J. Geom. 102 (2011), 81–84

0047-2468/11/010081-4
published online January 3, 2012
DOI 10.1007/s00022-011-0097-8

Journal of Geometry

Acute triangulation of a triangle in a general setting revisited

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Abstract. Axiom A16 from Pambuccian (Can. Math. Bull. 53, 534–541, 2010) is shown to be superfluous as it depends on axioms A1–A15. This provides a surprisingly simple axiom system in which the acute triangulation with seven triangles can be proved for any triangle, consisting only of A1–A15 in Pambuccian (Can. Math. Bull. 53, 534–541, 2010).

Mathematics Subject Classification (2010). 51G05, 51F20, 51F05.

Keywords. Acute triangulation, Axiom independence.

In [4] the second author had proved that any triangle can be triangulated into seven acute triangles and no less than seven will do based on an axiom system for ordered geometry to which certain very basic axioms for orthogonality were added. With one exception, they state, in essence, that line orthogonality is a symmetric relation, that one can drop and raise perpendiculars uniquely from a point (outside of and on a line respectively) to that line, and that a right angle cannot be contained within another right angle with the same vertex. One of them, that looked least like an axiom, A16, turns out to be superfluous, and it is the aim of this note to prove this fact. This leads to the surprising conclusion that the theorem regarding the acute triangulation with 7 triangles of any (obtuse) triangle can be proved inside an axiom system making only the very weak statements listed above.

The first-order language in which the axiom system was expressed in [4] contains one sort of individuals, to be interpreted as *points*, and two ternary predicates: B, with $B(abc)$ to be read as 'b lies between a and c', and $\perp$, with $\perp(abc)$ to be read as 'ab is orthogonal to ac' (or 'triangle abc has a right angle in a'). To improve the readability of the axioms, we introduced some abbreviations (defined notions): L, with $L(abc)$ to be read as 'a, b, and c are collinear points', Z, with $Z(abc)$ to be read as 'b lies strictly between a and c', ι, with

The work of the first author was supported by an NCUIRE Research Assistantship from Arizona State University, West campus.

$\iota_x(abc)$ to be read as 'x belongs to the interior of the angle formed by the rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$' (as it will be used only when the points a, b, and c are not collinear), and α, with $\alpha(abc)$ to be read as 'the angle $\widehat{bac}$ formed by the rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$ is acute'. For definitions and the axioms A1–A15 see [4]. The statements of the two lemmas cited in the text are Lemma 3.1: $\neg L(abc) \wedge \neg\alpha(bca) \rightarrow \alpha(cba)$ and Lemma 3.2: $\neg L(abc) \wedge \alpha(bca) \wedge \alpha(cba) \wedge \perp(uab) \wedge L(bcu) \rightarrow Z(buc)$.

Before turning to the proof of our main result, we shall prove the following

Lemma 1. $\perp(cab) \wedge Z(bcd) \wedge B(ade) \rightarrow \alpha(eba)$.

Proof. Let u be such that $\perp(dbu)$ (such a u exists by A9). We cannot have $\iota_u(dca)$, for else, by the axioms of ordered geometry (A1–A7), line du would have to intersect the open segment ac in some point p, from where there would be two perpendiculars to bd, namely pc and pd, contradicting A12. If $e = d$, we are done. If $Z(ade)$, then we must have $\iota_{u'}(dbe)$, for some u' with $L(duu')$ (see Fig. 1), so $\neg\alpha(dbe)$, and thus, by Lemma 3.1 from [4] applied to triangle deb, we must have $\alpha(ebd)$ and thus $\alpha(eba)$. □

Our aim is thus to prove the following

Theorem 2. *The statement (axiom A16 from [4])*

$$(\forall abcdd't)(\exists u)(\forall v)\ \perp(bac) \wedge \perp(cbd) \wedge Z(dcd') \wedge Z(atd') \wedge Z(btc) \rightarrow [Z(buc) \wedge (B(bvu) \rightarrow \alpha(vad))] \quad (0.1)$$

follows from A1–A15 in [4].

Proof. As in [4], we will denote, for x, y, z with $\neg L(xyz)$, the foot of the perpendicular from x to yz, by $F(yzx)$. Assume that a, b, c, and d are as in the hypothesis of (0.1). Let $r = F(bda)$. Notice that $r \neq b$, for if r were b, there would be two perpendiculars raised in b to ba, namely bd and bc, contradicting A14. Neither is it possible to have $Z(rbd)$, for if $Z(rbd)$ were the case, then, with e a point such that $\perp(bde)$ (such a point exists by A9), we have that, by A15, no point p with $L(bep)$ could be such that $\iota_p(bac)$, thus [by the axioms of ordered geometry (A1–A7)] line be must intersect the segment ar in some point q, so from q there would be two perpendiculars to line bd, namely qr and qb, contradicting A12. Thus there are only two possibilities for r (see Fig. 2): (i) $Z(brd)$ or (ii) $B(bdr)$.

