.

In projective geometry the order relation on a (projective) line can be described by the notion of separation which is a quaternary relation // on the set of points of a line with $AB//CD$ to be read as *'the point-pair (A, B) separates the point-pair $(C, D)'$*. For axiomatizations of this relation we refer to Pambuccian [19].

We use the following theorem to define a relation of separation on the set of ends.

Theorem 2.2. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be four different ends. Then the following conditions are equivalent:*

(a) *There exists a point E with $\mathcal{A}^E = \mathcal{B}$ and $\mathcal{C}^E = \mathcal{D}$.*
(b) *The joining lines of $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}, \mathcal{D}$ have a common point E.*
(c) *The joining lines of $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}, \mathcal{D}$ have no common perpendicular.*

Proof. (b) and (c) are equivalent since two lines of a hyperbolic plane have either a common point or a common line or are elements of the same end. The equivalence of (a) and (b) holds according to Bachmann [2, §14,2]. □

Definition 2.3. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be four different ends. The pair $(\mathcal{A}, \mathcal{B})$ separates $(\mathcal{C}, \mathcal{D})$ (notation $\mathcal{AB}//\mathcal{CD}$) if (one of) the conditions of Theorem 2.2 hold.*

Theorem 2.4. *The relation of separation is invariant under motions of G and hence under projective collineations.*

Proof. The separation relation is defined in terms of incidence and orthogonality and hence is invariant under motions of G and under projective collineations (since according to Theorem 2.1 the groups G and $PGL_2(K)$ are isomorphic). □

We show that Definition 2.3 satisfies the axioms of a separation relation given in Pambuccian [19, Section2.7].

Theorem 2.5. *If* $(\mathcal{A},\mathcal{B})$ *separates* $(\mathcal{C},\mathcal{D})$ *then* $(\mathcal{A},\mathcal{C})$ *does not separate* $(\mathcal{B},\mathcal{D})$.

Proof. Let $a\in\mathcal{A},\mathcal{C}$ and $b\in\mathcal{B},\mathcal{D}$ and $g\in\mathcal{A},\mathcal{B}$ and $h\in\mathcal{C},\mathcal{D}$. Since $\mathcal{AB}//\mathcal{CD}$ there exists a point $X\mid g,h$. Suppose $\mathcal{AC}//\mathcal{BD}$, i.e. there exists a point $E\mid a,b$. Then $\mathcal{A}^E=\mathcal{C}$ and $\mathcal{B}^E=\mathcal{D}$ (according to Theorem 2.2) and hence $g^E=h$ and $X^E=X$, i.e. $X=E$, which is a contradiction to $\mathcal{A}^X=\mathcal{B}$ and $\mathcal{A}^E=\mathcal{C}$ and $\mathcal{B}\neq\mathcal{C}$. □

Theorem 2.6. $(\mathcal{M},\mathcal{N})$ *separates two or none of the pairs* $(\mathcal{A},\mathcal{B}),(\mathcal{A},\mathcal{C}),(\mathcal{B},\mathcal{C})$.

Theorem 2.6 is a configurational statement which says (see Theorem 2.2): Let $\mathcal{A},\mathcal{B},\mathcal{C}$ be three different ends and a,b,c their joining lines (an *asymptotic triangle*; see [2, §14,2]). If a line g (which connects ends $\mathcal{M},\mathcal{N}\neq\mathcal{A},\mathcal{B},\mathcal{C}$) intersects one of the sides of the asymptotic triangle then g intersects another side as well.

This is the *Axiom of Pasch* formulated for an asymptotic triangle.

Theorem 2.6 is equivalent to the following theorem.

Theorem 2.7. *Let* $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{M},\mathcal{N}$ *be five different ends and* $a\in\mathcal{B},\mathcal{C}$ *and* $b\in\mathcal{A},\mathcal{C}$ *and* $c\in\mathcal{A},\mathcal{B}$ *and* $g\in\mathcal{M},\mathcal{N}$. *Then* g *has a common perpendicular with at least one of the lines* a,b,c. *If* g *has a common perpendicular with two of the lines* a,b,c *then* g *has a common perpendicular with each of the lines* a,b,c.

