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Non-euclidean geometries: the Cayley-Klein approach

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Abstract. A. Cayley and F. Klein discovered in the nineteenth century that euclidean and non-euclidean geometries can be considered as mathematical structures living inside projective-metric spaces. They outlined this idea with respect to the real projective plane and established ("begründeten") in this way the hyperbolic and elliptic geometry. The generalization of this approach to projective spaces over arbitrary fields and of arbitrary dimensions requires two steps, the introduction of a metric in a pappian projective space and the definition of substructures as Cayley-Klein geometries. While the first step is taken in H. Struve and R. Struve (J Geom 81:155–167, 2004), the second step is made in this article. We show that the concept of a Cayley-Klein geometry leads to a unified description and classification of a wide range of non-euclidean geometries including the main geometries studied in the foundations of geometry by D. Hilbert, J. Hjelmslev, F. Bachmann, R. Lingenberg, H. Karzel et al.

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1. Introduction

In 1859 A. Cayley discovered that euclidean geometry can be considered as a special case of projective geometry[1] which led him to the famous statement that "descriptive geometry (his term for projective geometry) is *all* geometry" [9]. Ten years later F. Klein [22] took up the ideas of A. Cayley and showed that projective geometry as well provides a framework for the development of hyperbolic and elliptic geometry.[2] Thus from a projective point of view

[1] He introduced in the real projective plane an euclidean metric by specializing two imaginary points known as the circular points at infinity.

[2] F. Klein replaced the two imaginary circular points by a real and by an imaginary non-degenerated conic to get a model of the hyperbolic and elliptic plane, respectively.

the euclidean, the hyperbolic and the elliptic geometry are independent and self-subsistent geometries[3].

F. Klein restricted his investigations mainly to the real euclidean, hyperbolic and elliptic geometry since he was of the opinion that only these geometries can describe the physical universe [24, p. 211] and even today the term 'non-euclidean geometry' frequently denotes just hyperbolic geometry (cp. e.g. K. Borsuk and W. Smielew [8]) or hyperbolic and elliptic geometry (cp. e.g. H.S.M. Coxeter [10]).

On the other side, in the twentieth century further non-euclidean geometries were discovered, e.g. the minkowskian and galilean geometry, the spacetimes of de Sitter and Newton-Hooke or Einstein's cylinder universe (we refer to F. Bachmann [1,2], O. Giering [13], D.M.Y. Sommerville [38], H.S.M. Coxeter [11], K. Strubecker [40], W. Benz [4], W. Blaschke [6], J.A. Lester [28] and the Russian school of I.M. Yaglom and B.A. Rosenfeld [34,35,44,45]). However all of these works are of a very specialized character and the general concept of 'non-euclidean geometry'—based on the framework of A. Cayley and F. Klein—was not systematically explored and elaborated and remained "an abstract system with hardly any concrete content" (I.M. Yaglom [44, p. 50]).

It is the aim of this paper to elaborate the Cayley-Klein approach to non-euclidean geometry and to define a concept of 'Cayley-Klein geometries' in an algebraically elegant way which includes all of the geometries mentioned above. We thereby generalize the investigations of A. Cayley and F. Klein and study geometries of arbitrary (finite) dimension over arbitrary fields of characteristic $\neq 2$.

Following F. Klein this undertaking requires two steps, namely (1) the introduction of a metric in a pappian projective space and (2) the definition of substructures of 'proper' subspaces (often called 'Eigentlichkeitsbereich', see F. Klein [23], F. Bachmann [1], P. Klopsch [25], G. Hessenberg and J.Diller [15], H. Karzel and H.-J. Kroll [20]) which we call *Cayley-Klein geometries*, a term coined by I.M. Yaglom [44].

We start in Sect. 2 with step (1) and introduce a projective metric in a projective space. These *projective-metric spaces* can be described algebraically as finite dimensional vector spaces endowed with a sequence of forms, each defined on the radical of the preceding one (cp. H. Struve and R. Struve [42] and [43]).

We then proceed with step (2) and define Cayley-Klein geometries as substructures of projective-metric spaces. We distinguish between two kinds of Cayley-Klein geometries which we call ordinary and singular. Ordinary Cayley-Klein geometries are generalizations of F. Klein's famous model of the real hyperbolic plane. The classical example of a singular Cayley-Klein geometry is the

[3]F. Klein [23, p. 380] emphasizes this point. We refer in this context (including historical and philosophical issues) also to R. Bonola [7] and the often overlooked book on the foundations of geometry [36] of B. Russell (reprint with a foreword of M. Kline).

real euclidean plane, which can be defined as a substructure of the real projective plane where a projective metric is given by an elliptic polarity on the line ‘at infinity’.

In Sects. 3 and 4 we consider Cayley-Klein geometries over the field of real numbers which are of particular interest (see F. Klein [24] and D.M.Y. Sommerville [38]). We show that there are 3^n real n-dimensional Cayley-Klein geometries and 2^n ordinary ones (see Theorem 4.8 and 3.8). The dual of a Cayley-Klein geometry is a Cayley-Klein geometry (see Theorem 3.10 and 4.6). Thus the principle of duality, which is well known in lattice theory and in incidence geometry, can be extended to metric geometry. Moreover in a proper subspace of a Cayley-Klein geometry there exists one and only one projective reflection (see Theorem 3.4 and 4.4). Thus also in this general concept of non-euclidean geometry the calculus of reflections which was developed by J. Hjelmslev [17], D. Hilbert [16], F. Bachmann [1], R. Lingenberg [29] et.al. can be applied.

In Sect. 5 we generalize the real case and define Cayley-Klein geometries over ordered and halfordered fields. To indicate the extension of the concept of a Cayley-Klein geometry we show that all complete planes of absolute geometry (i.e. the euclidean, elliptic, hyperbolic and halfelliptic planes over arbitrary fields of characteristic $\neq 2$ which F. Bachmann studied in [1, section 18,3]) are Cayley-Klein geometries.

2. Projective spaces with Cayley-Klein metric

In this section we give an overview of projective spaces with Cayley-Klein metric (see H. Struve and R. Struve [42] and [43]).

Following D.R. Hughes and F.C. Piper [18] we consider ‘classical’ projective geometries which can be represented as the lattice $\mathcal{P}(V)$ of subspaces of a finite dimensional vector space V over a commutative field K of characteristic $\neq 2$ (i.e. pappian projective spaces). For the lattice-theoretic approach to projective geometry we refer to K. Menger [30], G. Birkhoff [5], R. Baer [3], D. R. Hughes and F. C. Piper [18] and C.-A. Faure and A. Frölicher [12].

The partial ordering of the lattice $\mathcal{P}(V)$ is denoted by $\leq$ and infimum and supremum by $\wedge$ and $\vee$, respectively. We denote the universal bounds by 0 (zero-subspace) and 1 (the entire space). The dimension $dim(\alpha)$ of a subspace α is the geometric dimension (if not noted otherwise) which is one lower than the algebraic dimension. The map dim from $\mathcal{P}(V)$ onto $\{-1, 0, 1, \ldots, n\}$ is called the *dimension function*.

