J. Geom. 87 (2007) 50 – 54
0047–2468/07/020050 – 5

DOI 10.1007/s00022-007-1805-2

Journal of Geometry

On projective spaces $PG(r, q)$ with $r \geq 4$

Nicola Durante, Vito Napolitano and Domenico Olanda*

Abstract. In this paper a characterization of $PG(r, q)$, $r \geq 4$, in terms of planar spaces is given.

Mathematics Subject Classification (2000): 51Exx.
Key words: Planar space, projective space.

1. Introduction

In a recent paper [3] finite planar spaces $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ with no disjoint planes and (k, q)-regular, that is such that any line has cardinality $k + 1$ and any pencil of lines has cardinality $q + 1$, have been studied.

Obviously in such a planar space, besides lines, also planes all have the same cardinality and hence through every line there is a constant number, say $n + 1$, of planes. Under these hypotheses, in [3] the authors prove that $|\mathcal{L}| \geq |\mathcal{P}|$ and equality holds if and only if $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is $PG(4, q)$.

Hence in [3] the authors obtain a characterization of $PG(4, q)$ as a (k, q)-regular planar space.

Now, in order to get a characterization of $PG(r, q)$, $r \geq 4$, we again investigate finite planar spaces $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ without assuming that lines and planes all have the same cardinality but supposing only that:

(i) *Every pencil of lines has size* $q + 1$.
(ii) *Every pencil of planes has size* $n + 1$.

Such a planar space is said *regular with parameters* q *and* n.

Recall that if the planes of the planar space $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ pairwise intersect in the empty set or in a line, then $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is said a *locally projective threedimensional space*.

The following remarks are useful to illustrate the ideas behind this paper.

*This research was carried out within the activity of INdAM-GNSAGA and supported by the Italian Ministry M.I.U.R.

The projective spaces $PG(r, q)$, $r \geq 3$, are examples of regular planar spaces with parameters q and n (with $n = q^{r-2} + q^{r-3} + \ldots + q$). Moreover if $r \geq 4$, then in the projective space there are lines and planes disjoint from each other and for such pairs the following property holds.

(iii) *For any given line ℓ and plane π disjoint from each other, the number* $\mathbf{s} = \chi(\ell, \pi)$ *of planes through ℓ disjoint from π is a constant.*

Indeed it is

$$\mathbf{s} = \chi(\ell, \pi) = (q^{r-2} + q^{r-3} + \ldots + q + 1) - (q^2 + q + 1).$$

The integer $\mathbf{s}$ is obtained subtracting the number of planes through ℓ meeting π from the total number of planes through ℓ.

Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be the planar space obtained from a projective space $PG(r, q)$, $r \geq 4$, by deleting a point p_0. Then $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is again a regular planar space with the same parameters q and n as $PG(r, q)$ but Property (iii) does not hold. Indeed, in such a planar space, if ℓ is a line of length $q + 1$, then either

$$\chi(\ell, \pi) = (q^{r-2} + q^{r-3} + \ldots + q + 1) - (q^2 + q + 1)$$

or

$$\chi(\ell, \pi) = [(q^{r-2} + q^{r-3} + \ldots + q + 1) - (q^2 + q + 1)] + \mathbf{1}$$

according to $|\pi| = q^2 + q + 1$ or $|\pi| = q^2 + q$.

Above observations show that if a planar space satisfies Property (iii), then it cannot be obtained from a projective space $PG(r, q)$ by deleting "some" points. Hence if it is embeddable in $PG(r, q)$, then it is $PG(r, q)$.

In this paper, as a verification of this idea, we obtain the following result.

THEOREM I. *Let $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ be a finite regular planar space with parameters q and n satisfying Property* (iii)*. Then it is $n \geq q$. If $n = q$, then $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a locally projective threedimensional space. If $n > q$, then $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is $PG(r, q)$, $r \geq 4$, if and only if it contains a projective line, that is a line of maximal length $q + 1$.*

Hence in this paper we obtain a characterization of $PG(r, q)$, $r \geq 4$, in terms of regular planar spaces with parameters q and n.