Proof. Suppose g has no common perpendicular with a,b and c. Then there are points $A\mid a,g$ and $B\mid b,g$ and $C\mid c,g$ and $BAC=E\in P$ and $E\mid g$ (according to *Axiom A4*). Since $\mathcal{A}^E=\mathcal{A}^{BAC}=\mathcal{C}^{AC}=\mathcal{B}^C=\mathcal{A}$ we have $E\in\mathcal{A}$ (according to [2, Proposition 5 in §11,2]) which is a contradiction to $\mathcal{A}\subseteq S$ and $S\cap P=\varnothing$.

Now suppose g has a common perpendicular with a and b. Let $l_a\mid a,g$ and $l_b\mid b,g$ and $h_a,h_b,h_c\mid g$ with $h_a\in\mathcal{A}$ and $h_b\in\mathcal{B}$ and $h_c\in\mathcal{C}$ (which exist according to [2, Proposition 2 in §14,4]). Since $l_b,h_c,l_a\mid g$ we have $l_bh_cl_a=:h\in S$ and $h\mid g$ and $\mathcal{A}^h=\mathcal{A}^{l_bh_cl_a}=\mathcal{C}^{h_cl_a}=\mathcal{C}^{l_a}=\mathcal{B}$. According to [2, Proposition 9 in §11,3] we have $h\mid c$. Hence h is a common perpendicular of c and g. □

Theorem 2.8. *If* $\mathcal{AC}//\mathcal{BD}$ *then* $\mathcal{AC}//\mathcal{XD}$ *or* $\mathcal{AC}//\mathcal{BX}$.

Proof. Suppose $(\mathcal{A},\mathcal{C})$ separates $(\mathcal{B},\mathcal{D})$ but not $(\mathcal{X},\mathcal{D})$. Then $(\mathcal{A},\mathcal{C})$ separates one of the pairs $(\mathcal{B},\mathcal{D})$, $(\mathcal{B},\mathcal{X})$, $(\mathcal{X},\mathcal{D})$ and hence according to Theorem 2.6 two of the pairs, i.e. $\mathcal{AC}//\mathcal{BX}$. □

Theorem 2.9. *If neither* $\mathcal{AB}//\mathcal{CD}$ *nor* $\mathcal{AD}//\mathcal{BC}$ *then* $\mathcal{AC}//\mathcal{BD}$.

Proof. Let a, b, c, d, e, f be the joining lines of the ends $\mathcal{A}, \mathcal{B}$ resp. $\mathcal{B}, \mathcal{C}$ resp. $\mathcal{C}, \mathcal{D}$ resp. $\mathcal{A}, \mathcal{D}$ resp. $\mathcal{A}, \mathcal{C}$ resp. $\mathcal{B}, \mathcal{D}$. Under the assumptions of the theorem the lines a, c (resp. d, b) have a common perpendicular g resp. h.

According to [2, §14,2] holds $\mathcal{A}^g = \mathcal{B}$ and $\mathcal{A}^{hgh} = \mathcal{B}$. Hence $g, g^h \mid a$ and $g = h$ or $h \mid a$. Since $\mathcal{A}^h = \mathcal{D} \neq \mathcal{B}$ it is $h \nmid a$. Hence $g = h$ and $E := gh$ is a point with $\mathcal{A}^E = \mathcal{A}^{gh} = \mathcal{C}$ and $\mathcal{B}^E = \mathcal{A}^{gh} = \mathcal{D}$, i.e. E is a common point of e and f. □

The set of ends can be considered as the set of points of a projective line (the points of the projective coordinate line $\mathcal{P}^1(K)$ over the field K are the 1-dimensional subspaces of the two-dimensional vector space $V_2(K)$ of pairs over K and a point of $\mathcal{P}^1(K)$ thus consists of all pairs in K proportional to a pair $(x, 1)$ with $x \in K$ or to $(1, 0)$ (the point at 'infinity') which correspond to the end $\mathcal{I}$ and to the ends $\neq \mathcal{I}$; see Theorem 2.1).

