J. Geom. 82 (2005) 036 – 049
0047–2468/05/020036 – 14

DOI 10.1007/s00022-005-1630-4

Journal of Geometry

On warped product immersions

Bang-Yen Chen

Abstract. Warped product immersions appeared naturally in several recent studies. In this article we study fundamental geometric properties of such immersions. In addition we prove that every non-flat complex space form of complex dimension greater than one does not admit a warped product representation. As an application we obtain an improvement of an earlier result in [3] concerning warped products in real space forms.

Mathematics Subject Classifications (2000): 53C40, 53C42, 53B24.
Key words: Warped product immersion, totally umbilical submanifold, minimal submanifold, shape operator, warped product representation, geometric inequality.

1. Introduction

Let N_1 and N_2 be two Riemannian manifolds of positive dimension equipped with Riemannian metrics g_1 and g_2, respectively, and let f be a positive differentiable function on N_1. The warped product $N_1 \times_f N_2$ is defined to be the product manifold $N_1 \times N_2$ equipped with the Riemannian metric given by $g_1 + f^2 g_2$. We denote the dimension of N_1 and N_2 by n_1 and n_2, respectively. It is well-known that the notion of warped products plays some important roles in differential geometry as well as in physics (cf. [7]).

Let $\pi_1 : N_1 \times_f N_2 \to N_1$ and $\pi_2 : N_1 \times_f N_2 \to N_2$ denote the natural projections. For a vector field X on N_1, the lift of X to $N_1 \times_f N_2$ is the vector field $\tilde{X}$ whose value at each (p, q) is the lift X_p to (p, q). Thus the lift of X is the unique vector field on $N_1 \times_f N_2$ that is π_1-related to X and π_2-related to the zero vector field on N_2. The set of all such lifts of vector fields on N_1 is denoted by $\mathfrak{L}(N_1)$. Similarly, we denote by $\mathfrak{L}(N_2)$ the lifts of vector fields from vector fields on N_2.

Let $\phi : N_1 \times_f N_2 \to \tilde{M}$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into a Riemannian manifold $\tilde{M}$ and h be its second fundamental form. Denote by h_1 and h_2 the restriction of h to $\mathfrak{L}(N_1)$ and $\mathfrak{L}(N_2)$, respectively. We define the *partial mean curvature vectors* $\vec{H}_1$ and $\vec{H}_2$ by the following partial traces:

$$\vec{H}_1 = \frac{1}{n_1} \sum_{\alpha=1}^{n_1} h(e_\alpha, e_\alpha), \quad \vec{H}_2 = \frac{1}{n_2} \sum_{t=n_1+1}^{n_1+n_2} h(e_t, e_t) \tag{1.1}$$

for some orthonormal frame fields $e_1, \ldots, e_{n_1}$ and $e_{n_1+1}, \ldots, e_{n_1+n_2}$ of $\mathfrak{L}(N_1)$ and $\mathfrak{L}(N_2)$, respectively.

For $j = 1$ or 2, an immersion $\phi : N_1 \times_f N_2 \to \tilde{M}$ is called N_j-*totally geodesic* if the partial second fundamental form h_j vanishes identically. It is called N_j-*minimal* if the partial mean curvature vector $\overrightarrow{H}_j$ vanishes. And ϕ is called *mixed totally geodesic* if its second fundamental form h satisfies $h(X, Z) = 0$ for any $X \in \mathfrak{L}(N_1)$ and $Z \in \mathfrak{L}(N_2)$.

Let $\psi : N \to M$ be an isometric immersion and φ be a differentiable function on M, we denote by $\nabla\varphi$ and $D\varphi$ the gradient of φ and the normal component of $\nabla\varphi$ restricted on N, respectively.

If $M_1 \times_\rho M_2$ is a warped product and $\phi_i : N_i \to M_i$, $i = 1, 2$, are isometric immersions between Riemannian manifolds. Define a positive function f on N_1 by $f = \rho \circ \phi_1$. Then the map

$$\phi : N_1 \times_f N_2 \to M_1 \times_\rho M_2 \tag{1.2}$$

given by $\phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$ is an isometric immersion, which is called a *warped product immersion* (see [2, 6]).

The notion of warped product immersions appeared naturally in several recent studies related to different geometric aspects. For examples, it appeared in the study of multi-rotation surfaces in [5], a decomposition problem in [6], and a geometric inequality and minimal immersion problem in [3]. Hence it is natural and desired to investigate the fundamental properties of warped product immersions between warped product manifolds.

The main purpose of this article is thus to prove the following basic results for warped product immersions.