2. Regular finite planar spaces with parameters q and n

In this section $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ will denote a finite planar space. If π is a plane of the planar space and p is a point of π, the set of lines of π through p is called *pencil of lines* with center p. If ℓ is a line, the set of planes containing ℓ is called *pencil of planes* with axis ℓ.

From now on, we suppose that $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ fulfils the following properties of regularity:

(i) *Every pencil of lines has size* $q + 1$.
(ii) *Every pencil of planes has size* $n + 1$.

Such a planar space is called, as previously said, a *regular planar space with parameters* q *and* n.

Next we investigate some properties of finite regular planar spaces with parameters q and n.

From Property (i) the following result follows easily.

PROPOSITION 2.1. $|\ell| \leq q + 1$ *for every line* ℓ, *and hence* $|\pi| \leq q^2 + q + 1$ *for any plane* π.

A line of maximal size $q + 1$ is called *a projective line*.

PROPOSITION 2.2. *Let* $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ *be a finite regular planar space with parameters* q *and* n. *Then* $n \geq q$, *and equality holds if, and only if, any two planes meet in the empty set or in a line.*

Proof. Let π be a plane, and let p be a point of π and r be a line through p which is not contained in π.

Let $\ell_0, \ell_1, \ldots, \ell_q$ be the $q + 1$ lines of π through p. Since through the line r there are the $q+1$ planes spanned by r and by the lines $\ell_0, \ell_1, \ldots, \ell_q$, respectively, we have $n+1 \geq q+1$, and the assertion follows easily. □

Next we prove the following properties.

PROPOSITION 2.3. *Through any point* p *there is a constant number* θ *of lines.*

Proof. Let p be a point and r be a line through p. The lines through p, different from r, are partitioned on the $n + 1$ planes through r, and so the number θ of lines through p is:

$$\theta = (n + 1)q + 1. \tag{2.1}$$

PROPOSITION 2.4. *Through any point* p *there is a constant number* η *of planes.*

Proof. Let p be a point. Counting in two ways the line-plane pairs (ℓ, π) both passing through p and with ℓ contained in π gives $\theta(n + 1) = \eta(q + 1)$. It follows that

$$\eta = \frac{\theta(n + 1)}{q + 1} = \frac{((n + 1)q + 1)(n + 1)}{q + 1}. \tag{2.2}$$

□

Notice that, as already seen in Proposition 2.2, if $n = q$, then $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a locally projective threedimensional planar space, and we refer the reader to the literature on this topic [1],

[2], [4], [5], [7], [8]. Thus, from now on we may suppose that

$$\mathbf{n} > \mathbf{q}. \tag{2.3}$$

Under this assumption there are pairs of planes intersecting each other just in a point. Moreover the following result holds.

PROPOSITION 2.5. *Let π be a plane and p be a point of π. The number k of planes through p and intersecting π exactly in p is a positive number and it is independent of p and π.*

Proof. The planes through p different from π and intersecting π in a line are, by using (i) and (ii), $(q + 1)n$, and so

$$k = \eta - 1 - (q + 1)n. \tag{2.4}$$

□

Let π and π' be a pair of planes intersecting each other just in a point p. Then every line contained in π' and not containing p is disjoint from the plane π. Hence there are disjoint line-plane pairs (ℓ, π). For such pairs we assume that the following property holds.

(iii) *For any given line ℓ and plane π disjoint from each other, the number $\mathbf{s} = \chi(\ell, \pi)$ of planes through ℓ disjoint from π is a constant.*

In the next section we will prove that if $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is a regular planar space with parameters q and n with $n > q$, satisfying Property (iii) and containing a *projective* line, then the space is $PG(r, q)$ with $r \geq 4$ and hence Theorem I will be completely proved.

3. The characterization theorem. The proof.

In this section $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ will denote a finite regular planar space with parameters q and n with $n > q$ and satisfying Property (iii).

