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On a measure of asymmetry for Reuleaux polygons

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Abstract. In a previous paper, we showed that for regular Reuleaux polygons R_n the equality $\mathrm{as}_\infty(R_n) = 1/(2\cos\frac{\pi}{2n} - 1)$ holds, where $\mathrm{as}_\infty(\cdot)$ denotes the Minkowski measure of asymmetry for convex bodies, and $\mathrm{as}_\infty(K) \leq \frac{1}{2}(\sqrt{3}+1)$ for all convex domains K of constant width, with equality holds iff K is a Reuleaux triangle. In this paper, we investigate the Minkowski measures of asymmetry among all Reuleaux polygons of order n and show that regular Reuleaux polygons of order n ($n \geq 3$ and odd) have the minimal Minkowski measure of asymmetry.

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1. Introduction

The asymmetry (or symmetry) of convex bodies has been studied by mathematicians for a long time, and several kinds of asymmetry measures are introduced and studied (see [5,6,8,11] and the references there). Among such subjects, one topic is to find the extremal bodies, i.e., the bodies with maximal or minimal value for a given asymmetry measure, within some class of convex bodies.

In 2-dimensional space $\mathbb{R}^2$, the convex domains of constant width, in particular the Reuleaux polygons, got much attention (see [1,3,4,7,10,12–14]). It was known that, for most kinds of measures of asymmetry, the Reuleaux triangle is the most asymmetric one among all domains of constant width. Recently, we proved that this is true also for the well-known Minkowski measure (see [10]). In this paper, we continue to investigate the asymmetry of Reuleaux polygons and prove that, for the Minkowski measure of asymmetry, the regular Reuleaux

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polygons of order n are the extremal bodies among all Reuleaux polygons of order n. Precisely, we get the following

Main Theorem. *If $n \geq 3$ is odd, then*

$$\mathrm{as}_\infty(RP_n) \geq \mathrm{as}_\infty(R_n)$$

where $RP_n(R_n)$ denotes the class of (regular) Reuleaux polygons of order n and $\mathrm{as}_\infty(\cdot)$ denotes the Minkowski measure of asymmetry for convex bodies.

Remark 1. All regular Reuleaux polygons of order n have clearly the same measure of asymmetry since they are affine equivalent.

2. Preliminary

$\mathbb{R}^n$ denotes the usual n-dimensional Euclidean space with the canonical inner product $\langle \cdot, \cdot \rangle$. A compact convex set $C \subset \mathbb{R}^n$ is called a *convex body* (*convex domain* for $n = 2$) if it has non-empty interior (int for brevity). The family of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$.

A convex body C is said to be of *constant width* if its width function, i.e., the support function of $C + (-C)$, is a constant (see [2,9]). Equivalently, one can show that C is of constant width iff each boundary point of C is incident with (at least) one diameter (= chord of maximal length) of C.

Assume that $C \in \mathcal{K}^n$ is of constant width and that $V \subset \mathrm{bd}(C)$, the boundary of C. The set V is called a *pinching set* if each diameter of C is incident with (at least) one point of V. A convex body C of constant width is called a *Reuleaux polygon* if it admits a finite pinching set.

Given a convex body $C \in \mathcal{K}^n$ and $x \in \mathrm{int}(C)$, for a hyperplane H through x and the pair H_1, H_2 of support hyperplanes of C parallel to H, let $r(H, x)$ be the ratio, not less than 1, in which H divides the distance between H_1 and H_2. Denote

$$r(C, x) = \max\{r(H, x) : H \ni x\}.$$

Then the *Minkowski measure* $\mathrm{as}_\infty(C)$ *of asymmetry* of C is defined as (see [5,6,8])

$$\mathrm{as}_\infty(C) = \min_{x \in \mathrm{int}(C)} r(C, x).$$

A point $x \in \mathrm{int}(C)$ satisfying $r(C, x) = \mathrm{as}_\infty(C)$ is called a *critical point* (of C). The set of all critical points of C is denoted by C^*.