THEOREM 1. *Let* $\phi = (\phi_1, \phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ *be a warped product immersion between two warped product manifolds. Then we have:*

(a) ϕ *is mixed totally geodesic.*
(b) *The squared norm of the second fundamental form of* ϕ *satisfies*

$$\|h\|^2 \geq n_2 \|D \ln \rho\|^2, \quad n_2 = \dim N_2, \tag{1.3}$$

with the equality holding if and only if $\phi_1 : N_1 \to M_1$ *and* $\phi_2 : N_2 \to M_2$ *are both totally geodesic immersions.*
(c) ϕ *is* N_1*-totally geodesic if and only if* $\phi_1 : N_1 \to M_1$ *is totally geodesic.*
(d) ϕ *is* N_2*-totally geodesic if and only if* $\phi_2 : N_2 \to M_2$ *is totally geodesic and* $(\nabla \ln \rho)|_{N_1} = \nabla \ln f$ *holds, i.e., the restriction of the gradient of* $\ln \rho$ *to* N_1 *is the gradient of* $\ln f$*, or equivalently,* $D \ln \rho = 0$.
(e) ϕ *is a totally geodesic immersion if and only if* ϕ *is both* N_1*-totally geodesic and* N_2*-totally geodesic.*

THEOREM 2. *A warped product immersion* $\phi = (\phi_1, \phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ *between two warped product manifolds is totally umbilical if and only if we have:*

(1) $\phi_1 : N_1 \to M_1$ *is a totally umbilical immersion with mean curvature vector given by* $-D \ln \rho$*, and*
(2) $\phi_2 : N_2 \to M_2$ *is a totally geodesic immersion.*

THEOREM 3. *Let* $\phi = (\phi_1, \phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ *be a warped product immersion between two warped product manifolds. Then we have:*

(a) *The partial mean curvature vector* $\vec{H}_1$ *is equal to the mean curvature vector of* $\phi_1 : N_1 \to M_1$*; thus,* ϕ *is* N_1*-minimal if and only if* $\phi_1 : N_1 \to M_1$ *is a minimal immersion.*
(b) ϕ *is* N_2*-minimal if and only if* $\phi_2 : N_2 \to M_2$ *is a minimal immersion and* $(\nabla \ln \rho)|_{N_1} = \nabla \ln f$ *holds.*
(c) $\phi = (\phi_1, \phi_2)$ *is a minimal immersion if and only if* $\phi_2 : N_2 \to M_2$ *is a minimal immersion and the mean curvature vector of* $\phi_1 : N_1 \to M_1$ *is given by* $n_1^{-1} n_2 D \ln \rho$.

An isometric immersion $\psi : N \to M$ from a Riemannian manifold into another Riemannian manifold is called *pseudo-umbilical* if its shape operator $A_{\vec{H}}$ at the mean curvature vector $\vec{H}$ satisfies $A_{\vec{H}} X = \lambda X$ for any vector tangent to N, where λ is a function on N. Similarly, an immersion $\phi : N_1 \times_f N_2 \to M$ is called N_2-*pseudo-umbilical* if its shape operator $A_{\vec{H}}$ satisfies $A_{\vec{H}} Z = \lambda Z$ for tangent vectors Z in $\mathfrak{L}(N_2)$.

A warped product manifold $M_1 \times_\rho M_2$ is called a *warped product representation* of a real space form $R^m(c)$ of constant sectional curvature c if the warped product $M_1 \times_\rho M_2$ is an open dense subset of $R^m(c)$.

For warped product immersions into a real space form, we have the following.

THEOREM 4. *Let* $\phi = (\phi_1, \phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ *be a warped product immersion from a warped product* $N_1 \times_f N_2$ *into a warped product representation* $M_1 \times_\rho M_2$ *of a real space form* $R^m(c)$*. Then we have:*

(1) *The shape operator of* ϕ *satisfies*

$$A_{\vec{H}_1} Z = \left\{ \frac{\Delta f}{n_1 f} - c \right\} Z \tag{1.4}$$

for Z *in* $\mathfrak{L}(N_2)$*, where* Δ *is the Laplacian operator of* N_1.
(2) *For any* $X, Y \in \mathfrak{L}(N_1)$ *and* $Z \in \mathfrak{L}(N_2)$*,* $D_Z h(X, Y) = 0$ *holds, where* D *is the normal connection of* ϕ*. In particular, we have* $D_Z \vec{H}_1 = 0$.

(3) *The two partial mean curvature vectors* $\vec{H}_1$ *and* $\vec{H}_2$ *are perpendicular to each other if and only if the warping function* f *is an eigenfunction of the Laplacian operator* Δ *with eigenvalue* n_1c.

(4) *The warping function* f *is an eigenfunction of* Δ *with eigenvalue* n_1c *if and only if either* $\phi_1 : N_1 \to M_1$ *is a minimal immersion or* $(\nabla \ln \rho)|_{N_1} = \nabla \ln f$ *holds.*

(5) *When* $c = 0$*, the two partial mean curvature vectors* $\vec{H}_1$ *and* $\vec{H}_2$ *are perpendicular to each other if and only if the warping function* f *is a harmonic function.*

(6) *If* $\phi_1 : N_1 \to M_1$ *is a non-minimal immersion and the two partial mean curvature vectors* $\vec{H}_1$ *and* $\vec{H}_2$ *are parallel at each point, then* ϕ *is* N_2*-pseudo-umbilical and* $\phi_2 : N_2 \to M_2$ *is a minimal immersion.*

By applying the above results we are able to make the following refinement of Theorem 1.4 of [3].