We call the subspaces of dimension 0, 1, 2, $n-1$ points, lines, planes and hyperplanes respectively. If α, β are subspaces of V with $\alpha \leq \beta$ then the interval $[\alpha, \beta] = \{\gamma \in V : \alpha \leq \gamma \leq \beta\}$ is again a projective space.

To any projective space $\mathcal{P}(V)$ one can associate the dual projective space given by the dual (opposite) lattice $\mathcal{P}^*(V)$ which is defined as the same set with the opposite order $\alpha \leq_{op} \beta$ iff $\beta \leq \alpha$ and hence $\alpha \wedge_{op} \beta = \alpha \vee \beta$ and $\alpha \vee_{op} \beta = \alpha \wedge \beta$.

The dual projective space $\mathcal{P}^*(V)$ is isomorphic to $\mathcal{P}(V^*)$ where V^* denotes the dual vector space of V (the vector space of linear functionals on V) which is isomorphic to V. This establishes the well known result that for projective spaces the principle of duality holds (cp. D. R. Hughes and F. C. Piper [18, p. 78]).

Automorphisms (collineations) and anti-automorphisms (correlations) of $\mathcal{P}(V)$ are bijective mappings which preserve or reverse the partial ordering $\leq$ of the projective lattice $\mathcal{P}(V)$ (see D.R. Hughes and F.C. Piper [18, p. 21]). We mention that collineations fix 0 and 1 while correlations interchange 0 and 1. Automorphisms (anti-automorphisms) of $\mathcal{P}(V)$ which are induced by a linear mapping of V are called *projective collineations (correlations)*.

A projective correlation π of order 2 is called an *elliptic polarity* if it has no self-conjugate points (i.e. if α is of dimension 0, then $\alpha \wedge \alpha\pi = 0$). A projective correlation π of order 2 is called a *hyperbolic polarity*, if there are points α, β with $\alpha \wedge \alpha\pi = 0$ and $\beta \wedge \beta\pi \neq 0$. The polarity π can be described by a quadratic form q (or by an associated bilinear form f) which are non-degenerate, i.e. the null vector is the only element of the radical. Following O.T. O'Meara [31] we call (V, q) a *quadratic space*.

With these notations in mind we now define the concept of a projective space with a Cayley-Klein metric (cp. H. Struve and R. Struve [42]).

Definition 2.1. $\mathcal{CK} = (\mathcal{P}(V), (([\epsilon_0, \epsilon_1], \pi_1), \ldots, ([\epsilon_r, \epsilon_{r+1}], \pi_{r+1})))$ with $r \geq 0$ is a *projective space with Cayley-Klein metric* (or a *projective-metric space*) if the following assumptions hold:

(1) $\mathcal{P}(V)$ is a lattice of subspaces of a vector space V over a commutative field of characteristic $\neq 2$ (i.e. pappian projective space).
(2) $(\epsilon_0, \epsilon_1, \ldots, \epsilon_{r+1})$ is a flag, i.e. a chain of subspaces of V with $0 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_{r+1} = 1$.
(3) π_k (with $1 \leq k \leq r+1$) is a hyperbolic or an elliptic projective polarity on the interval $[\epsilon_{k-1}, \epsilon_k]$.

For a purely synthetic definition of projective-metric spaces we refer to H. Struve and R. Struve [43].

Isomorphisms of projective-metric spaces are defined in the obvious way (see H. Struve and R. Struve [42, definition 3.2]).

For notational convenience we sometimes denote a projective-metric space by $\mathcal{CK}(\epsilon_0, \ldots, \epsilon_{r+1})$ if the underlying polarities π_k are of no special concern.

The spaces $\mathcal{CK}(\epsilon_0, \epsilon_1)$ are called *ordinary*. Projective-metric spaces which are not ordinary are called *singular*.

A projective-metric space of dimension 0 is a projective point (a projective space with exactly two subspaces 0 and 1) with an elliptic polarity π which interchanges 0 and 1.

An ordinary projective-metric space $\mathcal{CK}$ of dimension 1 is a projective line with an elliptic or hyperbolic polarity. We call $\mathcal{CK}$ a projectice line *with an elliptic or hyperbolic metric*.

A singular projective-metric space $\mathcal{CK}(\epsilon_0, \epsilon_1, \epsilon_2)$ of dimension 1 is a projective line in which a projective-metric space $\mathcal{CK}(\epsilon_0, \epsilon_1)$ of dimension 0 is specialized. Following I.M. Yaglom [45] we call the metric of this projective-metric space *euclidean* (F. Klein [24] uses the term *parabolic*).

There are seven projective-metric spaces of dimension 2 and 18 of dimension 3. For a detailed classification see H. Struve and R. Struve [42].

Let $\mathcal{CK} = (\mathcal{P}(V), (([\epsilon_0, \epsilon_1], \pi_1), \ldots, ([\epsilon_r, \epsilon_{r+1}], \pi_{r+1})))$ be a projective-metric space and $\mathcal{P}^*(V)$ the dual projective space, i.e. the opposite lattice. Then $\mathcal{CK}^* = (\mathcal{P}^*(V), (([\epsilon_{r+1}, \epsilon_r], \pi_{r+1}), \ldots, ([\epsilon_1, \epsilon_0], \pi_1)))$ is again a projective-metric space which we call the *dual projective-metric space*. Thus the *principle of duality* can be extended from projective geometry to projective metric geometry. Obviously the following proposition holds:

Theorem 2.2. *Every ordinary projective-metric space $\mathcal{CK}(\epsilon_0, \epsilon_1)$ is self-dual. The polarity π is an anti-isomorphism from $\mathcal{CK}$ onto $\mathcal{CK}^* = (\mathcal{P}^*(V), ([1, 0], \pi))$.*

The well known pole-polar-theory of quadratic spaces can be extended to projective-metric spaces:

Definition 2.3. Let $\mathcal{CK} = (\mathcal{P}(V), (([\epsilon_0, \epsilon_1], \pi_1), \ldots, ([\epsilon_r, \epsilon_{r+1}], \pi_{r+1})))$ be a projective-metric space and φ_k the projection from $\mathcal{P}(V)$ into the interval $[\epsilon_{k-1}, \epsilon_k]$ with $\alpha\varphi_k = (\alpha \wedge \epsilon_k) \vee \epsilon_{k-1}$.

(1) β is called a *polar* of α if $\beta\varphi_k = (\alpha\varphi_k)\pi_k$ holds for $1 \leq k \leq r+1$.
(2) A subspace α is called *regular*[4] if α has one and only one polar.