The following is an equivalent definition of Minkowski measure (see [8,11]). Let $C \in \mathcal{K}^n$ and $x \in \mathrm{int}(C)$. For a chord L through x with endpoints p, q of C, let $r_1(L, x) = \max\{\frac{|xp|}{|xq|}, \frac{|xq|}{|xp|}\}$, where $|xp|$ denotes the length of xp, and denote

$$r_1(C, x) = \max\{r_1(L, x) : L \ni x\}.$$

Then the Minkowski measure is given by

$$\mathrm{as}_\infty(C) = \min_{x \in \mathrm{int}(C)} r_1(C, x).$$

It is known that if $x \in \mathrm{int}(C)$ is a critical point, then $r_1(C, x) = \mathrm{as}_\infty(C)$ (see [11]). A chord L satisfying $r_1(L, x) = \mathrm{as}_\infty(C)$ is called a *critical chord* (of C). Denote

$$S_C(x) = \left\{ p \in \mathrm{bd}(C) : x \in pq, \frac{|xp|}{|xq|} = r_1(C, x) \right\}.$$

3. Proof of Main Theorem

First we state some known results as lemmas here (see references for proofs).

Lemma 3.1. ([10]) *Let R_n be a regular Reuleaux polygon of order n. Then*

$$\mathrm{as}_\infty(R_n) = 1/\left(2\cos\frac{\pi}{2n} - 1\right).$$

Lemma 3.2. ([10]) *Let K be a convex domain of constant width. Then*

$$1 \le \mathrm{as}_\infty(K) \le \frac{1}{2}(\sqrt{3} + 1).$$

Moreover, equality holds on the left-hand side precisely iff K is a circular disc, and on the right-hand side precisely iff K is a Reuleaux triangle.

Lemma 3.3. ([11]) *Suppose $x \in \mathrm{ri}(C^*)$ (the relative interior) and $y \in S_C(x)$. Then*
$y + \frac{\mathrm{as}_\infty(C)+1}{\mathrm{as}_\infty(C)}(C^* - y) \subset \mathrm{bd}(C)$ *and $y \in S_C(p)$ for each $p \in C^*$.*

Remark 2. This lemma shows that the set $S_C(p)$ does not vary as p ranges over $\mathrm{ri}(C^*)$. We denote this set by $C^\dagger$.

Lemma 3.4. ([11]) *For $C \in \mathcal{K}^n, \mathrm{as}_\infty(C) + \dim C^* \le n$. If $\mathrm{as}_\infty(C) + \dim C^* \ge n - 1$, then $\mathrm{ri}(C^*) \subset \mathrm{int}(\mathrm{conv}(C^* \cup C^\dagger))$, where* dim *denotes dimension and* conv *denotes convex hull.*

Lemma 3.5. ([11]) *$C^\dagger$ contains at least $\mathrm{as}_\infty(C) + 1$ points.*

Next, we prove two propositions which will be needed in the proof of the Main Theorem.

Proposition 3.6. *If RP_n is a Reuleaux polygon of order n with vertices $e_1, \ldots, e_n$, (unique) critical point o and width ω, then there are three vertices e_{i1}, e_{i2}, e_{i3} such that $|oe_{i1}| = |oe_{i2}| = |oe_{i3}| = a$ and $\mathrm{as}_\infty(RP_n) = \frac{a}{\omega - a}$, where $a := \max_i\{|oe_i|\}$.*

Proof. By Lemma 3.5, there are three critical chords of RP_n (notice $\mathrm{as}_\infty(RP_n) > 1$), denoted by $p_i q_i, i = 1, 2, 3$, such that $\mathrm{as}_\infty(RP_n) = \frac{|op_i|}{|oq_i|}$. By Theorem 3.2 of [11], there are three pairs of line supporting RP_n at p_i, q_i respectively. Since

RP_n is of constant width, $|p_iq_i| = \omega$. Thus $\mathrm{as}_\infty(RP_n) = \frac{|op_i|}{\omega - |op_i|}$, which leads to $|op_i| = \frac{\mathrm{as}_\infty(RP_n)}{\mathrm{as}_\infty(RP_n)+1}\omega,\ i = 1, 2, 3$.

Now we shall show that each p_i is a vertex of RP_n and $|op_i| = \max_i\{|oe_i|\}$.