Hence our results can be summarized in the following way:

Theorem 2.10. *The projective line of ends, with the relation of separation defined in Definition 2.3, is an ordered projective line and the coordinate field K is orderable.*

Proof. According to the Theorems 2.4 to 2.9 the axioms of separation of a projective line given in Pambuccian [19, Section 2.7] are satisfied and hence the associated coordinate field is orderable. □

Since the field K of ends of a hyperbolic plane is the coordinate field of the associated projective ideal plane and since K is according to Theorem 2.10 an orderable field, the ideal plane can be ordered and hence induces a relation of order in the hyperbolic plane. Since a hyperbolic plane contains with two points of the ideal plane their joining line and with each point all lines through this point, Hilbert's axioms of incidence and order are satisfied, i.e., a hyperbolic plane is an ordered plane in the sense of Pambuccian [19, Section 3.1].

We close this section and show how the relation of betweenness on the set of points of a hyperbolic plane can be defined purely in terms of the calculus of reflections.

Let κ be the conic of the projective ideal plane whose points are the ends of the given hyperbolic plane. We choose parametric coordinates of κ which are induced by the pencil of lines through a point $\mathcal{I}$ of κ. The cross-ratio $CR(\mathcal{ABCD})$ of four points $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \neq \mathcal{I}$ of κ is defined as the cross-ratio of the lines $a, b, c, d \in \mathcal{I}$ through $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ respectively and does not depend on the choice of the point $\mathcal{I}$ of κ (by Steiners theorem). $\mathcal{AB}//\mathcal{CD}$ holds if and only if $CR(\mathcal{ABCD}) < 0$ (see Karzel and Kroll [9] or Lenz [11]).

Since a line cuts a pencil in sets of points with the same cross-ratio as the intercepted lines, the projection π of κ from one of its points $\mathcal{I}$ on a line l of the hyperbolic plane with $l \notin \mathcal{I}$ preserves the cross-ratio and hence induces a relation of separation on l (if A, B, C, D are points on l with $A = \mathcal{A}\pi$ and $B = \mathcal{B}\pi$ and $C = \mathcal{C}\pi$ and $D = \mathcal{D}\pi$ then $AB//CD$ if $\mathcal{AB}//\mathcal{CD}$).

The relation of separation on κ induces on the set of ends $\neq \mathcal{I}$ a relation of betweenness (if $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are ends $\neq \mathcal{I}$ then $\mathcal{B}$ lies between $\mathcal{A}$ and $\mathcal{C}$ if $\mathcal{AC}//\mathcal{BI}$) and the projection π induces an associated relation on l: *If A, B, C are points on l (with $A = \mathcal{A}\pi$ and $B = \mathcal{B}\pi$ and $C = \mathcal{C}\pi$ and $D = \mathcal{D}\pi$) then B lies between A and C if $\mathcal{AC}//\mathcal{BI}$.*

More explicitly the following holds: Let A, B, C be points on a line l and $\mathcal{I}$ an end with $l \notin \mathcal{I}$. Let $a, b, c \in \mathcal{I}$ with $a \mid A$ and $b \mid B$ and $c \mid C$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ the unique ends $\neq \mathcal{I}$ with $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $c \in \mathcal{C}$. Then B lies between A and C if there exists a point E with $\mathcal{A}^E = \mathcal{C}$ and $\mathcal{B}^E = \mathcal{I}$ (i.e. if $\mathcal{AC}//\mathcal{BI}$).

In terms of the calculus of reflections we get the following theorem which can be used as a *definition* of a relation of betweenness in a hyperbolic plane.