We will prove Theorem I by showing that if the planar space contains a projective line, that is a line of length $q + 1$, then it is $PG(r, q)$, with $r \geq 4$.

We will prove the following proposition.

PROPOSITION 3.1. *If there is a projective line, then $(\mathcal{S}, \mathcal{L}, \mathcal{P})$ is $PG(r, q)$ with $r \geq 4$.*

Proof. Let L be a projective line and let ℓ be a line disjoint with L. The line ℓ is skew with L and since $n > q$ there exists at least a plane π through ℓ disjoint with L.

Since the line L has length $q + 1$, by (i), it meets every coplanar line, and so each plane through L meeting π, intersects π just in a point. It follows that there are as many planes through L meeting π as the points of π, hence $|\pi| = n + 1 - \mathbf{s}$.

We will now prove that ℓ has size $q + 1$.

Assume, by way of contradiction, that $|\ell| \leq q$. Let π' be a plane through ℓ, different from π, let p' be a point of $\pi' \setminus \ell$ and let ℓ' be a line through p' parallel to ℓ and contained in π'. Such a line does exist from (i) since $|\ell| \leq q$. In $\pi \setminus \ell$ there are at most $n - 1 - \mathbf{s}$ points and all of them, connected with ℓ', give at most $n - 1 - \mathbf{s}$ planes through ℓ', different from π' and meeting π. It follows that there would be at least $\mathbf{s} + 1$ planes through ℓ' disjoint from π, that contradicts (iii).

Next let ℓ be a line meeting L and let π be the plane spanned by the lines ℓ and L. Let p be a point of π and let π_0 be a plane through p intersecting π just in p. There is such a plane since $n > q$. Let now t be a line of π_0 not through p. Since the line t is disjoint with π, it is disjoint with both L and ℓ.

From the above argument it follows that $t \cap L = \emptyset$ and thus $|t| = q + 1$, hence from $t \cap \ell = \emptyset$ it follows again $|\ell| = q + 1$.

This proves that every line of the space is a projective line. So every plane is a projective plane and then the space is a projective space $PG(r, q)$ with $r \geq 4$ since it is $n > q$. $\square$

References

[1] A. Beutelspacher, Embedding finite planar spaces in projective spaces, Finite Geometries. Lecture Notes in Pure and Appl. Math. Dekker–New York **103** (1985) 9–17.

[2] J. Doyen and X. Hubaut, Finite regular locally projective spaces, Math. Z. **119** (1971) 83–88.

[3] N. Durante, P.M. Lo Re and D. Olanda, On regular planar spaces of type (k, n), Discrete Math. **301** (2005) 66–73.

[4] W.M. Kantor, Dimension and embedding theorems for geometric lattices, J. Comb. Th. (A) **17** (1974) 1731–195.

[5] A. Kreuzer, Locally projective spaces which satisfy the bundle theorem, J. Geom. **56** (1996) 87–98.

[6] A. Kreuzer, Linear spaces with projective lines, Combinatorics ’98 (Palermo). Discrete Math. **255** (1-3) (2002) 249–258.

[7] K. Metsch, Embedding theorems for locally projective three dimensional linear spaces, Proc. of Combinatorics ’94 – Discrete Math. **174** (1997) 227–245.

[8] L. Teirlinck, Combinatorial properties of planar spaces and embeddability, J. Comb. Th. (A) **43** (2) (1986) 291–302.

Nicola Durante and Domenico Olanda
Dipartimento di Matematica e Applicazioni
Università di Napoli “Federico II”
Complesso di Monte S. Angelo - Edificio T
via Cintia
80126 Napoli, Italy
e-mail: ndurante@unina.it
olanda@unina.it

Vito Napolitano
Dipartimento di Matematica
Università della Basilicata
Viale dell’Ateneo Lucano
85100 Potenza, Italy
e-mail: vnapolitano@unibas.it

Received 26 July 2005; revised 22 November 2005.