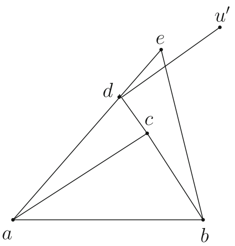

Figure 1 If the angle $\widehat{acb}$ is right, then $\widehat{aeb}$ is acute

Suppose (i) holds. Then there must be, by the Pasch axiom A6, a point p with $Z(arp) \wedge (Z(bpc) \vee Z(cpd) \vee p = c)$. If $Z(cpd) \vee p = c)$ holds, then we can take the u in (0.1) to be c, and (0.1) is satisfied by Lemma 1. If $Z(bpc)$ holds, then we can take the u in (0.1) to be p, and (0.1) is satisfied by Lemma 1.

Suppose (ii) holds. Then we can take the u in (0.1) to be c, and (0.1) is satisfied. To see this, notice that, if $r = d$, then applying Lemma 3.1 of [4] to triangle arv for any v with $B(bvc)$ we get that, since $\neg\alpha(rav)$, we must have $\alpha(var)$. If r is such that $Z(bdr)$, then, for any v with $B(bvc)$, ray $\overrightarrow{vd}$ must, by the axioms of ordered geometry (A1–A7), intersect the open segment ar in a point q. By Lemma 3.1 applied to triangle qrd, we must have $\alpha(qrd)$, and thus $\neg\alpha(qva)$, so, applying Lemma 3.1 to triangle qva, we get $\alpha(vaq)$, i. e. $\alpha(vad)$. $\square$

It is worth mentioning that a more general form of Lemma 3.3 from [4] receives a significantly shorter proof by using the above Lemma 1. In effect, one can prove that $\alpha(cab) \wedge B(bcd) \wedge Z(ade) \rightarrow \alpha(eab)$ by combining the fact that $\alpha(cab) \rightarrow (Z(bF(bca)c) \vee Z(aF(acb)c))$—which can be seen by noticing that if $B(cbF(bca))$ (which is the only alternative to $Z(bF(bca)c)$ since $B(bcF(bca))$ is excluded by the definition of $\alpha(cab)$ and by Lemma 3.1 from [4]), then we must have, by Lemma 3.1 from [4], $\alpha(acF(bca))$, and thus, given $B(cbF(bca))$, also $\alpha(abc)$, so, by Lemma 3.2 from [4], we must have $Z(aF(acb)c))$—with the above Lemma 1.

In Section 4 of [4, p. 540], while surveying the classical geometries that satisfy the axioms A1–A15, it was mentioned that all ordered metric planes and all ordered geometries with a Euclidean metric satisfy these axioms. The correct statement would have been that all ordered metric planes and all ordered geometries with a Euclidean metric *that satisfy A15* satisfy the axioms A1–A15. That ordered metric planes need not satisfy A15 (which is equivalent to the statement that in a right triangle the foot of the altitude to the hypotenuse lies between its endpoints), not even if the metric is Euclidean (i. e. if there is a rectangle in the plane), can be seen from the following example: the point-set of the model is $\mathbb{Q} \times \mathbb{Q}$, with the usual betweenness relation (i. e. point $\mathbf{c}$ lies between points $\mathbf{a}$ and $\mathbf{b}$ if and only if $\mathbf{c} = t\mathbf{a} + (1-t)\mathbf{b}$, with $0 < t < 1$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are in $\mathbb{Q} \times \mathbb{Q}$ and t is in $\mathbb{Q}$), and with segment congruence $\equiv$ given

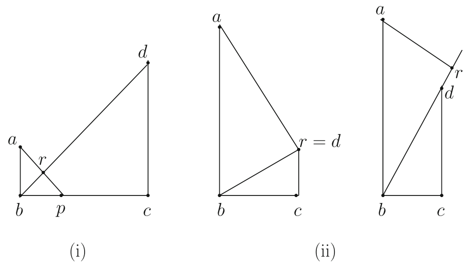

FIGURE 2 The various positions r could be in

by $\mathbf{ab} \equiv \mathbf{cd}$ if and only if $\|\mathbf{a} - \mathbf{b}\| = \|\mathbf{c} - \mathbf{d}\|$, where $\|\mathbf{x}\|$ stands for $x_1^2 - 2x_2^2$ where $\mathbf{x} = (x_1, x_2)$. Two lines $ux + vy + w = 0$ and $u'x + v'y + w = 0$ are orthogonal if and only if $-2uu' + vv' = 0$, and -2 is called the *orthogonality constant* of the Euclidean plane (see [1,2,5,6] and [3] for more on Euclidean planes). If $\mathbf{o} = (0, 1)$, $\mathbf{a} = (1, 0)$, and $\mathbf{b} = (2, 0)$, then the $\widehat{\mathbf{aob}}$ is a right angle, and the foot $\mathbf{c} = (0, 0)$ of the perpendicular from $\mathbf{o}$ to line $\mathbf{ab}$ does not lie between $\mathbf{a}$ and $\mathbf{b}$.

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Received: 3 May 2011.

Revised: 18 October 2011.