Theorem 2.11. *Let A, B, C be points on a line l and $\mathcal{I}$ an end with $l \notin \mathcal{I}$. Then B lies between A and C if there exists a point E with $(\mathcal{I}^A)^E = \mathcal{I}^C$ and $\mathcal{I}^E = \mathcal{I}^B$.*

Proof. Using the notations of the theorem we get $\mathcal{I}^A = \mathcal{A}$ and $\mathcal{I}^B = \mathcal{B}$ and $\mathcal{I}^C = \mathcal{C}$ (according to (5) in Sect. 2) and from this the theorem follows immediately. □

3. Ordered projective lines

Pambuccian gives in [19] an extensive overview of the axiomatics of ordered geometry. Starting with the one-dimensional case he considers projective lines over fields K of characteristic $\neq 2$. Their group $PGL_2(K)$ of projective collineations was axiomatized by Bachmann [1] and it is natural to ask, whether Bachmann's axiom system—in which individuals are Möbius transformations and the only primitive notion is the binary operation of composition of Möbius transformations—allows us to bring a projective order relation into the picture (see [19, Section 2.7]).

Bachmann (see [1] and [2, §11]) and Lingenberg [12] characterize the projective general linear group $PGL_2(K)$ as an abstract group $\mathcal{H}$ whose generators are involutory and satisfy the following axioms (involutory elements of $\mathcal{H}$ are denoted by lowercase Latin letters; involutory elements a, b are called *connected* if there is an involutory element v such that av and bv are involutory).

Basic assumption. $\mathcal{H}$ is a group and each element of $\mathcal{H}$ can be represented as the product of two involutions.

Axiom 1. If $a \neq b$ and abc, abd are involutory then acd is involutory.
Axiom 2. There exist elements a, b which are not connected.
Axiom 3. If neither a, b nor c, d are connected then there exists an element v such that abv and cdv are involutory.

A group which satisfies this axiom system is called a *H-group*.

According to [2, Theorem 9] the following theorem holds.

Theorem 3.1. *A group $PGL_2(K)$ over a field K of characteristic $\neq 2$ is a H-group and each H-group can be represented as a group $PGL_2(K)$ over a field K of characteristic $\neq 2$.*

The group $PGL_2(K)$—associated to a projective line—allows a second geometric interpretation (see Veblen and Young [24, Chapter viii] and Bachmann [2, §10,4]): $PGL_2(K)$ is isomorphic to the group of projective collineations of a Pappian projective plane which leave a non-empty (non-degenerated) conic κ invariant. The involutory elements of $\mathcal{H}$ are in this interpretation the harmonic homologies (projective reflections) whose axis and center are not incident and pole and polar with respect to the conic.

If K is an ordered field then the points which are in the interior of κ and the lines which are incident with at least one interior point are the elements of Klein's model of a hyperbolic plane. The set S of projective reflections in lines of the hyperbolic plane generate the group G of motions and (G, S) satisfies the axioms of a hyperbolic plane given in Sect. 2.

According to [2, Theorem 11] the following theorem holds.

Theorem 3.2. *A H-group over a field K of characteristic $\neq 2$—and hence a group $PGL_2(K)$—is the group of motions of a hyperbolic plane if and only if K can be ordered.*

Since a projective line can be ordered if and only if the underlying field can be ordered (see Karzel and Kroll [9] or Lenz [11]) the following theorem holds.

Theorem 3.3. *The group of projective collineations of an ordered projective line is isomorphic to the group of motions of a hyperbolic plane. These groups can be characterized by the axiom system of Bachmann given in Sect. 2.*

Hence the question of Pambuccian mentioned above can be answered in the affirmative: A projective line over a field of characteristic $\neq 2$ can be ordered if and only if the group G of projective collineations has a subset S of involutory elements such that (G, S) satisfies the axiom system given in Sect. 2 and the relation of separation can be introduced as in Definition 2.3.