With regard to the existence and uniqueness of a polar the following proposition holds (see [43, proposition 4.7 and 4.8]):

Theorem 2.4. *In a projective-metric space holds:*

(1) *Each subspace has at least one polar.*
(2) *α is regular if and only if α has a polar β with $\beta \in [\epsilon_{k-1}, \epsilon_k]$ for an integer k with $1 \leq k \leq r+1$.*

Remark. The universal bounds 0 and 1 are regular. 0 is the polar of 1 and vice versa.

Definition 2.5. Let β be a polar of α with $\alpha \wedge \beta = 0$. Then the harmonic homology with α and β as center and axis is called a *(projective) reflection* in α or in β, respectively. The identity is the projective reflection associated

[4] regular in the sense of Giering [13], not in the sense of the theory of metric vector spaces [1] or [31].

to the subspaces 0, 1 (cp. R. Lingenberg [29, p.195]). The set of all projective reflections generates the *group of motions*.[5]

Ordered projective-metric spaces

A projective space can be ordered or halfordered if the underlying field K is ordered or halfordered.

A *halforder* of K is a homomorphism *sgn* from the multiplicative group K^* of K into the cyclic group $(\{1, -1\}, \cdot)$ of order two (see E. Sperner [39]). The elements of $K^+ = \{x : sgn(x) = 1\}$ are called positive, the elements of $K^* \backslash K^+$ negative. A halforder of K is called *trivial*, if $K^+ = K^*$. Every field can be trivially halfordered. To every subgroup of index 2 of K^* there exists a non-trivial halforder and vice versa. If $a + b \in K^+$ for all $a, b \in K^+$ then K^+ is a positive domain and K an ordered field.

An order (or halforder) of K induces an order (or halforder) of the projective space $\mathcal{P}(V)$ (see H. Karzel and H.-J. Kroll [19] and H. Lenz [27]). Let A, B, C, D collinear points with $A, B \neq C, D$. We say that the point-pair (A, B) separates the point-pair (C, D) (which we denote by $AB//CD$) if the cross ratio of A, B, C, D is positive. Three collinear points A, B, C determine the segments $]A, B[^+ := \{X : AB//CX\}$ and $]A, B[^- := \{Y : notAB//CY\}$ such that the set of all points of the line joining A, B is the disjoint union of $]A, B[^+$ and $]A, B[^-$ and $\{A, B\}$. We refer with respect to the theory of ordered and half-ordered projective spaces to E. Sperner [39], H. Lenz [27] and H. Karzel and H.-J. Kroll [19].

From an algebraic point of view a projective-metric space is a vector space (over a commutative field of characteristic $\neq 2$) with a metric structure given by a sequence of forms $q_1, q_2, \ldots, q_{r+1}$ each defined on the radical of the preceding one. We say that a halforder of K is *compatible with the metric* of a projective-metric space if Sylvester's law of inertia holds for all associated forms $q_1, q_2, \ldots, q_{r+1}$. This is the case if K is ordered (see S. Lang [26]). However Sylvester's law does not necessarily hold if K is halfordered (see W. Pejas [33]). We define:

Definition 2.6. A projective-metric space $\mathcal{CK}$ is called *ordered*, if the underlying field is ordered. $\mathcal{CK}$ is called *halfordered*, if the underlying field is halfordered and if the halforder is compatible with the metric of $\mathcal{CK}$.

3. Real ordinary Cayley-Klein geometries

F. Klein defined his famous model of (plane) hyperbolic geometry as a substructure of the real projective plane $\mathbf{P^2}$ where a projective metric is given

[5] which may be—as in the euclidean case—a proper subgroup of the group of automorphisms of the projective-metric space (cp. F. Bachmann [1, section 5,5]).

by a hyperbolic polarity π. Points of the hyperbolic plane $\mathbf{H^2}$ are the points of $\mathbf{P^2}$ which are interior of the 'fundamental conic' of self-conjugate points of π. Lines of $\mathbf{H^2}$ are the lines of $\mathbf{P^2}$ which are incident with at least one interior point. The projective reflections in points and in lines of $\mathbf{H^2}$ generate the group of motions of $\mathbf{H^2}$ (which is in fact isomorphic to the full group of automorphisms of $\mathbf{P^2}$).

We generalize the approach of F. Klein and define in this section ordinary Cayley-Klein geometries as substructures (in the sense indicated above) of ordinary projective-metric spaces.

Let $(K, \leq)$ be the field of real numbers with the natural order, (V, q) the associated quadratic space and f the associated (non-degenerate) symmetric bilinear form.

A subspace T of V is called *positive definite* if $q(x) > 0$, *negative definite* if $q(x) < 0$ and *totally isotropic* if $q(x) = 0$ for every non-zero vector $x \in T$.

According to E. Witt all maximal totally isotropic subspaces of V have the same dimension which is called the *(Witt) index* of the quadratic space (V, q).

According to Sylvester's law of inertia all maximal positive definite subspaces of V have the same dimension, which is called the *signature* of (V, q) (the term 'signature' is used in the literature in different ways; we follow E. Snapper and R.J. Troyer [37]).

Since proportional forms λq with $\lambda \in K\backslash\{0\}$ induce the same polarity on V, we can assume that the dimension of maximal positive definite subspaces is not less than the dimension of maximal negative definite subspaces of V.

Let T be a subspace of V and let $T^\perp = \{v \in V : f(v, w) = 0 \text{ for all } w \in T\}$ be the orthogonal complement of T. Then T is called *non-degenerate* if $T \cap T^\perp = 0$, i.e. if the null vector is the only vector of T which is orthogonal to all vectors of T. Let T be a non-degenerate subspace of V and q^* the restriction of q to T. Then (T, q^*) is a quadratic space with an associated index and signature which we denote by $ind(T)$ and $sig(T)$, respectively.

If T is a non-degenerate subspace of V, which is generated by $v \in T$ with $v \neq 0$ then $sig(T) = 1$ or $sig(T) = 0$ according as $q(v) > 0$ or $q(v) < 0$. This example shows that if $T \neq V$ then $sig(T)$ may be less than the dimension of a maximal negative definite subspace of T.

Theorem 3.1. *Let T be a non-degenerate subspace of V. If $sig(V) = s$ and $sig(T) = m$ then $sig(T^\perp) = s - m$.*

Proof. Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal basis of T and $\{v_{k+1}, \ldots, v_n\}$ an orthogonal basis of $T^\perp$. Then $\{v_1, \ldots, v_n\}$ is a basis of V and according to our assumptions and Sylvester's law of inertia there are s vectors $v \in \{v_1, \ldots, v_n\}$ with $q(v) > 0$ and m vectors $w \in \{v_1, \ldots, v_k\}$ with $q(w) > 0$. Hence there are s - m elements $u \in \{v_{k+1}, \ldots, v_n\}$ with $q(u) > 0$ and $sig(T^\perp) = s - m$. □

Definition 3.2. Let $\mathcal{CK}$ be a real ordinary projective-metric space and (V, q) the associated quadratic space. A set $\mathbf{S}$ of non-degenerate subspaces is called a *set of proper subspaces*[6] if the following conditions hold:

(a) In $\mathbf{S}$ there are subspaces of all dimensions.
(b) Let $\alpha \in \mathbf{S}$ and $dim(\alpha) = dim(\beta)$. Then $\beta \in \mathbf{S}$ iff $sig(\alpha) = sig(\beta)$.
(c) Let $\alpha, \beta \in \mathbf{S}$ and $dim(\alpha) - dim(\beta) = 1$. Then $sig(\alpha) - sig(\beta) \in \{0, 1\}$.

