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The necessary condition for the discrete L_0-Minkowski problem in $\mathbb{R}^2$

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Abstract. We prove that the sufficiency condition employed to show the existence and, in certain cases the uniqueness, of solutions to the discrete, planar L_0-Minkowski problem with data containing, at least, a pair of opposite vectors is also a necessary condition.

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The classical Minkowski problem deals with the existence, uniqueness, regularity and stability of closed, convex hypersurfaces whose Gauss curvature, viewed as a function on the unit sphere, is preassigned. For an atomic measure on the unit sphere, the question concerns the existence and uniqueness of polytopes with facets of fixed normal directions and fixed surface areas. In the planar setting, the Minkowski problem consists of a sufficient and necessary condition for the existence of a convex polygon whose sides have preassigned lengths and orientations:

Let $\mathcal{U} = \{\bar{u}_1, ..., \bar{u}_N\}$ be an ordered set of directions in $\mathbb{S}^1$, not all in a half-disk, and let $\mathcal{L} = \{l_1, ..., l_N\}$ be an ordered set of strictly positive numbers. There exists a convex N-gon whose i-th side has outer normal $\bar{u}_i$ and, respectively, length l_i if and only if $\sum_{i=1}^{N} l_i\bar{u}_i = \vec{0}_{\mathbb{R}^2}$.

This is the simplest and the trivial case of Minkowski's problem as the aforementioned condition represents simply the closure of a polygonal line with the desired properties. One should note that the convex polygon so obtained is unique up to translation. See [16] for a detailed discussion on the full extent of Minkowski's problem.

Due to Lutwak [10], a significantly more difficult question is whether a measure on the unit sphere $\mathbb{S}^n$ can be realized as the L_p-surface area measure of a convex body, where $p \neq n$ is some fixed real number. If so, is this body unique?
Lutwak showed within the Brunn-Minkowski-Firey theory that the classical problem, corresponding to $p = 1$, generalizes naturally to the L_p-Minkowski problem stated above. In [10] a solution to the *even* L_p-Minkowski problem in $\mathbb{R}^{n+1}$ was given for all $p \geq 1$ except for $p = n$ when it was shown that no solution is possible. The problem is called even if the measure takes equal values on opposite directions of $\mathbb{S}^n$. It is conjectured that in the even case the convex body, if it exists, is centrally symmetric, [10].

Since its birth a decade ago, the L_p-problem has developed rapidly and it connects to a number of other areas. For example, the solution to the even L_p-Minkowski problem was one of the critical ingredients needed to obtain sharp affine Sobolev inequalities [13], [12] and, for $p = 2$, it has implications to the Cramer-Rao inequality, one of the basic inequalities in information theory [14]. Recent progress has been made in a variety of cases depending on p and n with a plethora of methods mostly from PDE, [1], [2], [3], [4], [5], [6], [7], [8], [9], [11], [15], [18], [19], [20] to cite just a few. Yet, much of the problem still presents a real challenge when $p < 1$.

In the planar, discrete setting, the L_0-Minkowski problem reduces to the following statement:

Let $\mathcal{U} = \{\bar{u}_1, ..., \bar{u}_N\}$ be an ordered set of directions in $\mathbb{S}^1$, not all in a half-disk, and let $\Gamma = \{\gamma_1, ..., \gamma_N\}$ be an ordered set of strictly positive numbers. Does there exist a convex N-gon such that the side i has outer normal $\bar{u}_i$ and the triangle formed by this side and the origin has area γ_i? If so, is this polygon unique?

Despite their simple formulations, the existence and the uniqueness of the planar L_0-Minkowski problem, which are entwined with the asymptotic behavior of a semiflow on the space of N-gons with given outer normals to the sides, are not trivial, [18], [19].

A sufficiency condition was employed to conclude:

THEOREM 1. *[18], [19] Let $\mathcal{U} = \{\bar{u}_1, ..., \bar{u}_N\}$ be an ordered family of pairwise distinct unitary directions in $\mathbb{S}^1$ and let $\Gamma = \{\gamma_1, ..., \gamma_N\}$ be an ordered set of strictly positive values. Assume that one of the following holds:*

(i) *$N \geq 4$ and $\mathcal{U}$ consists of pairwise linearly independent vectors, not all in a half-disk.*
(ii) *$N > 4$ and $\mathcal{U}$ contains, at least, two linearly dependent vectors. For any j, k with $\bar{u}_j = -\bar{u}_k$, we have*

$$\gamma_j + \gamma_k < \sum_{i=1, i\neq j,k}^{N} \gamma_i. \tag{1}$$

(iii) *$N = 4$, $\mathcal{U}$ contains a unique pair of opposite vectors, $\bar{u}_1 = -\bar{u}_3$, and*

$$\gamma_1 + \gamma_3 < \gamma_2 + \gamma_4. \tag{2}$$

Then there exists a solution to the discrete planar L_0-Minkowski problem.