4. A simple axiom system for plane hyperbolic geometry

After Hilbert's [6] axiomatization of plane hyperbolic geometry there have been various attempts to simplify his axiom system (for a survey see Pambuccian [20]). Tarski [23] used only one sort of individual variables (points) and only two primitive notions (betweenness and equidistance) which led to a remarkable simplification of the language and of the axioms of Hilbert. In [13] and [14] Menger showed that it is sufficient to use the notion of point-line incidence to axiomatize hyperbolic geometry and claimed that for this reason hyperbolic geometry is simpler than Euclidean geometry (see Skala [21] for a formulation of Menger's axiom system in a first-order language).

Both axiom systems mentioned above are axiomatizations of 'classical' hyperbolic planes with free mobility which can be coordinatized over Euclidean fields. According to Bachmann [2, §15,2] these are the hyperbolic planes in which every line is element of an end.

Given that an axiom system for hyperbolic geometry should be able to define elementary geometric concepts such as incidence, order and congruence, the results of Sect. 2 allow the formulation of a simple axiom system $\mathcal{H}$ for classical hyperbolic planes with only one sort of individual variables (elements $\alpha, \beta, \gamma, \ldots$ of a set G) and only one binary operation $\cdot$ on G. To improve the readability of the axioms, we introduce the following abbreviations:

$\varepsilon(\alpha) \quad \Leftrightarrow \quad \alpha^2 = \alpha$ (to be interpreted as α is an *idempotent element*)

$\iota(\alpha) \quad \Leftrightarrow \quad \varepsilon(\alpha^2) \wedge \neg\, \varepsilon(\alpha)$ (to be interpreted as α is an *involution* of $(G, \cdot)$)

$\alpha \,|\beta \quad \Leftrightarrow \quad \iota(\alpha) \wedge \iota(\beta) \wedge \iota(\alpha \cdot \beta)$ (we write $\alpha, \beta \mid \gamma$ if $\alpha \mid \gamma \wedge \beta \mid \gamma$)

$\alpha \backsim \beta \quad \Leftrightarrow \quad \iota(\alpha) \wedge \iota(\beta) \wedge (\exists \gamma)\, (\iota(\gamma) \wedge \alpha, \beta \mid \gamma)$ (in the negated case we write $\nsim$)

We present the axioms in informal language (their formalization being straightforward) and define two subsets P and S of G (which correspond in a first-order language to unary predicates).

(1) $\alpha \in P \quad \Leftrightarrow \quad \iota(\alpha) \wedge (\forall \beta)\, (\iota(\beta) \rightarrow \alpha \sim \beta)$
(2) $\alpha \in S \quad \Leftrightarrow \quad \iota(\alpha) \wedge (\exists \beta)\, (\iota(\beta) \wedge \alpha \nsim \beta)$

As in Sect. 2 elements of S are denoted by lowercase Latin variables $a, b, \ldots$ and elements of P by uppercase variables $A, B, \ldots$. The axiom system consists of the following axioms:

Axiom H1. If $\alpha, \beta, \gamma \in G$ then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$
Axiom H2. If $\alpha, \beta \in G$ and $\alpha^2 = \alpha$ then $\alpha \cdot \beta = \beta = \beta \cdot \alpha$
Axiom H3. If $\alpha \in G$ then there are a, b with $\alpha = a \cdot b$ or a, A with $\alpha = a \cdot A$.
Axiom H4. If $a \mid b$ then $a \cdot b \in P$.
Axiom H5. If $A \cdot B = B \cdot A$ then $A = B$.
Axiom H6. If $A, B \mid c, d$ then $A = B$ or $c = d$.
Axiom H7. If $a, b, c \mid e$ then $abc \in S$.
Axiom H8. If $a, b, c \mid E$ then $abc \in S$.
Axiom H9. If $\alpha, \beta, \gamma \,|\, \sigma$ and $\iota(\tau)$ and $\alpha, \beta, \gamma \nsim \tau$ then $\alpha = \beta$ or $\alpha = \gamma$ or $\beta = \gamma$.
Axiom H10. There exist A, b with $A \nmid b$.