The elements of $\mathbf{S}$ are called *proper subspaces* and the set of elements of $\mathbf{S}$ with $dim(\alpha) = d$ is denoted by $\mathbf{S}_d$.

According to the first condition 0 and 1 are elements of every set $\mathbf{S}$ of proper subspaces (as in every projective space). According to the second condition $\mathbf{S}$ contains with a subspace α all subspaces which have the same dimension and signature as α. In Klein's model of the hyperbolic plane these are the subspaces of dimension 0 and signature 0 (the points interior of the fundamental conic) and the subspaces of dimension 1 and signature 1 (the secants with respect to the fundamental conic). According to condition c) there are incident subspaces of arbitrary dimensions. For further examples we refer to the end of this section.

Obviously, all sets of proper subspaces can be constructed in a simple way:

Theorem 3.3. *Let $\mathcal{CK}$ be a real ordinary projective-metric space of dimension n and (V, q) the associated quadratic space. Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n+1} = 1$ be a flag of subspaces (a maximal flag). Then the set of subspaces β which have the same dimension and signature as one of the elements α_k (with $k \in \{0, \ldots, n+1\}$) is a set of proper subspaces.*

Theorem 3.4. *Let α be a proper subspace (i.e., an element of a set $\mathbf{S}$ of proper subspaces) of an ordinary projective-metric space. Then α is regular and in α there exists one and only one projective reflection.*

Proof. Let α be a proper subspace of an ordinary projective-metric space $([0, 1], \pi)$ and β the polar of α. Then $\beta \in [0, 1]$ and α is according to Theorem 2.4 regular. Since α is non-degenerate (according to Definition 3.2) $\alpha \wedge \beta = 0$, i.e. there exists one and only one projective reflection in α. □

To any set $\mathbf{S}$ of proper subspaces of a real ordinary projective-metric space $\mathcal{CK}$ we assign a geometric structure which we call a real ordinary Cayley-Klein geometry. Following D. Hilbert's axiomatization of the euclidean geometry [16] we define to this end the relations of incidence, order and congruence:

- A m-dimensional subspace of $\mathbf{S}$ is a m-dimensional subspace of the Cayley-Klein geometry.
- Subspaces $\alpha, \beta \in \mathbf{S}$ are incident iff $\alpha, \beta \in \mathcal{CK}$ are incident, i.e. the incidence relation is a restriction of the incidence relation of the projective-metric space.

[6] or 'Eigentlichkeitsbereich' (see e.g. F. Klein [23], F. Bachmann [1, section 20,10], G. Hessenberg and J.Diller [15], H. Karzel and H.-J. Kroll [20]).

- A pair (A, B) of points of $\mathbf{S}$ separates a pair (C, D) of collinear points of $\mathbf{S}$ with $A, B \neq C, D$ if (A, B) separates (C, D) in $\mathcal{CK}$, i.e. the relation of separation is a restriction of the corresponding relation of $\mathcal{CK}$. The relation of separation allows it to define further concepts (see E. Sperner [39], H. Lenz [27] and H. Karzel and H.-J. Kroll [19]). We mention: A *segment* $]A, B[$ of $\mathcal{CK}$ (see Sect. 2) is a segment of $\mathbf{S}$ if $]A, B[\subset \mathbf{S}$. If only one of the segments $]A, B[^+$ and $]A, B[^-$ is a segment of $\mathbf{S}$, then we say that the points of this segment lie *between* A and B. Two points A, B are on the same *side of a hyperplane* $\alpha \in \mathbf{S}$ iff there is a segment $]A, B[$ with $\alpha \cap\,]A, B[\, = \{\}$.
- The projective reflections in elements of $\mathbf{S}$ generate the group G of motions. Two sets $\mathcal{F}_1, \mathcal{F}_2$ of subspaces of $\mathbf{S}$ (two 'figures') are called *congruent* iff there is a motion which maps the elements of $\mathcal{F}_1$ onto the elements of $\mathcal{F}_2$.

Definition 3.5. Let $\mathcal{CK}$ be a real ordinary projective-metric space. A set $\mathbf{S}$ of proper subspaces of $\mathcal{CK}$ with the dimension function *dim* restricted to $\mathbf{S}$, the incidence relation restricted to the cartesian product $\mathbf{S} \times \mathbf{S}$, the relation of separation restricted to quadrupels of points of $\mathbf{S}$ and the relation of congruence restricted to $\mathbf{S}_0^2 \times \mathbf{S}_0^2$ (where $\mathbf{S}_0$ denotes the set of points of $\mathbf{S}$) is called *a real ordinary Cayley-Klein geometry.*

Isomorphisms of ordinary Cayley-Klein geometries are defined as for any other mathematical structures (cp. G. Graetzer [14]): Cayley-Klein geometries $(\mathbf{S}, dim, \mathsf{I}, //, \equiv)$ and $(\mathbf{S}^*, dim^*, \mathsf{I}^*, //^*, \equiv^*)$ are isomorphic, if a bijective map φ from $\mathbf{S}$ onto $\mathbf{S}^*$ exists with $dim(\alpha) = dim^*(\varphi(\alpha))$ for all $\alpha \in \mathbf{S}$ such that φ and φ^{-1} preserve the relations of incidence, separation and congruence.

Definition 3.6. The *type* of a real ordinary Cayley-Klein geometry of dimension n is the tupel $(s_0, s_1, \ldots, s_n)$ where s_k denotes the signature of the subspaces of dimension k.

Theorem 3.7. *A (n+1)-tupel* $(s_0, s_1, \ldots, s_n)$ *of natural numbers* ≥ 0 *is the type of a real ordinary Cayley-Klein geometry if and only if the following conditions hold:*

(a) $s_0 \geq 0$ *and* $s_n \geq (n+1)/2$
(b) $s_{i-1} = s_i$ *or* $s_{i-1} = s_i - 1$ *for* $1 \leq i \leq n$
(c) $i - (n - s_n) \leq s_i \leq i + 1$ *for* $0 \leq i \leq n.$

Proof. Let $(s_0, s_1, \ldots, s_n)$ be the type of a real ordinary Cayley-Klein geometry and T a subspace of V of (geometric) dimension i. Obviously $s_0 \geq 0$ and according to our assumption that the dimension of a maximal positive subspace of V is not less than the dimension of a maximal negative subspace of V it holds $s_n \geq (n+1)/2$.