THEOREM 5. *Let* $\phi : N_1 \times_f N_2 \to R^m(c)$ *be an isometric immersion of a warped product* $N_1 \times_f N_2$ *into a real space form* $R^m(c)$ *of constant curvature* c*. Then the squared mean curvature* H^2 *of* ϕ *satisfies the inequality:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} H^2 + n_1 c, \tag{1.5}$$

where $n_j = \dim N_j$ *and* $n = n_1 + n_2$.

The equality sign of (1.5) *holds identically if and only if exactly one of the following two cases occurs:*

(1) *The warping function* f *is an eigenfunction of the Laplacian operator* Δ *with eigenvalue* n_1c *and* ϕ *is a minimal immersion;*

(2) $\Delta f \neq (n_1c)f$ *and locally* ϕ *is a non-minimal warped product immersion* $(\phi_1, \phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ *of* $N_1 \times_f N_2$ *into some warped product representation* $M_1 \times_\rho M_2$ *of* $R^m(c)$ *such that* $\phi_2 : N_2 \to M_2$ *is a minimal immersion and the mean curvature vector of* $\phi_1 : N_1 \to M_1$ *is given by* $-(n_2/n_1)D \ln \rho$.

In the last section, we provide examples to illustrate that both case (1) and case (2) of Theorem 5 do occur for $c > 0$, $c = 0$ and $c < 0$.

2. Proofs of Theorems 1, 2 and 3

Let N be an n-dimensional Riemannian manifold isometrically immersed in another Riemannian manifold $\tilde{M}$. Denote by $\langle\, ,\rangle$ the inner product for N as well as for $\tilde{M}$.

Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections of N and $\tilde{M}$, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to N and ξ normal to N, where h denotes the second fundamental form, D the normal connection, and A the Weingarten map of the submanifold.

The mean curvature vector $\vec{H}$ is defined by

$$\vec{H} = \frac{1}{n} \operatorname{trace} h = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i), \tag{2.3}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the tangent bundle TN of N. The squared mean curvature is given by $H^2 = \langle \vec{H}, \vec{H} \rangle$.

Let (N_j, g_j) and $(M_j, \tilde{g}_j)$, $j = 1, 2$, be Riemannian manifolds and $\phi_j : N_j \to M_j$, $j = 1, 2$, be isometric immersions. Assume that

$$\phi = (\phi_1, \phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$$

is a warped product immersion of the warped product manifold $N_1 \times_f N_2$ into the warped product manifold $M_1 \times_\rho M_2$.

Denote by ∇^1 and ∇^f the Levi-Civita connections of $N_1 \times N_2$ equipped with the direct product metric $g_0 = g_1 + g_2$ and with the warped product metric $g = g_1 + f^2 g_2$, respectively. Similarly, denote by $\tilde{\nabla}^1$ and $\tilde{\nabla}^\rho$ the Levi-Civita connections of $M_1 \times M_2$ equipped with the direct product metric $\tilde{g}_0 = \tilde{g}_1 + \tilde{g}_2$ and with the warped product metric $\tilde{g} = \tilde{g}_1 + \rho^2 \tilde{g}_2$, respectively.

For vector fields U and V on $M_1 \times_\rho M_2$, it is known that the connections $\tilde{\nabla}^1$ and $\tilde{\nabla}^\rho$ are related by (see [1, 6]):

$$\tilde{\nabla}^\rho_U V = \tilde{\nabla}^1_U V - \langle U_2, V_2 \rangle (\nabla \ln \rho) + \langle \nabla \ln \rho, U \rangle V_2 + \langle \nabla \ln \rho, V \rangle U_2, \tag{2.4}$$

where $\langle\, , \rangle$ is the inner product with respect to $\tilde{g}$, $\nabla \ln \rho$ is the gradient of $\ln \rho$ on M_1, and U_2 and V_2 are the natural projections of U and V onto $\mathfrak{L}(M_2)$, respectively.

From (2.4) we obtain

$$\tilde{\nabla}^\rho_X Y = \tilde{\nabla}^1_X Y, \tag{2.5}$$

$$\tilde{\nabla}^\rho_Z W = \tilde{\nabla}^1_Z W - \langle Z, W \rangle (\nabla \ln \rho), \tag{2.6}$$

$$\tilde{\nabla}^\rho_X Z = \tilde{\nabla}^\rho_Z X = (X \ln \rho) Z \tag{2.7}$$

for X, Y in $\mathfrak{L}(M_1)$ and Z, W in $\mathfrak{L}(M_2)$. Hence, by applying (2.5) and (2.6), we obtain

$$h(X, Y) = h^0(X, Y), \tag{2.8}$$

$$h(Z, W) = h^0(Z, W) - \langle Z, W\rangle D \ln \rho, \tag{2.9}$$

$$h(X, Z) = 0 \tag{2.10}$$

for X, Y in $\mathfrak{L}(N_1)$ and Z, W in $\mathfrak{L}(N_2)$, where h is the second fundamental form of the warped product immersion $(\phi_1, \phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$, and h^0 is the second fundamental form of the direct product immersion $(\phi_1, \phi_2) : N_1 \times_1 N_2 \to M_1 \times_1 M_2$ between two direct Riemannian products.