Theorem 4.1. *The axioms H1–H10 axiomatize classical hyperbolic planes.*

Proof. We show that the axioms of a hyperbolic plane given in Sect. 2 are satisfied.

According to *Axiom H1* the operation $\cdot$ is a binary associative operation on G. Let $\alpha \in S$. Then $\alpha^4 = \alpha^2$ and (according to *Axiom H2*) α^2 is a neutral element which we denote by 1. This definition does not depend on the choice of α: If $\beta \in S$ then $(\alpha\alpha)\beta = \beta$ (since $\alpha\alpha$ is a neutral element) and $((\alpha\alpha)\beta)\beta = \beta\beta$ which implies $\alpha\alpha = \beta\beta$ (since $\beta\beta$ is a neutral element). We show that every element $\alpha \in G$ has an inverse element with respect to 1: If $\alpha = ab$ or $\alpha = aA$ (see *Axiom H3*) then $\alpha^{-1} = ba$ resp. $\alpha^{-1} = Aa$ since $ab \cdot ba = a(bb)a = aa = 1$ resp. $aA \cdot Aa = aa = 1$. This proves that $(G, \cdot)$ is a group with 1 as identity element. Hence the elements $\alpha \in G$ with $\iota(\alpha)$ are the involutions of G and according to (1) and (2) these are the elements of P and of S. According to *Axiom H3* the set of involutions generates G.

According to *Axiom H*4 involutory elements of S^2 are elements of P. Let E be an arbitrary element of P. According to *Axiom H3* there are a, b with $E = a \cdot b$ or there are a, A with $E = a \cdot A$. If $E = a \cdot A$ then $E \mid A$ which is a contradiction to *Axiom H5*. Hence there are a, b with $E = a \cdot b$ which shows that P is the set of involutions of S^2 and that the Basic Assumption of Sect. 2 holds.

We show that *Axiom A1* of Sect. 2 is satisfied. Let $A, B \in P$. According to (1) there is an element $e \in S$ with $A, B \mid e$ or an element $E \in P$ with $A, B \mid E$. Since there are no elements C, D with $C \mid D$ (according to *Axiom H5*) there is an element e with $A, B \mid e$ and *Axiom A1* holds.

The *Axioms A2, A3, A4* of Sect. 2 hold since the *Axioms H6, H7, H8* hold. *Axiom* $\neg V^*$ is satisfied since there exists an element of S (see *Axiom H10*).

Axiom H9 is a generalization of *Axiom H* (see Sect. 2), which was introduced by Struve and Struve [22] for a common characterization of hyperbolic and co-Minkowskian geometry. If $\alpha, \beta, \gamma, \sigma, \tau$ satisfy the assumptions of *Axiom H9* then $\alpha, \beta, \gamma, \tau \in S$ (since $\alpha, \beta, \gamma, \tau$ are involutions with $\alpha, \beta, \gamma \nsim \tau$) and $\alpha, \beta, \gamma \mid \sigma$. If $\sigma \in P$ *Axiom H9* is equivalent with *Axiom H*. If $\sigma \in S$ axiom H9 holds as well (see Struve and Struve [22]).

It remains to show that Axiom D of Sect. 2 holds. According to *Axiom H10* there exist A, b with $A \nmid b$ and according to *Axiom A1* there is an element $c \in S$ with $c \mid A, A^b$ and $b \mid c$ (because of *Axiom H6*). If d is an element with $d \nsim b$ (which exists according to (2)) then b, c, d show that Axiom D is satisfied.

The axiom system characterizes classical hyperbolic planes since according to (2) every line is element of an end. □

There is no general accepted definition for the simplicity of an axiom system (see Pambuccian [15]). However axiom system $\mathcal{H}$ uses the simplest possible language with only one sort of individual variables and only one (binary) operation. In this sense it is the simplest possible one.