From Definition 3.2, (c) we get $s_{i-1} = s_i$ or $s_{i-1} = s_i - 1$. Since $sig(T) \leq 1 + dim(T)$ it is $s_i \leq i+1$. Since the dimension of a maximal negative definite subspace of V is not less than the dimension of a maximal negative definite subspace of T one gets $(n+1)-sig(V) \geq (i+1)-sig(T)$ and $i-(n-s_n) \leq s_i$.

One easily verifies that if $\mathcal{CK}$ is a real ordinary projective-metric space and $(s_0, s_1, \ldots, s_n)$ a tupel which fulfills the conditions of Theorem 3.7 then there is an associated set of proper subspaces and an associated ordinary Cayley-Klein geometry. □

We now give a complete list of real ordinary Cayley-Klein geometries of dimension < 3. The Cayley-Klein geometry of dimension 0 is the elliptic point $\mathcal{CK}([0,1],\pi)$ where $[0,1]$ is a projective space of dimension 0 and π an elliptic polarity (which interchanges 0 and 1). The type of this Cayley-Klein geometry is (1).

There are two ordinary Cayley-Klein geometries of dimension 1, namely the elliptic line of type $(1,2)$ and the hyperbolic line of type $(0,1)$ (which is isomorphic to the Cayley-Klein geometry of type $(1,1)$).

Now let $\mathcal{CK}([0,1],\pi)$ be a real ordinary projective-metric space of dimension 2. If π is an elliptic polarity then $\mathcal{CK}([0,1],\pi)$ is an elliptic Cayley-Klein geometry. If π is a hyperbolic polarity then the set of points is the union of the set of isotropic points (which form a quadric Q) and the set of non-isotropic points with signature 0 or 1 which are—with reference to the quadric Q-internal and external points, respectively. The set of lines of $\mathcal{CK}([0,1],\pi)$ is the union of the set of tangents of Q (which are degenerate subspaces) and the set of non-tangential lines. Non-tangential lines have signature 1 or 2 and are incident with two or none of the points of Q. Thus we get three ordinary plane Cayley-Klein geometries of type $(0,1,2), (1,2,2)$ and $(1,1,2)$ which are called hyperbolic, cohyperbolic and double-hyperbolic, respectively (see I.M. Yaglom [44] and H. Struve and R. Struve [42]).

In all these cases the group of motions of the Cayley-Klein geometry is the full group of automorphisms of the associated projective-metric space.

In total we get $2^2 = 4$ ordinary Cayley-Klein planes. This result can be generalized:

Theorem 3.8. *The number of real ordinary Cayley-Klein geometries of dimension n is 2^n.*

Proof (by induction on the dimension n). Let $\mathcal{CK}([0,1],\pi)$ be a real n-dimensional projective-metric space and (V,q) the associated quadratic space.

Since we assume that the dimension of maximal positive definite subspaces of (V,q) is not less than the dimension of maximal negative definite subspaces of V we get for the signature s of the associated quadratic form $s \geq [n/2]$ with $[n/2] = n/2$ if n is even and $[n/2] = (n+1)/2$ if n is odd.

If $n = 0$ or $n = 1$ or $n = 2$ then the theorem holds, as we have shown.

Let us assume that the theorem holds for $n-1$. Let $A(m)$ be the number of m-dimensional ordinary Cayley-Klein geometries and $A(m,s)$ the number of Cayley-Klein geometries of this type where the associated quadratic form has the signature s. With these definitions we get $A(n) = A(n,n+1) + A(n,n) + \cdots + A(n,[n/2])$.

In a n-dimensional ordinary Cayley-Klein geometry with an associated quadratic form q of signature $s \neq n+1$ a hyperplane has the signature s or $s-1$. Hence $A(n,s) = A(n-1,s) + A(n-1,s-1)$ and $A(n) = A(n,n+1)$ $+\sum_{s=[n/2]}^{n} A(n-1,s) + \sum_{s=[n/2]}^{n} A(n-1,s-1)$. Since according to our assumption the proposition holds for $n-1$ we have $\sum_{s=[n/2]}^{n} A(n-1,s) = 2^{n-1}$ and $A(n) = A(n,n+1) + 2^{n-1} + (2^{n-1} - A(n-1,n))$.

Since there is one and only one n-dimensional Cayley-Klein geometry, if $q(x)$ has the signature $n+1$ (the elliptic one), we get $A(n,n+1) = 1$ and similar $A(n-1,n) = 1$. Hence $A(n) = 1 + 2^{n-1} + 2^{n-1} - 1 = 2^n$. □

Definition 3.6 is the key for a classification of n-dimensional ordinary Cayley-Klein geometries and for a study of their relationships. Well known geometries are:

- the Cayley-Klein geometry of type $(1,2,\ldots,n,n+1)$ is the *elliptic geometry*;
- the Cayley-Klein geometry of type $(0,1,2,\ldots,n-1,n)$ is the *hyperbolic geometry*;
- the Cayley-Klein geometry of type $(1,2,\ldots,n,n)$ is the *cohyperbolic geometry*;
- the geometry of type $(0,1,2,\ldots,n-1,n-1)$ is the *hyperbolic geometry of index* 2.

Duality

Let $\mathcal{CK} = (\mathcal{P}(V),([0,1],\pi))$ be a real ordinary projective-metric space. The polarity π is an anti-isomorphism from $\mathcal{CK}$ onto the dual projective-metric space $\mathcal{CK}^*$ (see Theorem 2.2). We show that π maps a set of proper subspaces of $\mathcal{CK}$ onto a set of proper subspaces of $\mathcal{CK}^*$.

Theorem 3.9. *Let $\mathcal{CK} = (\mathcal{P}(V),([0,1],\pi))$ be a real ordinary projective-metric space and $\mathbf{S}$ a set of proper subspaces of $\mathcal{CK}$. Then the set $\mathbf{S}^* = \{\alpha\pi : \alpha \in \mathbf{S}\}$ of polars of elements of $\mathbf{S}$ is a set of proper subspaces of $\mathcal{CK}^*$.*

Proof. Let $\mathcal{CK} = (\mathcal{P}(V),(([\epsilon_0,\epsilon_1],\pi))$ be a real n-dimensional projective-metric space. For $\alpha \in \mathbf{S}$ let $\alpha^* = \alpha\pi$ and $\mathbf{S}^* = \{\alpha^* : \alpha \in \mathbf{S}\}$. We show that $\mathbf{S}^*$ satisfies the conditions (a), (b), (c) of Definition 3.2.

The polarity π is an anti-automorphism of $\mathcal{CK}$ (see Theorem 2.2) such that for all $\alpha \in \mathbf{S}$ hold: (i) $dim(\alpha^*) = (n-1) - dim(\alpha)$; (ii) $sig(\alpha^*) = sig(V) - sig(\alpha)$.

To (a): Since there are subspaces of all dimensions in $\mathbf{S}$ there are according to (i) subspaces of all dimensions in $\mathbf{S}^*$.