Let $x \in \mathrm{bd}(RP_n)$, and x' be the intersection of $\mathrm{bd}(RP_n)$ and the ray $\overrightarrow{xo}$. By the definition of $r_1(RP_n, o)$, $\frac{|ox|}{|ox'|} \leq \mathrm{as}_\infty(RP_n)$, which implies that $|ox| \leq \frac{as_\infty(RP_n)}{as_\infty(RP_n)+1}|xx'|$.
Notice that $|xx'| \leq \omega$ implies $|ox| \leq |op_i|$, so $RP_n \subset B$, where $B := B(o, |op_1|)$ is the disc with radius $|op_1|$ and centered at o.

If one of these points p_i, say p_1, is not a vertex, then q_1 must be a vertex by Theorem 3.2 of [11]. Let e_{j1}, e_{j2} be the vertices such that $|q_1e_{j1}| = |q_1e_{j2}| = |q_1p_1| = \omega$. Thus, if the set $B_1 = B(q_1, \omega)$, then $e_{j1}, e_{j2} \notin B$ and $B_1 \supset B \supset RP_n$, a contradiction. □

Lemma 3.7. *Under the constraint $s^2 + t^2 = d^2, s, t > 0$, where $d \geq 0$ is a constant, the function $f(s,t) := s^2 + t^2 - 2cst$, where $c < 0$ is a constant, attains its maximum $(1-c)d^2$ when $s = t = \frac{d}{\sqrt{2}}$. In particular, the maximum is increasing with respect to d.*

Proof. The proof is a routine calculation. □

Proposition 3.8. *Let $x, y, x', y' \in \mathbb{R}^n$, viewed as vectors. If $|x - y| = |x' - y'|, |x| = |y|$ and $|x| \geq |x'|, |y| \geq |y'|, \theta_{xy} > \frac{\pi}{2}$, then $\cos\theta_{xy} \geq \cos\theta_{x'y'}$ and in turn $\theta_{xy} \leq \theta_{x'y'}$, where θ_{xy} denotes the angle between (the vectors) x and y.*

Proof. It's clear that $\theta_{x'y'} > \frac{\pi}{2}$. By $|x - y| = |x' - y'|$, we get

$$\langle x - y, x - y\rangle = \langle x' - y', x' - y'\rangle.$$

Thus, if setting x'', y'' such that $|x''| = |y''|$ and $|x''|^2 + |y''|^2 = |x'|^2 + |y'|^2$, then by Lemma 3.7

$$\begin{aligned} |x|^2 + |y|^2 - 2|x||y|\cos\theta_{xy} &= |x'|^2 + |y'|^2 - 2|x'||y'|\cos\theta_{x'y'} \\ &\leq |x''|^2 + |y''|^2 - 2|x''||y''|\cos\theta_{x'y'} \leq |x|^2 + |y|^2 - 2|x||y|\cos\theta_{x'y'}, \end{aligned}$$

which leads clearly to $\cos\theta_{xy} \geq \cos\theta_{x'y'}$ and in turn $\theta_{xy} \leq \theta_{x'y'}$. □

Corollary 3.9. *Let $\triangle abc$ and $\triangle a'b'c'$ be triangles in $\mathbb{R}^2$. If $|ac| = |bc|, |ab| = |a'b'|, \angle c > \frac{\pi}{2}, \angle a = \angle b, |a'c'| \leq |ac|, |b'c'| \leq |bc|$, then $\angle a' + \angle b' \leq 2\angle a = 2\angle b$.*

Proof. Taking $x = a - c, y = b - c, x' = a' - c', y' = b' - c'$ in Proposition 3.8, we get the desired conclusion. □

Now we present the proof for our Main Theorem. Since all Reuleaux triangles are regular, we need only to prove it for $n \geq 5$.

Proof. of Main Theorem. For simplicity, we start with the Reuleaux polygons of order 5.