The equality in (1) is not sufficient unless $N = 4$ and $\mathcal{U}$ of the form $\{\bar{u}_1, \bar{u}_2, -\bar{u}_1, -\bar{u}_2\}$. In this case, expressing the area in two different ways, one notes that $\gamma_1 + \gamma_3 = \gamma_2 + \gamma_4$ is also a necessary condition. In fact, whether a condition of type (1) or (2) is also necessary was not known except for the even case:

THEOREM 2. *[18] The discrete planar L_0-Minkowski problem associated to the sets* $\mathcal{U} = \{\bar{u}_1, ..., \bar{u}_M, -\bar{u}_1, ..., -\bar{u}_M\}$ *and* $\Gamma = \{\gamma_1, ..., \gamma_M, \gamma_1, ..., \gamma_M\}$, $M > 2$, *has a solution if and only if*

$$\gamma_i < \sum_{j \neq i} \gamma_j, \quad \textit{for any } i = 1, ..., M. \tag{3}$$

Moreover, the solution is unique and it is symmetric with respect to the origin.

It is precisely the existence of a centrally symmetric solution which makes the strict inequality (1) necessary. This is an immediate consequence of the fact that each parallelogram formed by the vertices of parallel equal sides is included in the L_0-polygonal body with more than four sides. Thus the area of each such parallelogram which is equal to $4\gamma_i$ must be strictly less than the area of the entire body, $2\sum_{i=1}^{N} \gamma_i$. Theorem 2. answers also the polygonal case of Lutwak's conjecture which states that if an L_0-solution to even data exists, then it must be centrally symmetric. This was known for the L_0-polygons with 4 or 6 sides. The latter has been positively answered by A. Soranzo, [11], whose method is likely to hold also for $N = 8$, but it does not in the general case.

In this paper, we show that the condition (1) (or (2) if $N = 4$) is necessary for the existence of L_0-solutions.

THEOREM 3. *Let* $\mathcal{U} = \{\bar{u}_1, ..., \bar{u}_N\}$ *be an ordered family of pairwise distinct unitary directions in* $\mathbb{S}^1$ *and let* $\Gamma = \{\gamma_1, ..., \gamma_N\}$ *be an ordered set of strictly positive values,* $N > 4$.

If there exists an L_0-polygon associated to $(\mathcal{U}, \Gamma)$, *then for any pair* (i, j) *with* $\bar{u}_i = -\bar{u}_j$, *if such a pair exists, the inequality*

$$\gamma_j + \gamma_k < \sum_{i=1, i \neq j,k}^{N} \gamma_i$$

holds.

Proof. Let K be an L_0-polygon for $(\mathcal{U}, \Gamma)$ as above. Suppose that there exist $\bar{u}_i, \bar{u}_j \in \mathcal{U}$ such that $\bar{u}_i = -\bar{u}_j$.
Denote by l_i the length of the i-th side and by h_i the distance from the origin to that side. Note that for each $i = 1, ..., N$, one has $\frac{h_i l_i}{2} = \gamma_i$.

Furthermore denote by P_i, P_{i+1}, P_j, P_{j+1} the vertices of the sides of outer normals $\bar{u}_i$, respectively $\bar{u}_j$. As, at least, two of its sides are parallel, $P_i P_{i+1} P_j P_{j+1}$ is at least a trapezoid if not a parallelogram. We express its area in two ways:

$$\text{Area}(P_i P_{i+1} P_j P_{j+1}) = \frac{(h_i + h_j)(l_i + l_j)}{2} = \gamma_i + \gamma_j + a + b, \tag{4}$$

where $a = \text{Area}(P_i O P_{j+1})$, $b = \text{Area}(P_{i+1} O P_j)$, $\gamma_i = \text{Area}(P_i O P_{i+1})$, $\gamma_j = \text{Area}(P_j O P_{j+1})$, and O denotes the origin, which belongs to the interior of K by the definition of the L_0-solution.