To (b): Let $\alpha^* \in \mathbf{S}^*$ and $dim(\alpha^*) = dim(\beta^*)$. Then according to (i) $dim(\alpha) = dim(\beta)$ and as $\mathbf{S}$ is a set of proper subspaces: $\beta \in \mathbf{S}$ iff $sig(\alpha) = sig(\beta)$. Hence $\beta^* \in \mathbf{S}^*$ iff $sig(\alpha) = sig(\beta)$ which is according to (ii) equivalent with $sig(\alpha^*) = sig(\beta^*)$.

To (c): Let $\alpha^*, \beta^* \in \mathbf{S}^*$ and $dim(\alpha^*) - dim(\beta^*) = 1$. Then $\alpha, \beta \in \mathbf{S}$ and $dim(\alpha) = (n-1) - dim(\alpha^*)$ and $dim(\beta) = (n-1) - dim(\beta^*)$ and hence $dim(\beta) - dim(\alpha) = 1$. Since $\mathbf{S}$ is a set of proper subspaces we get $sig(\beta) - sig(\alpha) \in \{0, 1\}$ and with (ii) $sig(\alpha^*) - sig(\beta^*) \in \{0, 1\}$. □

Hence the dual of a set of proper subspaces of $\mathcal{CK}$ is a set of proper subspaces of $\mathcal{CK}^*$ and the dual of an ordinary Cayley-Klein geometry is the *dual ordinary Cayley-Klein geometry*. Hence the *principle of duality*— which is well known in incidence geometry—can be extended to metric geometry. An immediate consequence of Theorem 3.9 is:

Theorem 3.10. *The dual geometry of a real ordinary Cayley-Klein geometry of type* $(s_0, s_1, \ldots, s_n)$ *is the Cayley-Klein geometry of type* $(s_n - s_{n-1}, s_n - s_{n-2}, \ldots, s_n - s_0, s_n)$. *An ordinary Cayley-Klein geometry is* self-dual *if* $(s_0, s_1, \ldots, s_n) = (s_n - s_{n-1}, s_n - s_{n-2}, \ldots, s_n - s_0, s_n)$.

Examples. The dual geometry of n-dimensional hyperbolic geometry is n-dimensional cohyperbolic geometry. The elliptic geometry is self-dual as well as the double-hyperbolic plane of type $(1, 1, 2)$.

4. Real singular Cayley-Klein geometries

The classical example of a singular Cayley-Klein geometry is the real euclidean plane $\mathbf{E^2}$, which can be defined as a substructure of the real projective plane $\mathbf{P^2}$ where a projective metric is given by an elliptic polarity on the line ω "at infinity". Points of the euclidean plane $\mathbf{E^2}$ are the points of $\mathbf{P^2}$ which are not incident with ω and lines of $\mathbf{E^2}$ are the lines of $\mathbf{P^2}$ which are different from ω. The projective reflections in points and in lines of $\mathbf{E^2}$ generate the group of motions of $\mathbf{E^2}$.

We generalize this approach and define in this section singular Cayley-Klein geometries as substructures of singular projective-metric spaces. The geometric relations and morphisms of a Cayley-Klein geometry are restrictions of the corresponding relations and morphisms of the associated projective-metric space.

Definition 4.1. Let $\mathcal{CK} = (\mathcal{P}(V), (([\epsilon_0, \epsilon_1], \pi_1), \ldots, ([\epsilon_r, \epsilon_{r+1}], \pi_{r+1})))$ be a real singular projective-metric space and $\mathbf{S}_k$ a set of proper subspaces of an ordinary Cayley-Klein geometry defined on the intervall $([\epsilon_{k-1}, \epsilon_k], \pi_k)$ for $1 \leq k \leq r+1$.

Then the set $\mathbf{S}$ of subspaces α which have a polar $\beta \in \mathbf{S}_k$ for a number $k \in \{1, \ldots, r+1\}$ is called a *set of proper subspaces*. The elements of $\mathbf{S}$ are called *proper subspaces* and the set of elements of $\mathbf{S}$ with $dim(\alpha) = d$ is denoted by $\mathbf{S}_d$.

An immediate consequence of Theorems 3.3, 3.9 and Definition 4.1 is the following theorem, which shows that all sets of proper subspaces of a (singular or ordinary) projective-metric space can be constructed in a unified simple way:

Theorem 4.2. *Let $\mathcal{CK}$ be a real projective-metric space of dimension n. Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n+1} = 1$ be a flag of subspaces of V (a maximal flag) which contains $\epsilon_0, \epsilon_1, \ldots, \epsilon_{r+1}$ as elements. Then the set of subspaces α which have a polar β with the same dimension and signature as one of the elements α_k (with $k \in \{0, \ldots, n+1\}$) is a set of proper subspaces of $\mathcal{CK}$.*

Theorem 4.3. *A proper subspace of a singular projective-metric space is regular.*

Proof. Let α be a proper subspace of a singular projective-metric space. Then α has a polar $\beta \in \mathbf{S}_k$ for a number $k \in \{1, \ldots, r+1\}$ (see Definition 4.1) and hence $\beta \in [\epsilon_{k-1}, \epsilon_k]$. According to Theorem 2.4 α is regular. □

Theorem 4.4. *Let α be a proper subspace of a singular projective-metric space and β the polar of α. Then $\alpha \wedge \beta = 0$, i.e. there exists one and only one projective reflection in α.*

Proof. Let α be a proper subspace of a singular projective-metric space and β the polar of α. Then β is a subspace of an ordinary Cayley-Klein geometry defined on an interval $([\epsilon_{k-1}, \epsilon_k], \pi_k)$ or $\beta = \epsilon_k$ for a number $k \in \{1, \ldots, r+1\}$.

If $\beta = \epsilon_k$ then $\epsilon_i \leq \beta$ for $1 \leq i \leq k$ and $(\beta\varphi_i)\pi_i = \epsilon_i\pi_i = \epsilon_{i-1}$. Since β is a polar of α it is $\alpha\varphi_i = (\beta\varphi_i)\pi_i = \epsilon_{i-1}$ for $1 \leq i \leq k$ and hence $\alpha\varphi_1 = \alpha \wedge \epsilon_1 = 0$ and $\alpha \wedge \epsilon_2 = 0$ (since $\alpha\varphi_2 = (\alpha \wedge \epsilon_2) \vee \epsilon_1 = \epsilon_1$ and $\alpha \wedge \epsilon_1 = 0$) and by iteration one finally gets $\alpha \wedge \epsilon_k = \alpha \wedge \beta = 0$.