The restriction of h^0 to $\mathfrak{L}(N_1)$ and to $\mathfrak{L}(N_2)$ are the second fundamental form of $\phi_1 : N_1 \to M_1$ and $\phi_2 : N_2 \to M_2$, respectively. Hence, $h^0(X, Y)$ and $h^0(Z, W)$ are orthogonal for X, Y in $\mathfrak{L}(N_1)$ and Z, W in $\mathfrak{L}(N_2)$.

Equation (2.10) is nothing but statement (a) of Theorem 1.

Statements (b) and (c) of Theorem 1 follows from (2.8) and (2.9).

If $\phi : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ is a N_2-totally geodesic immersion, then it follows from (2.9) that $h^0(Z, W) = \langle Z, W\rangle(D \ln \rho)$ for Z and W in $\mathfrak{L}(N_2)$. Since $D \ln \rho$ and $h^0(Z, W)$ are orthogonal, we have $h^0(Z, W) = 0$ and $D \ln \rho = 0$. The first equation implies that ϕ_2 is totally geodesic and the second equation implies that $(\nabla \ln \rho)|_{N_1} = \nabla \ln f$.

Conversely, if ϕ_2 is totally geodesic and $(\nabla \ln \rho)|_{N_1} = \nabla \ln f$ holds, then it follows from (2.9) that $h(Z, W) = 0$ for $Z, W \in \mathfrak{L}(N_2)$. Hence ϕ_2 is a totally geodesic immersion. This proves statement (d) of Theorem 1.

Statement (e) of Theorem 1 follows from statements (c) and (d) of Theorem 1 and equation (2.10).

To prove Theorem 2, let us assume that $\phi : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ is a totally umbilical immersion. Then we have

$$h(X, Y) = \langle X, Y\rangle \vec{H}, \quad h(Z, W) = \langle Z, W\rangle \vec{H} \tag{2.11}$$

for X, Y in $\mathfrak{L}(N_1)$ and Z, W in $\mathfrak{L}(N_2)$.

On the other hand, equations (2.8) and (2.11) imply that $\vec{H}$ is tangent to the first factor M_1. Hence, it follows from (2.9) and (2.11) that $h^0(Z, W) = 0$ for Z, W in $\mathfrak{L}(N_2)$, since $h^0(Z, W)$ is always tangent to the second factor M_2. Therefore $\phi_2 : N_2 \to M_2$ is a totally geodesic immersion. Consequently, we obtain condition (2) of Theorem 2.

Also from (2.8), (2.9), and (2.11) we find

$$h^0(X, Y) = \langle X, Y\rangle \vec{H}, \quad \vec{H} = -D \ln \rho \tag{2.12}$$

for X, Y in $\mathfrak{L}(N_1)$ which implies condition (1) of Theorem 2.

Conversely, it is easy to verify that if both conditions (1) and (2) of Theorem 2 hold, then ϕ is a totally umbilical immersion. This gives Theorem 2.

Since each lift of N_1 is a totally geodesic submanifold of the warped product manifold $N_1 \times_f N_2$ (cf. [1, 7]), equation (2.8) implies that $\vec{H}_1$ is nothing but the mean curvature vector of $\phi_1 : N_1 \to M_1$. This gives statement (a) of Theorem 3.

If $\phi : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ is N_2-minimal, then equation (2.9) implies that

$$f^2 \operatorname{trace} h_2^0 = n_2(D \ln \rho). \tag{2.13}$$

Because $D \ln \rho$ and trace h_2^0 are orthogonal, we know that ϕ_2 is a minimal immersion and $(\nabla \ln \rho)|_{N_1} = \nabla \ln f$ holds.

Conversely, if ϕ_2 is a minimal immersion and $(\nabla \ln \rho)|_{N_1} = \nabla \ln f$ holds, then it follows from equation (2.9) that trace $h_2 = 0$. Hence we have statement (b) of Theorem 3.

Finally, let us suppose that $\phi : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ is a minimal immersion. Then we have trace $h = 0$. Thus, by applying (2.8) and (2.9), we find

$$0 = \operatorname{trace} h_1^0 + f^2 \operatorname{trace} h_2^0 - n_2(D \ln \rho). \tag{2.14}$$

Since trace h_1^0 and $D \ln \rho$ are both tangent to the first factor M_1 and trace h_2^0 is tangent to M_2, equation (2.14) implies that

$$\operatorname{trace} h_1^0 = n_2(D \ln \rho), \quad \operatorname{trace} h_2^0 = 0 \tag{2.15}$$

which implies that $\phi_2 : N_2 \to M_2$ is a minimal immersion and the mean curvature vector of $\phi_1 : N_1 \to M_1$ is given by $n_1^{-1} n_2 D \ln \rho$.

The converse is easy to verify.