By the definition of Reuleaux polygons, $|e_1e_3| = |e_1e_4| = |e_2e_4| = |e_2e_5| = |e_3e_5| = \omega$ (see Fig. 1). By Proposition 3.6 and Lemma 3.4, we may assume $|oe_1| = |oe_3| = |oe_4| = a = \frac{\mathrm{as}_\infty(RP_5)}{\mathrm{as}_\infty(RP_5)+1}\omega$, where $a = \max_i\{|oe_i|\}$. Since

$\text{as}_\infty(RP_5) \leq \frac{\sqrt{3}+1}{2}$, we get $a = |oe_1| \leq \frac{\sqrt{3}}{3}\omega$ and $\angle e_1oe_4 > \frac{\pi}{2}$. Denote $\alpha := \angle oe_1e_4 = \angle oe_4e_1$, and denote by $\alpha_i(i = 1, 2..., 10)$ the ten base angles (e.g. $\angle oe_1e_3$) of these five triangles formed by taking the convex hull of $e_1e_3, e_1e_4, e_2e_4, e_2e_5$ or e_3e_5 in turn with the point o.

By Corollary 3.9, we have $\Sigma_{i=1}^{10}\alpha_i \leq 10\alpha$. Thus, since each $\angle e_i$ could be expressed as the sum or difference of two of these α_i's uniquely, we get $\pi = \Sigma_{i=1}^{5}\angle e_i \leq \Sigma_{i=1}^{10}\alpha_i \leq 10\alpha$, which leads to $\alpha \geq \frac{\pi}{10}$. Hence, by the fact that $\omega = 2|oe_1|\cos\alpha$ and Lemma 3.1,

$$\text{as}_\infty(RP_5) = \frac{|oe_1|}{\omega - |oe_1|} = \frac{1}{2\cos\alpha - 1} \geq \frac{1}{2\cos\frac{\pi}{10} - 1} = \text{as}_\infty(R_5).$$

As for general n, let $e_i, i = 1, ..., n$ be the vertices of RP_n, then there exist at least three vertices, denoted by e_{i1}, e_{i2}, e_{i3}, such that $|oe_{i1}| = |oe_{i2}| = |oe_{i3}| = a = \frac{\text{as}_\infty(RP_n)}{\text{as}_\infty(RP_n)+1}\omega$, where $a := \max_i\{|oe_i|\}$. We consider the triangle $\text{conv}\{x, y, z\}$, where $|xy| = |xz| = a, |yz| = \omega$, and denote $\angle xyz$ by α (see Fig. 1).

By the same argument as used above, we obtain that $\angle yxz > \frac{\pi}{2}$. Similarly, denoting by $\alpha_i(i = 1, 2..., n)$ the $2n$ base angles of these n triangles formed by taking in turn the convex hull of e_ie_j and o for these n chords with $|e_ie_j| = \omega$, then by applying Corollary 3.9 repeatedly, we have $\Sigma_{i=1}^{2n}\alpha_i \leq 2n\alpha$. Since in this case each $\angle e_i$ could also be expressed as the sum or difference of two of these α_i's uniquely, we have

$$\pi = \Sigma_{i=1}^{n}\angle e_i \leq \Sigma_{i=1}^{2n}\alpha_i \leq 2n\alpha,$$

which leads to $\alpha \geq \frac{\pi}{2n}$ and

$$\text{as}_\infty(RP_n) = \frac{a}{\omega - a} = \frac{1}{2\cos\alpha - 1} \geq \frac{1}{2\cos\frac{\pi}{2n} - 1} = \text{as}_\infty(R_n).$$

□

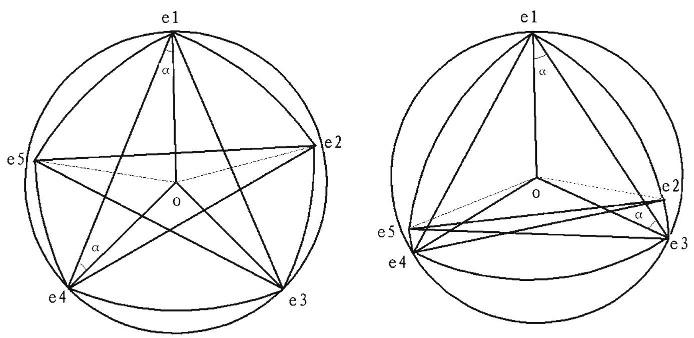

FIGURE 1

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