Beside the many different ways to look at simplicity using syntactic criterions (cf. Pambuccian [16]) we want to emphasize a *semantic* one: Axiom system $\mathcal{H}$ allows the elementary definition of geometric relations such as incidence, order and congruence in a simple way: If the elements of S are called *lines* and the elements of P *points* (cf. Sect. 2) then a point A and a line b are *incident* if $A \mid b$; lines a, b are *orthogonal* if $a \mid b$; three lines a, b, c lie in a pencil if $\iota(abc)$; segments (A, B) and (C, D) (i.e. pairs of points) are *congruent* if there is an element $\alpha \in G$ with $\alpha^{-1}A\alpha = C$ and $\alpha^{-1}B\alpha = D$; angles $\measuredangle(a, b)$ and $\measuredangle(c, d)$

(i.e. pairs of lines) with vertices E resp. F are *congruent* if there is an element $\alpha \in G$ with $\alpha^{-1}E\alpha = F$ and $\alpha^{-1}a\alpha = c$ and $\alpha^{-1}b\alpha = d$.

If A, B, C are points on a line g then B lies between A and C if the condition of Theorem 2.11 holds. This condition is equivalent to the following one (since in a classical hyperbolic plane every line is element of an end): If a, b, c are lines through A, B, C respectively which are orthogonal to g then B lies between A and C if there is a point E on b with $EaE \not\sim c$.

Moreover, the axiom system $\mathcal{H}$ can be considered as an axiomatization of the group of motions of a (classical) hyperbolic plane with $\alpha \in G$ interpreted as a motion, the binary operation $\cdot$ as the composition of motions and $\alpha^{-1}A\alpha$ as the image of the point A under α.

This is remarkable since in a first-order language quantifiers can bind only individual variables, but not sets of individual variables which are normally used to define the concept of a rigid motion (as bijections of the set of points and of the set of lines). However, this does not mean that motions of a hyperbolic plane cannot be axiomatized by a first-order language—as axiom system $\mathcal{H}$ shows.

Remark Written in prenex normal form (a formula of first-order logic is in prenex normal form if it is written as a string of quantifiers followed by a quantifier-free part) axiom system $\mathcal{H}$ is a $\forall\exists\forall$-axiom system whereas an axiomatization in terms of incidence alone (as Menger's axiom system) must contain at least one $\forall\exists\forall\exists$-statement (as shown in Pambuccian [18]). This shows that axiom system $\mathcal{H}$ is simpler than Menger's axiom system as far as quantifier complexity is concerned.

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Appendix

The referee raised the question whether there does not exist a $\forall\exists$-axiom system in this language for this theory (the existence of an $\forall\exists$-axiom system for elementary hyperbolic geometry was shown by V. Pambuccian [17]).

To this end we introduce an axiom system $\mathcal{H}^*$ for classical hyperbolic planes with one sort of individual variables (elements $\alpha, \beta, \gamma, \ldots$ of a set G), one binary operation $\cdot$ on G and two constant symbols λ_A and λ_b. Axiom system $\mathcal{H}^*$ is a slightly modified version of axiom system $\mathcal{H}$: The sets P and S are defined as conjugate classes of G (see Bachmann [2, §15,2]) which can be represented by the elements λ_A and λ_b of G.

$\varepsilon(\alpha) \;\Leftrightarrow\; \alpha^2 = \alpha$ (to be interpreted as α is an *idempotent element*)

$\iota(\alpha) \quad \Leftrightarrow \quad \varepsilon(\alpha^2) \wedge \neg\, \varepsilon(\alpha)$ (to be interpreted as α is an *involution* of $(G, \cdot)$)

$\alpha \,|\beta \quad \Leftrightarrow \quad \iota(\alpha) \wedge \iota(\beta) \wedge \iota(\alpha \cdot \beta)$ (we write $\alpha, \beta \mid \gamma$ if $\alpha \mid \gamma \wedge \beta \mid \gamma$)