3. Proof of Theorem 4

Let φ a differentiable function on a Riemannian n-manifold N. Then the Laplacian of φ is given by

$$\Delta\varphi = \sum_{j=1}^{n} \{(\nabla_{e_j} e_j)\varphi - e_j e_j \varphi\}, \tag{3.1}$$

where $e_1, \ldots, e_n$ is an orthonormal frame field on N. For each plane section of N, we denoted by $K(\pi)$ the sectional curvature of the plane section π.

Suppose that $M_1 \times_\rho M_2$ is a warped product representation of a real space form $R^m(c)$ and $\phi : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ is a warped product immersion. Then, from statement (a) of

Theorem 1, we have

$$h(X, Z) = 0 \tag{3.2}$$

for X in $\mathfrak{L}(N_1)$ and Z in $\mathfrak{L}(N_2)$.

Since $N_1 \times_f N_2$ is a warped product, we get [7]

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \quad \langle \nabla_X Y, Z\rangle = 0 \tag{3.3}$$

for unit vector fields X, Y in $\mathfrak{L}(N_1)$ and Z in $\mathfrak{L}(N_2)$. By applying (3.3) we find

$$K(X \wedge Z) = \langle \nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z\rangle = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \tag{3.4}$$

If we choose a local orthonormal frame $e_1, \ldots, e_{n_1+n_2}$ in such way that $e_1, \ldots, e_{n_1}$ are in $\mathfrak{L}(N_1)$ and $e_{n_1+1}, \ldots, e_{n_1+n_2}$ in $\mathfrak{L}(N_2)$, then (3.4) yields

$$\frac{\Delta f}{f} = \sum_{\alpha=1}^{n_1} K(e_\alpha \wedge e_s), \quad s = n_1 + 1, \ldots, n_1 + n_2. \tag{3.5}$$

On the other hand, from the equation of Gauss, we know that the curvature tensor R of $N_1 \times_f N_2$ satisfies

$$\begin{aligned} \langle R(X, Y)Z, W\rangle \;=\; & \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle \\ & + c\,\{\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle\} \end{aligned} \tag{3.6}$$

for vectors X, Y, Z, W tangent to $N_1 \times_f N_2$. Using (3.2), (3.5), and (3.6), we obtain

$$\langle \vec{H}_1, h(Z, Z)\,\rangle = \frac{\Delta f}{n_1 f} - c \tag{3.7}$$

for any unit Z in $\mathfrak{L}(N_2)$. Thus, by applying polarization, we find

$$\langle \vec{H}_1, h(Z, W)\,\rangle = 0 \tag{3.8}$$

for orthonormal vectors Z, W in $\mathfrak{L}(N_2)$. Equations (3.7) and (3.8) imply that the shape operator at $\vec{H}_1$ satisfies

$$A_{\vec{H}_1} Z = \left\{\frac{\Delta f}{n_1 f} - c\right\} Z \tag{3.9}$$

for Z in $\mathfrak{L}(N_2)$. Thus we have statement (1).

It follows from (3.2) and (3.3) that the covariant derivative of the second fundamental form satisfies

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \qquad (3.10)$$
$$= -h(\nabla_X Y, Z) = 0,$$

due to the fact that N_1 is totally geodesic in $N_1 \times_f N_2$.

On the other hand, by applying (3.2) and (3.3), we also find

$$(\bar{\nabla}_Z h)(X, Y) = D_X h(X, Y). \qquad (3.11)$$

Therefore, after applying (3.10), (3.11), and the equation of Codazzi, we obtain statement (2).

By applying equations (1.1) and (3.7) we obtain

$$\langle \vec{H}_1, \vec{H}_2 \rangle = \frac{\Delta f}{n_1 f} - c \qquad (3.12)$$

which gives statement (3).

It follows from equations (2.8) and (2.9) that the partial mean curvature vectors $\vec{H}_1$ and $\vec{H}_2$ are perpendicular to each other if and only if we have either

(i) $\vec{H}_1 = 0$ or
(ii) $(\nabla \ln \rho)|_{N_1} = \nabla \ln f$.

According to statement (a) of Theorem 3, the first case occurs when and only when ϕ_1 is a minimal immersion. By combining these results with statement (3) of Theorem 4, we obtain statement (4) of Theorem 4.

Obviously, statement (5) is a special case of statement (3).

If ϕ_1 is a non-minimal immersion and if the two partial mean curvature vectors $\vec{H}_1$ and $\vec{H}_2$ are parallel, then there exists a function μ such that $\vec{H}_2 = \mu \vec{H}_1$. In such case the mean curvature vector of ϕ is related to the partial mean curvature vector $\vec{H}_1$ by $\vec{H} = ((n_1 + n_2\mu)/n) \vec{H}_1$. Therefore, after applying (3.9) we may conclude that ϕ is N_2-pseudo-umbilical.

Since ϕ_1 is assumed to be a non-minimal immersion, we have $\vec{H}_1 \neq 0$ according to statement (a) of Theorem 3. Therefore, by applying the parallelism of $\vec{H}_1$ and $\vec{H}_2$ and the orthogonality of $h^0(X, Y)$ and $h^0(Z, W)$ for $X, Y \in \mathfrak{L}(N_1)$ and $Z, W \in \mathfrak{L}(N_2)$, we may conclude from (2.8) and (2.9) that $\phi_2 : N_2 \to M_2$ is a minimal immersion.