Since $\gamma_i = h_i l_i/2$, $\gamma_j = h_j l_j/2$, denote by $\eta := l_j/l_i$ to rewrite the last equality of (4) in the form

$$\frac{1}{\eta}\gamma_j + \eta\gamma_i = a + b. \tag{5}$$

Similarly, if $\xi := h_j/h_i$, one has

$$\frac{1}{\xi}\gamma_j + \xi\gamma_i = a + b. \tag{6}$$

Thus η and ξ are the roots of the equation:

$$x^2\gamma_i - x(a+b) + \gamma_j = 0. \tag{7}$$

Therefore the equation has two real roots, possibly equal to each other,

$$x_{1,2} = \frac{(a+b) \pm \sqrt{\Delta}}{2\gamma_i} = \frac{2\gamma_j}{(a+b) \mp \sqrt{\Delta}}, \tag{8}$$

where $\Delta = (a+b)^2 - 4\gamma_i\gamma_j \geq 0$. Actually $(a+b)^2 - 4\gamma_i\gamma_j \geq 0$ implies already a necessary condition for the existence of K, namely

$$2\sqrt{\gamma_i\gamma_j} \leq \sum_{k \neq i,j} \gamma_k, \tag{9}$$

with equality if and only if $N = 4$. It is easy to see that if $N = 4$, the area of the trapezoid equals the area of the L_0-polygon.

Note however that this condition will be automatically satisfied if

$$\gamma_i + \gamma_j < \sum_{k=1, k \neq i,j}^{N} \gamma_k.$$

We will now show that $\gamma_i + \gamma_j \leq a + b$. We may assume, without any loss of generality, that $\gamma_j \leq \gamma_i$.

Case 1: $\gamma_i \leq (a+b)/2$. Since $\gamma_j \leq \gamma_i$, we have $\gamma_i + \gamma_j \leq a + b$.

Case 2: $\gamma_i > (a+b)/2$. As $\Delta = (a+b)^2 - 4\gamma_i\gamma_j \geq 0$, we also have $(a+b)/2 > \gamma_j$. Thus $\gamma_j < \dfrac{a+b}{2} < \gamma_i$. There exists a choice of origin, $O^* \in \text{Int}(P_i P_{i+1} P_j P_{j+1})$, such that $\text{Area}(P_i O^* P_{i+1}) =: \gamma_i^* = (a+b)/2 = \gamma_j^* := \text{Area}(P_j O^* P_{j+1})$. As the area of $P_iP_{i+1}P_jP_{j+1}$ did not change, one has for this particular choice of the origin, that $\text{Area}(P_iO^*P_{j+1}) + \text{Area}(P_{i+1}O^*P_j) =: a^* + b^* = \gamma_i + \gamma_j$. Using the same reasoning which led to equation (7), we deduce that η and $\xi^* = h_j^*/h_i^*$ are solutions of the equation

$$(a+b)x - 2(\gamma_i + \gamma_j) + (a+b)\frac{1}{x} = 0. \tag{10}$$

As a side remark, note that the existence of real roots implies

$$\Delta^* = 4\left((\gamma_i + \gamma_j)^2 - (a+b)^2\right) \geq 0, \tag{11}$$

or, $\gamma_i + \gamma_j \geq a + b$.

Since, by (8), η is one of the numbers $\dfrac{(a+b) \pm \sqrt{\Delta}}{2\gamma_i} = \dfrac{2\gamma_j}{(a+b) \mp \sqrt{\Delta}}$, by (10), either

$$(a+b)\frac{(a+b) - \sqrt{\Delta}}{2\gamma_i} - 2(\gamma_i + \gamma_j) + (a+b)\frac{(a+b) + \sqrt{\Delta}}{2\gamma_j} = 0 \tag{12}$$

or

$$(a+b)\frac{(a+b) + \sqrt{\Delta}}{2\gamma_i} - 2(\gamma_i + \gamma_j) + (a+b)\frac{(a+b) - \sqrt{\Delta}}{2\gamma_j} = 0 \tag{13}$$

holds.

The first equality leads to

$$\Delta(\gamma_i + \gamma_j) + (a+b)\sqrt{\Delta}(\gamma_i - \gamma_j) = 0, \tag{14}$$

while the second resumes to

$$\Delta(\gamma_i + \gamma_j) + (a+b)\sqrt{\Delta}(\gamma_j - \gamma_i) = 0. \tag{15}$$