If β is a subspace of an ordinary Cayley-Klein geometry defined on an interval $([\epsilon_{k-1}, \epsilon_k], \pi_k)$ then β is non-degenerate (see Definition 3.2), i.e. $\beta \wedge \beta\pi_k = \epsilon_{k-1}$. Since $\beta \in [\epsilon_{k-1}, \epsilon_k]$ it is $\epsilon_i \leq \beta$ for $1 \leq i \leq k-1$ and hence $\alpha \wedge \epsilon_{k-1} = 0$ (as we have shown some lines above). Hence $\alpha \wedge \beta \leq \alpha \wedge \epsilon_k \leq \alpha\varphi_k = \beta\varphi_k\pi_k = \beta\pi_k$ and $\alpha \wedge \beta \leq \beta\pi_k \wedge \beta \leq \epsilon_{k-1}$. Since $\alpha \wedge \epsilon_{k-1} = 0$ it is $\alpha \wedge \beta = 0$. □

To any set $\mathbf{S}$ of proper subspaces of a real singular projective-metric space $\mathcal{CK}$ we assign a geometric structure (which we call a singular Cayley-Klein geometry) in the same way as in the ordinary case (see Sect. 3): A m-dimensional subspace of $\mathbf{S}$ is a m-dimensional subspace of the Cayley-Klein geometry. Subspaces $\alpha, \beta \in \mathbf{S}$ are *incident* iff $\alpha, \beta \in \mathcal{CK}$ are incident. The relation of separation defined on the set of points of $\mathcal{CK}$ induces a *separation relation* on the set of points of $\mathbf{S}$. The projective reflections in elements of $\mathbf{S}$ generate the group G of motions. Two sets $\mathbf{F}_1, \mathbf{F}_2$ of subspaces of $\mathbf{S}$ (two 'figures') are called *congruent* iff there is a motion which maps the elements of $\mathbf{F}_1$ onto the elements of $\mathbf{F}_2$.

Hence by interchanging the words 'ordinary' and 'singular' we define as in Definition 3.5 (without any further changes) *real singular Cayley-Klein geometries* and isomorphisms of singular Cayley-Klein geometries.

Definition 4.5. Let $\mathcal{CK}(\epsilon_0, \ldots, \epsilon_{r+1})$ be a singular projective-metric space and let $t_1, t_2, \ldots, t_{r+1}$ be the types of ordinary Cayley-Klein geometries defined on the intervals of $\mathcal{CK}$. Then $(t_1, t_2, \ldots, t_{r+1})$ is the *type* of the associated singular Cayley-Klein geometry.

Let $\mathcal{CK}(\epsilon_0, \ldots, \epsilon_{r+1})$ be a singular projective-metric space and $\mathbf{S}_k$ a set of proper subspaces of $([\varepsilon_{k-1}, \epsilon_k], \pi_k)$ for $k \in \{1, \ldots, r+1\}$. Since the dual of $\mathcal{CK}$ is a singular projective-metric space $\mathcal{CK}^*$ (see Sect. 2) and the dual of a set of proper subspaces of an interval $[\epsilon_{k-1}, \epsilon_k]$ is a set of proper subspaces of the corresponding interval of $\mathcal{CK}^*$, we get with Theorem 3.10:

Theorem 4.6. *The dual geometry of a real singular Cayley-Klein geometry of type* $(t_1, t_2, \ldots, t_{r+1})$ *is a Cayley-Klein geometry of type* $(t^*_{r+1}, t^*_r, \ldots, t^*_1)$ *where* t^*_k *denotes the type of the Cayley-Klein geometry which is dual to the Cayley-Klein geometry of type* t_k *(see Theorem 3.10).*

An immediate consequence is:

Theorem 4.7. *A real singular Cayley-Klein geometry is selfdual if* $(t_1, \ldots, t_{r+1}) = (t^*_{r+1}, \ldots, t^*_1)$.

We now give a complete list of real singular Cayley-Klein geometries of dimension < 3.

A singular projective-metric space $\mathcal{CK}(\epsilon_0, \epsilon_1, \epsilon_2)$ of dimension 1 is a projective line in which a projective-metric space of dimension 0 (an elliptic point) is specialized. The associated singular Cayley-Klein geometry is the euclidean line (one-dimensional euclidean geometry) of type $((1), (1))$.

A singular projective-metric space $\mathcal{CK}(\epsilon_0, \ldots, \epsilon_{r+1})$ of dimension 2 is composed of the ordinary projective-metric spaces $\mathcal{CK}(\epsilon_0, \epsilon_1)$, $\mathcal{CK}(\epsilon_1, \epsilon_2), \ldots, \mathcal{CK}(\epsilon_r, \epsilon_{r+1})$. The enumeration of Table 1 shows that there are five singular Cayley-Klein planes.

The dual geometry of n-dimensional euclidean geometry is n-dimensional coeuclidean geometry. The galilean plane geometry is self-dual.

Table 1 Singular Cayley-Klein planes

Geometry	$\mathcal{CK}(\epsilon_0, \epsilon_1)$	$\mathcal{CK}(\epsilon_1, \epsilon_2)$	$\mathcal{CK}(\epsilon_2, \epsilon_3)$
Euclidean	*elliptic line*	*elliptic point*	–
Minkowskian	*hyperbolic line*	*elliptic point*	–
Galilean	*elliptic point*	*elliptic point*	*elliptic point*
Coeuclidean	*elliptic point*	*elliptic line*	–
Cominkowskian	*elliptic point*	*hyperbolic line*	–

Theorem 4.8. *The number of real n-dimensional Cayley-Klein geometries is* 3^n *(for* $n \geq 1$*).*

Proof (by induction on the dimension n). Let $\mathcal{CK}(\epsilon_0, \ldots, \epsilon_{r+1})$ be a projective-metric space of dimension n.

If $n = 1$ then there are 3 Cayley-Klein geometries and if $n = 2$ there are $9 = 3^2$ Cayley-Klein geometries.

Suppose the proposition holds for $k < n$. Let $r \geq 1$ and $[\epsilon_r, \epsilon_{r+1}]$ an intervall of length k (with $k < n$). Then there exist 2^{k-1} ordinary Cayley-Klein geometries on $[\epsilon_r, \epsilon_{r+1}]$ (according to Theorem 3.8) and according to our assumption 3^{n-k} Cayley-Klein geometries on $\mathcal{CK}(\epsilon_0, \dots \epsilon_r)$, i.e. $2^{k-1} \cdot 3^{n-k}$ Cayley-Klein geometries on $\mathcal{CK}(\epsilon_0, \dots, \epsilon_{r+1})$. If $r = 0$ there are 2^n Cayley-Klein geometries. Thus we get in total $\sum_{k=1}^{n}(2^{k-1} \cdot 3^{n-k}) + 2^n = 3^n$ Cayley-Klein geometries. □

5. Cayley-Klein geometries over arbitrary fields

In this section we define the general concept of a Cayley-Klein geometry, which can be done in nearly the same way as in the real case (see Sects. 3 and 4). Let $\mathcal{CK}$ be an ordered or halfordered projective-metric space (see Definition 2.6). We define a set of proper subspaces of $\mathcal{CK}$ as in Definition 3.2 (for the ordinary case) and Definition 4.1 (for the singular case). All sets of proper subspaces of $\mathcal{CK}$ can hence be constructed in a unified simple way (cp. Theorem 4.2):