4. Proof of Theorem 5

It follows from Theorem 1.4 of [3] that every isometric immersion $\phi : N_1 \times_f N_2 \to R^m(c)$ of a warped product manifold $N_1 \times_f N_2$ into a real space form $R^m(c)$ satisfies inequality (1.5), with the equality sign holding if and only if we have

(A) ϕ is a mixed totally geodesic immersion and

(B) $n_1 \vec{H}_1 = n_2 \vec{H}_2$ holds.

Now, let us assume that the immersion ϕ satisfies the equality case of (1.5) identically.

If the warping function f is an eigenfunction of the Laplacian operator Δ with eigenvalue $n_1 c$, then the equality case of (1.5) implies that ϕ is a minimal immersion. Thus we obtain case (1) of Theorem 5.

Next, assume that the warping function f is not an eigenfunction of Δ with eigenvalue $n_1 c$. Then ϕ is a non-minimal immersion. Since the immersion ϕ is mixed totally geodesic according to (A), Theorem 16 of [6] implies that locally there exists a warped product representation $M_1 \times_\rho M_2$ of $R^m(c)$ such that ϕ is a warped product immersion:

$$\phi = (\phi_1, \phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$$

from $N_1 \times_f N_2$ into $M_1 \times_\rho M_2$. Since $\Delta f \neq (n_1 c) f$, statement (1) of Theorem 4 implies that $\vec{H}_1$ is nonzero. Hence, by applying statement (a) of Theorem 3, we know that $\phi_1 : N_1 \to M_1$ is a non-minimal immersion.

Since the two partial mean curvature vectors $\vec{H}_1$ and $\vec{H}_2$ are parallel according to (B), we obtain from statement (6) of Theorem 4 that $\phi_2 : N_2 \to M_2$ is a minimal immersion. Hence, by applying (2.9), we find $\vec{H}_2 = -D \ln \rho$. Therefore, by using the condition: $n_1 \vec{H}_1 = n_2 \vec{H}_2$ from (B), we obtain $\vec{H}_1 = -(n_2/n_1) D \ln \rho$. Combining this with statement (a) of Theorem 3 shows that the mean curvature vector of $\phi_1 : N_1 \to M_1$ is given by $-(n_2/n_1) D \ln \rho$.

Obviously, when the warping function f is an eigenfunction with eigenvalue $n_1 c$ and ϕ is a minimal immersion, the equality case of (1.5) holds identically.

Finally, suppose that f is not an eigenfunction with eigenvalue $n_1 c$ and ϕ is a non-minimal warped product immersion: $\phi = (\phi_1, \phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ from $N_1 \times_f N_2$ into a warped product representation $M_1 \times_\rho M_2$ of $R^m(c)$ which satisfies the two conditions:

(2-a) $\phi_2 : N_2 \to M_2$ is a minimal immersion and

(2-b) the mean curvature vector of $\phi_1 : N_1 \to M_1$ is given by $-(n_2/n_1) D \ln \rho$.

Then, by applying statement (a) of Theorem 1, we know that ϕ is a mixed totally geodesic immersion.

Also, from (2-a) and (2.9) we have

$$\vec{H}_2 = -D\ln\rho. \tag{4.1}$$

On the other hand, by using (2-b) and statement (a) of Theorem 3, we also have

$$\vec{H}_1 = -\frac{n_2}{n_1} D\ln\rho. \tag{4.2}$$

Combining (4.1) and (4.2) gives $n_1\vec{H}_1 = n_2\vec{H}_2$. Therefore, by applying Theorem 1.4 of [3], we obtain the equality case of (1.5) holds identically.

5. An additional result

A Kaehler manifold $\tilde{M}^n(c)$ of constant holomorphic sectional curvature c is called a *complex space form.* There are three types of complex space forms: *elliptic, hyperbolic,* or *flat*, according as the holomorphic sectional curvature is positive, negative, or zero. Complex projective n-space CP^n, complex hyperbolic n-space CH^n, and complex Euclidean n-space $\mathbb{C}^n$ are complete and simply-connected complex space forms of elliptic, hyperbolic, and flat type, respectively.

Just like real space forms, a warped product manifold $M_1 \times_\rho M_2$ is called a *warped product representation* of a complex space form $\tilde{M}^n(c)$ if the warped product manifold $M_1 \times_\rho M_2$ is an open dense subset of $\tilde{M}^n(c)$.

Since the Euclidean $2n$-space $\mathbb{E}^{2n}$ admits warped product representations of the form $M_1 \times_\rho M_2$, the complex Euclidean n-space $\mathbb{C}^n$ with $n \geq 1$ also admits warped product representations of the form $M_1 \times_\rho M_2$. Moreover, it is known that $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos s} \mathbb{R}$ and $\mathbb{R} \times_{e^x} \mathbb{R}$ are warped product representations of $CP^1(1)$ and $CH^1(-1)$, respectively.

In contrast, for CP^n and CH^n with $n > 1$ we have the following non-existence result.