As $\gamma_i > \gamma_j$, the first equality can hold if and only if $\Delta = 0$, otherwise both terms are strictly positive. If this happens, $\eta = \xi = \sqrt{\gamma_j/\gamma_i}$. On the other hand, $\Delta = 0$ is equivalent to $a + b = 2\sqrt{\gamma_i\gamma_j}$, so the roots of the equation (10) are distinct $\sqrt{\gamma_i/\gamma_j}$, $\sqrt{\gamma_j/\gamma_i}$. Therefore $\Delta^* > 0$ and $\gamma_i + \gamma_j > a + b$. However, since η remains the same under different choices of origin, we may refine the previous procedure as follows. Consider origins $0(\lambda)$ such that $\text{Area}(P_iO(\lambda)P_{i+1}) =: \gamma_i(\lambda) = \lambda(a+b)$ and $\text{Area}(P_jO(\lambda)P_{j+1}) =: \gamma_j(\lambda) = (1-\lambda)(a+b)$, where λ belongs to some open interval containing $1/2$. Note that for this choice $\gamma_i(\lambda) + \gamma_j(\lambda) = a + b$ and $\text{Area}(P_iO(\lambda)P_{j+1}) + \text{Area}(P_jO(\lambda)P_{i+1}) =: a(\lambda) + b(\lambda) = \gamma_i + \gamma_j$. Thus η is also a solution of the equation

$$\lambda(a+b)x - (\gamma_i + \gamma_j) + (1-\lambda)(a+b)\frac{1}{x} = 0. \tag{16}$$

In the subcase considered here, recall that $a+b = 2\sqrt{\gamma_i\gamma_j}$ and $\eta = \sqrt{\gamma_j/\gamma_i}$. These reduce equation (16) to $(2\lambda - 1)(\gamma_i - \gamma_j) = 0$. As λ can be different than 1/2, we infer that $\gamma_i = \gamma_j$ contradicting the assumption of the case.

Thus $\Delta \neq 0$, and (15) must hold. We divide it by $\sqrt{\Delta}$ and re-write as

$$\sqrt{\Delta}(\gamma_i + \gamma_j) = (a+b)(\gamma_i - \gamma_j). \tag{17}$$

Squaring both sides we obtain, after simplification, $\gamma_i + \gamma_j = a + b$.

In conclusion, for all $N \geq 4$, the existence of an L_0-solution implies $\gamma_i + \gamma_j \leq a + b$. However, if $N > 4$, due to the convexity of K, the vertices $P_k,\ k \neq i,\ i+1,\ j,\ j+1$, belong to the exterior of $P_i P_{i+1} P_j P_{j+1}$. Thus $a + b < \sum_{k \neq i,j} \gamma_k$, or

$$\gamma_i + \gamma_j < \sum_{k \neq i,j} \gamma_k, \tag{18}$$

which proves that, if $N > 4$, our sufficiency condition is also necessary. □

REMARK. For simplicity, we have not included in Theorem 3. the case $N = 4$ which will address here. Following the same reasoning as above, it suffices to consider the equality case when $N = 4$. Suppose $\gamma_1 + \gamma_3 = \gamma_2 + \gamma_4 =: a + b$, and suppose that the sides 1 and 3 are parallel. Assume, without any loss of generality, that $\gamma_3 \leq \gamma_1$. Then $\{\eta := l_3/l_1,\ \xi := h_3/h_1\} = \{1, \gamma_3/\gamma_1\}$.

If $\eta = 1$, $l_1 = l_3$ and K is a parallelogram. In this case, it was already known that $\gamma_1 + \gamma_3 = \gamma_2 + \gamma_4$ is a necessary and sufficient condition for the existence of an L_0-solution. Same conclusion if $\gamma_1 = \gamma_3$, thus we may assume $\gamma_3 < \gamma_1$.

Therefore $\gamma_3 < \dfrac{\gamma_2 + \gamma_4}{2} < \gamma_1$. Choosing the origin O^* as before, (10) becomes $x^2 - 2x + 1 = 0$ as $\gamma_1 + \gamma_3 = \gamma_2 + \gamma_4 =: a + b$. Thus $\eta = 1$, contradicting $\gamma_1 \neq \gamma_3$.

We may conclude that, if $N = 4$, the equality case occurs if and only if the L_0-solution is a parallelogram. Otherwise, $\gamma_1 + \gamma_3 < \gamma_2 + \gamma_4$ which is the sufficient condition employed in the case of four directions with exactly one pair of opposite normal vectors. □

We end by noting the difference between the discrete planar case of $p = 0$ and the smooth planar L_0-case where neither a necessary nor a sufficient condition is related to the existence of solutions, [5]. The uniqueness was established only for the even problem, [6]. In fact, even more interesting is the fact that for any $p \geq 1$, and any $n \geq 1$, the discrete L_p-Minkowski problem has a unique solution independent of the structure of the sets $\mathcal{U} \subset \mathbb{S}^n$ and Γ, [9], where γ_i corresponds now to the surface area of the i-th facet of outer normal $\bar{u}_i$. We believe that the higher dimensional discrete case of the L_0-Minkowski problem will also require a sufficient and necessary condition and we would like to investigate this issue in a further paper.

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