$\alpha \backsim \beta \quad \Leftrightarrow \quad \iota(\alpha) \wedge \iota(\beta) \wedge (\exists\gamma)\,(\iota(\gamma) \wedge \alpha, \beta \mid \gamma)$ (in the negated case we write $\nsim$)

$\alpha \approx \beta \quad \Leftrightarrow \quad (\exists\sigma)\,\sigma^{-1}\alpha\sigma = \beta$ (to be interpreted as α is *conjugate* to β)

We define two subsets P and S of G in the following way:

(1*) $\alpha \in P \quad \Leftrightarrow \quad \alpha \approx \lambda_A$
(2*) $\alpha \in S \quad \Leftrightarrow \quad \alpha \approx \lambda_b$

As in Sect. 2 elements of S are denoted by lowercase Latin variables $a, b, \ldots$ and elements of P by uppercase variables $A, B, \ldots$. The axiom system consists of the following axioms:

Axiom H1.* If $\alpha, \beta, \gamma \in G$ then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$
Axiom H2.* If $\alpha, \beta \in G$ and $\alpha^2 = \alpha$ then $\alpha \cdot \beta = \beta = \beta \cdot \alpha$
Axiom H3.* $\iota(\lambda_A)$ and $\iota(\lambda_b)$ and $\lambda_A \neq \lambda_b$ and $\lambda_A \nmid \lambda_b$.
Axiom H4.* If $\alpha \in G$ and $\iota(\alpha)$ then $\alpha \backsim \lambda_A$.
Axiom H5.* There exists $\alpha \in G$ with $\iota(\alpha)$ and $\alpha \nsim \lambda_b$.
Axiom H6.* If $\alpha \in G$ and $\iota(\alpha)$ then either $\alpha \approx \lambda_A$ or $\alpha \approx \lambda_g$.
Axiom H7.* If $\alpha \in G$ then there are a, b with $\alpha = a \cdot b$ or a, A with $\alpha = a \cdot A$.
Axiom H8.* If $a \mid b$ then $a \cdot b \in P$.
Axiom H9.* If $A \cdot B = B \cdot A$ then $A = B$.
Axiom H10.* If $A, B \mid c, d$ then $A = B$ or $c = d$.
Axiom H11.* If $a, b, c \mid e$ then $abc \in S$.
Axiom H12.* If $a, b, c \mid E$ then $abc \in S$.
Axiom H13.* If $\alpha, \beta, \gamma \,|\, \sigma$ and $\iota(\tau)$ and $\alpha, \beta, \gamma \nsim \tau$ then $\alpha = \beta$, $\alpha = \gamma$ or $\beta = \gamma$.

Theorem 4.2. *The axioms H1*–H13* axiomatize classical hyperbolic planes.*

Proof. $(G, \cdot)$ is a group and the elements $\alpha \in G$ with $\iota(\alpha)$ are the involutions of G (since the *Axioms H1*, H2** and *H7** hold; see the proof of Theorem 4.1).

Because of (1*) and (2*) it is $\lambda_A \in P$ and $\lambda_b \in S$. The elements of S and P are involutions (since P and S are conjugate classes and λ_A and λ_b are involutions; see *Axiom H3**) and generate G (according to *Axiom H7**).

According to the *Axioms H4*, H5** and *H6** the definitions (1*) and (2*) of S and P are equivalent to the corresponding definitions (1) and (2) of axiom system $\mathcal{H}$.

To finish the proof we have to show that the *Axioms H1–H10* of axiom system $\mathcal{H}$ hold. The *Axioms H1–H9* hold since the *Axioms H1*, H2*, H7*, H8*, H9*, H10*, H11*, H12*, H13** hold. *Axiom H10* is satisfied because *Axiom H3** holds. □

Written in prenex normal form axiom system $\mathcal{H}^*$ is a $\forall\exists$-axiom system.

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