PROPOSITION 1. *Every non-flat complex space form $\tilde{M}^n(c)$ with $n > 1$ does not admit a warped product representation of the form: $M_1 \times_\rho M_2$.*

Proof. Assume that $M_1 \times_\rho M_2$ is a warped product representation of a complex space form $\tilde{M}^n(c)$ of constant holomorphic sectional curvature c. If $c \neq 0$, then $\tilde{M}^n(c)$ is irreducible. Thus the warping function ρ is a non-constant function.

For a given fixed point $p \in M_1$, the lift $\hat{M}_2^p = \{p\} \times M_2$ of M_2 is a totally umbilical submanifold of $\tilde{M}^n(c)$ whose mean curvature vector is given by $-(\nabla \ln\rho)(p)$ (see [1, p. 66]). Since ρ is non-constant, there exists a point $p \in M_1$ such that $\hat{M}_2^p$ is a non-totally geodesic submanifold. For each such point p, a result of [4] implies that $\hat{M}_2^p$ is a totally real submanifold whose dimension is less than n, unless M_2 is one-dimensional.

On the other hand, for each point $q \in M_2$, the lift $\hat{M}_1^q = M_1 \times \{q\}$ of M_1 is a totally geodesic submanifold of $\tilde{M}^n(c)$. Hence $\hat{M}_1^q$ is either a totally real submanifold or a holomorphic submanifold of $\tilde{M}^n(c)$.

Now, we divide the proof into two cases.

CASE 1 ($\dim M_2 > 1$). In this case, $\hat{M}_2^p$ is a totally real submanifold of dimension less than n. Since $M_1 \times_\rho M_2$ is a warped product representation of $\tilde{M}^n(c)$ and $\hat{M}_2^q$ is totally real, $\hat{M}_1^q$ cannot be a holomorphic submanifold. Hence $\hat{M}_1^q$ is a totally real submanifold since $\hat{M}_1^q$ is a totally geodesic submanifold of a non-flat complex space form. Thus we have $\dim M_1 \leq n$. Therefore the dimension of $M_1 \times_\rho M_2$ is less than the real dimension of $\tilde{M}^n(c)$ which is a contradiction. Consequently, this case cannot occur.

CASE 2 ($\dim M_2 = 1$). In this case, the lift $\hat{M}_1^q$ cannot be a holomorphic submanifold. Hence the lift $\hat{M}_1^1$ is a totally real totally geodesic submanifold. Therefore we must have $\dim M_1 = 1$. Consequently, we obtain $n = 1$ which is a contradiction. □

6. Examples

In this section we provide several simple examples of isometric immersions from warped products into real space forms to show that both case (1) or case (2) of Theorem 5 do occur for $c = 0,\ c > 0$ and $c < 0$.

EXAMPLE 1. There exist many minimal isometric immersions from some warped products $N_1 \times_f N_2$ with harmonic warping function f into a Euclidean space. For instance, if N_2 is a minimal submanifold of the unit $(m-1)$-hypersphere S^{m-1} in $\mathbb{E}^m$ centered at the origin, then the minimal cone $C(N_2)$ over N_2 with vertex at the origin of $\mathbb{E}^m$ is the warped product manifold $\mathbb{R}_+ \times_s N_2$ whose warping function $f = s$ is a harmonic function. Here s denotes the coordinate function of the positive real line $\mathbb{R}_+$.

This example provides us many examples of isometric immersions of warped products in a real space form which satisfy case (1) of Theorem 5.

EXAMPLE 2. Let (r, θ, z) denote the cylindrical coordinates of $\mathbb{E}^3$. Then the metric tensor $\tilde{g}$ of $\mathbb{E}^3$ is given by

$$\tilde{g} = dr^2 + dz^2 + r^2 d\theta^2. \tag{6.1}$$

Let $\mathbb{E}_+^2 = \mathbb{R}_+ \times \mathbb{R}$ denote the half plane defined by $\mathbb{E}_+^2 = \{(r, z) : r > 0\}$ equipped with the standard Euclidean metric $\tilde{g}_1 = dr^2 + dz^2$ and let S^1 be the unit circle with metric $\tilde{g}_2 = d\theta^2$. Then $\mathbb{E}_+^2 \times_r S^1$ equipped with metric $\tilde{g} = \tilde{g}_1 + r^2\tilde{g}_2$ is a warped product representation of $\mathbb{E}^3$.

Let s be the natural coordinate of the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Consider the warped product $N_1 \times_f N_2 =: (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos s} S^1$ and also consider the immersion:

$$\phi = (\phi_1, \phi_2) : (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times_{\cos s} S^1 \to \mathbb{E}^2_+ \times_r S^1, \tag{6.2}$$

where ϕ_1 is defined by $\phi_1(s) = (\cos s, \sin s)$ for $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\phi_2 : S^1 \to S^1$ is the identity map.

It is easy to verify that ϕ is an isometric warped product immersion whose image is an open dense subset of the standard unit sphere S^2 in $\mathbb{E}^3$. The squared mean curvature H^2 of ϕ is equal to one. Since the warping function f of $N_1 \times_f N_2$ is $\cos s$ and s is the arc length of ϕ_1, the Laplacian Δf of f is given by $-f''(s)$. Thus we find $\Delta f = f$ which shows that the equality sign of (1.5) holds identically.