Theorem 5.1. *Let $\mathcal{CK}$ be a halfordered projective-metric space of dimension n. Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n+1} = 1$ be a flag of subspaces (a maximal flag) which contains $\epsilon_0, \epsilon_1, \dots, \epsilon_{r+1}$ as elements. Then the set of subspaces β which have the same dimension and signature as one of the elements α_k (with $k \in \{0, \dots, n+1\}$) is a set of proper subspaces.*

As in Sects. 3 and 4 we assign to any set $\mathbf{S}$ of proper subspaces of an ordered (or halfordered) projective-metric space $\mathcal{CK}$ a geometric structure which we call a Cayley-Klein geometry: A m-dimensional subspace of $\mathbf{S}$ is a m-dimensional subspace of the Cayley-Klein geometry. Subspaces $\alpha, \beta \in \mathbf{S}$ are *incident* iff $\alpha, \beta \in \mathcal{CK}$ are incident. The order (separation) relation defined on the set of points of $\mathcal{CK}$ induces an *order relation* on the set of points of $\mathbf{S}$. The projective reflections in elements of $\mathbf{S}$ generate the group G of *motions*. Two sets $\mathbf{F}_1, \mathbf{F}_2$ of subspaces of $\mathbf{S}$ (two 'figures') are called congruent iff there is a motion which maps the elements of $\mathbf{F}_1$ onto the elements of $\mathbf{F}_2$.

Definition 5.2. Let $\mathcal{CK}$ be a halfordered projective-metric space. A set $\mathbf{S}$ of proper subspaces of $\mathcal{CK}$ with the dimension function *dim* restricted to $\mathbf{S}$, the incidence relation restricted to $\mathbf{S} \times \mathbf{S}$, the relation of separation restricted to quadrupels of points of $\mathbf{S}$ and the relation of congruence restricted to $\mathbf{S}_0^2 \times \mathbf{S}_0^2$ (where $\mathbf{S}_0$ denotes the set of points of $\mathbf{S}$) is called *a Cayley-Klein geometry.*

The Cayley-Klein geometry is called ordered or halfordered if the associated projective-metric space is ordered or halfordered, respectively.

Remark. Definition 5.2 (and the corresponding definitions in Sects. 3 and 4) are algebraic definitions of Cayley-Klein geometries. The general concept of a Cayley-Klein geometry can be defined by developing an axiom system whose models are the algebraic structures defined in Definition 5.2 or in the following alternative way (which does not require an axiomatization of Cayley-Klein geometries which is an open problem and in no way a simple task):

Definition 5.3. A quintuple $(\mathbf{S}, d, I, //, \equiv)$ consisting of a set $\mathbf{S}$ of 'subspaces', a dimension function d (i.e. a map d from $\mathbf{S}$ into $\{-1, 0, 1, \ldots, n\}$), an incidence relation I defined on $\mathbf{S} \times \mathbf{S}$, a relation $//$ of separation defined on the set of quadrupels of points of $\mathbf{S}$, a relation $\equiv$ of congruence defined on $\mathbf{S}_0^2 \times \mathbf{S}_0^2$ (where $\mathbf{S}_0$ denotes the set of points of $\mathbf{S}$) is called a *Cayley-Klein geometry*, if it is isomorphic to one of the algebraic structures (models) defined in Definition 5.2 (the isomorphisms of mathematical structures of this type was defined in Sect. 3).

Over the field of real numbers (or more general over ordered fields with the property that every positive element has a square root) the signature of a vector space V determines the index since $dim(V) = ind(V) + sig(V)$ (cp. O. Giering [13, p.65]). Since this equation does not hold over arbitrary fields of characteristic $\neq 2$ we have to modify the definition of the type of a Cayley-Klein geometry:

Definition 5.4. The *type* of an ordinary Cayley-Klein geometry of dimension n is the tupel $< (s_0, i_0), (s_1, i_1), \ldots, (s_n, i_n) >$ where s_k denotes the signature and i_k the index of the subspaces of dimension k.

Definition 5.5. Let $\mathcal{CK}$ be a singular projective-metric space and $t_1, t_2, \ldots, t_{r+1}$ the types of ordinary Cayley-Klein geometries defined on the intervalls of $\mathcal{CK}$. Then $(t_1, t_2, \ldots, t_{r+1})$ is the type of the associated singular Cayley-Klein geometry.

We close this article with examples of Cayley-Klein geometries.

Example 5.1. *Real Cayley-Klein geometries* are ordered Cayley-Klein geometries.

Example 5.2. The *hyperbolic planes* of absolute geometry (see F. Bachmann [1, section 15]) are ordered Cayley-Klein planes.

Example 5.3. The *ordered euclidean planes* (see F. Bachmann [1, section 13]) are ordered Cayley-Klein planes.

Example 5.4. The *Hilbert planes* (which satisfy D. Hilberts [16] axioms of plane incidence, order and congruence) in which the circle axiom holds (see F. Bachmann [1, section 20,13] and W. Pejas [32]) are the Cayley-Klein planes of Examples 5.1 and 5.2 which have the further property of free mobility (any two points have a midpoint; two lines with a common point have a bisector).

Example 5.5. *Ordered Cayley-Klein geometries* are halfordered Cayley-Klein geometries.

Example 5.6. The *halfelliptic planes* of absolute geometry (see F. Bachmann [1, section 20,14]) are halfordered Cayley-Klein planes.

Example 5.7. Let (V, q) be a quadratic space over a field K of characteristic $\neq 2$ and the null vector the only isotropic vector of V. With respect to the trivial halforder of K the quadratic space is positive definite and Sylvester's law of inertia holds. The associated Cayley-Klein geometry is an *elliptic geometry*. The *elliptic planes* of absolute geometry are the elliptic Cayley-Klein geometries of dimension 2 (see F. Bachmann [1, section 9,6]).

Example 5.8. The n-dimensional euclidean geometry over a field of characteristic $\neq 2$ (see F. Bachmann [1, section 13 and section 20,9] and H. Kinder [21]) is the singular Cayley-Klein space which is associated to a projective-metric space $\mathcal{CK}(\epsilon_0, \epsilon_1, \epsilon_2)$ with an elliptic hyperplane ϵ_1.

Example 5.9. All *complete planes of absolute geometry* (i.e. the euclidean, elliptic, hyperbolic and halfelliptic planes over arbitrary fields of characteristic $\neq 2$; see F. Bachmann [1, section 18,3]) are according to Examples 5.2, 5.6, 5.7 and 5.8 Cayley-Klein geometries.

Example 5.10. The n-dimensional *galilean geometry* over a field of characteristic $\neq 2$ (see I.M. Yaglom, I. M. [44], H. Struve [41] and F. Bachmann [2, section 7.6]) is the singular Cayley-Klein geometry which is associated to a projective-metric space $\mathcal{CK}(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$ with an elliptic hyperline ϵ_1 and elliptic intervalls $[\epsilon_1, \epsilon_2]$ and $[\epsilon_2, \epsilon_3]$ of length 1.

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