On the other hand, it is easy to verify that $\nabla \ln r = r^{-1}\partial/\partial r$. Hence we have

$$D \ln \rho = \cos s \frac{\partial}{\partial r} + \sin s \frac{\partial}{\partial z} \tag{6.3}$$

which gives $\overrightarrow{H}_1 = -D \ln \rho$. Therefore we also have condition (2-b). This provides us an example which satisfies case (2) of Theorem 5.

EXAMPLE 3. Let S^{2n_1} be the unit $2n_1$-sphere equipped with the metric:

$$g = du_1^2 + \cos^2 u_1 du_2^2 + \cdots + \prod_{k=1}^{2n_1-1} \cos^2 u_k \, du_{2n_1}^2. \tag{6.4}$$

If we put

$$g_1 = du_1^2 + \cos^2 u_1 du_2^2 + \cdots + \prod_{k=1}^{n_1-1} \cos^2 u_k \, du_{n_1}^2,$$

$$g_2 = du_{n_1+1}^2 + \cos^2 u_{n_1+1} du_{n_1+2}^2 + \cdots + \prod_{k=n_1+1}^{2n_1-1} \cos^2 u_k \, du_{2n_1}^2,$$

then S^{2n_1} is isometric to the warped product $N_1 \times_f N_2$, where $N_1 = (S^{n_1}, g_1)$ and $N_2 = (S^{n_1}, g_2)$, and $f = \cos u_1 \cdots \cos u_{n_1}$. A direct long computation shows that the warping function f satisfies $\Delta f = n_1 f$.

Let $\phi : N_1 \times_f N_2 \to \mathbb{E}^{2n_1+1}$ be the standard embedding of S^{2n_1} in $\mathbb{E}^{2n_1+1}$. Then the squared mean curvature H^2 of ϕ is equal to one. Therefore we obtain the equality case of (1.5). Since ϕ is a non-minimal immersion, Theorem 5 implies that the immersion ϕ satisfies case (2) of Theorem 5.

EXAMPLE 4. Let $N_1 \times_f N_2$ denote the warped product representation of the unit $2n_1$-sphere S^{2n_1} with $N_1 = (S^{n_1}, g_1)$, $N_2 = (S^{n_1}, g_2)$, and $f = \cos u_1 \cdots \cos u_{n_1}$ as given in Example 3.

Consider a totally umbilical immersion:

$$\phi : N_1 \times_f N_2 \to H^{2n_1+1}(c) \tag{6.5}$$

of the warped product manifold $N_1 \times_f N_2$ into the real hyperbolic $(2n_1+1)$-space $H^{2n_1+1}(c)$ of constant curvature $c < 0$. Then the squared mean curvature H^2 of ϕ is equal to $1 - c$. Since we have $(\Delta f)/f = n_1$, the equality case of (1.5) holds identically.

Because ϕ is a non-minimal immersion, the immersion $\phi : N_1 \times_f N_2 \to H^{2n_1+1}(c)$ satisfies case (2) of Theorem 5.

EXAMPLE 5. Let $N_1 \times_f N_2$ denote the same warped product representation of S^{2n_1} as given Examples 3 and 4.

Consider a totally umbilical immersion:

$$\phi : N_1 \times_f N_2 \to S^{2n_1+1}(c) \tag{6.6}$$

of $N_1 \times_f N_2$ into a $(2n_1 + 1)$-sphere $S^{2n_1+1}(c)$ of constant curvature $c < 1$. Then the squared mean curvature H^2 of ϕ is equal to $1-c$. Since we have $(\Delta f)/f = n_1$, the equality case of (1.5) holds identically.

It is easy to see that the immersion $\phi : N_1 \times_f N_2 \to S^{2n_1+1}(c)$ satisfies case (2) for $0 < c < 1$.

References

[1] B.Y. Chen, Geometry of Submanifolds and Its Applications, Science University of Tokyo, Tokyo, 1981.

[2] B.Y. Chen, Riemannian submanifolds, In: Handbook of Differential Geometry, (edited by F. Dillen and L. Verstraelen), volume 1, North Holland, Amsterdam 2000, 187–418.

[3] B.Y. Chen, On isometric minimal immersions from warped products into real space forms. Proc. Edinburgh Math. Soc. **45** (2002) 579–587.

[4] B.Y. Chen and K. Ogiue, Two theorems on Kaehler manifolds, Michigan Math. J. **21** (1974) 225–229.

[5] F. Dillen and S. Nölker, Semi-parallelity, multi-rotation surfaces and the helix-property, J. Reine Angew. Math. **435** (1993) 33–62.

[6] S. Nölker, Isometric immersions of warped products, Differential Geom. Appl. **6** (1996) 1–30.

[7] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1982.

B.Y. Chen
Department of Mathematics
Michigan State University
East Lansing, Michigan 48864-1027
USA
e-mail: bychen@math.msu.edu

Received 25 March 2002.