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Characterizations of cyclic polytopes

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Abstract. Let P denote a simplicial convex $2m$-polytope with n vertices. Then the following are equivalent: (i) P is cyclic; (ii) P satisfies Gale's Evenness Condition; (iii) Every subpolytope of P is cyclic; (iv) P has at least $2m+2$ cyclic subpolytopes with $n-1$ vertices if $n \geq 2m+5$; (v) P is neighbourly and has n universal edges.

We present an additional characterization based upon an easily described point arrangement property.

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1. Introduction

The importance of cyclic d-polytopes is well known and well documented. They have important combinatorial properties and numerous applications in various branches of mathematics and science. For example, they are solutions to extremum problems (The Upper Bound Conjecture) and they serve as bases for diverse constructions (from triangulations to bimatrix games). Their best known realization is with vertices on the moment curve $\{(t, t^2, \ldots, t^d) \mid t \in \mathbb{R}\} \subset \mathbb{R}^d$, $d \geq 2$, and their best known characterization is Gale's Evenness Condition (GEC).

Let X be a set with an ordered list of points $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^d$, $d \geq 2$. We say that X is *k-inseparable* ($k \geq 1$) if for $i = 1, 2, \ldots, n-k$, $\text{conv}\{x_i, x_{i+1}, \ldots, x_{i+k}\}$ is disjoint from every hyperplane spanned by points from $X \backslash \{x_i, x_{i+1}, \ldots, x_{i+k}\}$.

Our two main results are the following:

THEOREM A. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of $n \geq d+2$ points in general position in $\mathbb{R}^d$; that is, $n \in \{d+2k, d+2k+1\}$ for some $k \geq 1$. If there is an ordering on X such that X is k-inseparable then $P = \text{conv}X$ is a cyclic d-polytope with respect to that ordering.*

THEOREM B. *Let $d = 2m$, $k \geq 1$ and X be a set of $n \in \{d+2k, d+2k+1\}$ points in general position in $\mathbb{R}^d$. Then X is the vertex set of a cyclic $2m$-polytope if, and only if, X is k-inseparable.*

The necessity in Theorem B follows from GEC ($Y \subset X$ determines a facet of conv X if, and only if, $|Y| = d$ and any two vertices of $X \backslash Y$ have an even number of vertices of Y between them) and that every cyclic $2m$-subpolytope of a cyclic $2m$-polytope is also cyclic. One consequence of these is that (cf. [5]) the vertex set of a cyclic $2m$-polytope is contained in a $2m$-order curve and thus, it is well known to be 1-inseparable with respect to the ordering induced by either orientation of the curve. Clearly, a 1-inseparable set is k-inseparable for $k \geq 1$.

In regard to Theorem A, we assume that $X = \{x_1, \ldots, x_n\}$ is k-inseparable and note the following simplifications and plan of proof:

1.1 It is clear that conv$\{x_1, x_2, \ldots, x_d\}$, conv$\{x_n, x_1, \ldots, x_{d-1}\}$, conv$\{x_{n-1}, x_n, x_1, \ldots, x_{d-2}\}, \ldots,$ conv$\{x_{n-d+1}, \ldots, x_{n-1}, x_n\}$ are facets of P. With this starting point, it is easy to verify Theorem A in the cases $k = 1$ or $d = 2$. Hence, we assume that $k \geq 2$, $d \geq 3$ and proceed by induction on d.

1.2 Let $\bar{H} \subset \mathbb{R}^d \backslash \{x_1\}$ be a hyperplane and $\bar{x}_i$ be the projection of x_i from x_1 onto $\bar{H}$. Then $\bar{X} = \{\bar{x}_2, \ldots, \bar{x}_n\}$ is a k-inseparable set of points in general position in $\bar{H}$ and $|\bar{X}| \in \{(d-1) + 2k, (d-1) + 2k + 1\}$. From this and the induction hypothesis on d, it follows that conv $\bar{X}$ is a cyclic $(d-1)$-polytope with $n-1$ vertices and with respect to the induced ordering. It is clear that conv $\bar{X}$ is combinatorially equivalent to the vertex figure P/x_1 of P at x_1.
Similarly, P/x_n is also a cyclic $(d-1)$-polytope. By iteration and with $P/[x_i, x_j] := (P/x_i)/x_j = (P/x_j)/x_i$, the quotient polytopes $P/[x_1, x_2]$, $P/[x_1, x_n]$ and $P/[x_{n-1}, x_n]$ are cyclic $(d-2)$-polytopes with $n-2$ vertices for $d \geq 4$.

1.3 We recall (cf. [2], p. 63) that there are explicit formulae for the number $c(n, d)$ of facets of a cyclic d-polytope with n vertices, and that if d is odd then $c(n, d) = 2c(n-1, d-1) - c(n-2, d-2)$.
Let d be odd. From 1.2, each facet of P containing x_1 (respectively x_n) satisfies GEC and there are $c(n-1, d-1)$ of them. Also, there are $c(n-2, d-2)$ facets of P that contain both x_1 and x_n. It follows now by the Upper Bound Theorem that each facet of P contains x_1 or x_n, and as a consequence, P satisfies GEC. Thus P is a cyclic d-polytope and we may assume that d is even.

1.4 Let d be even, $k \geq 2$ and $|X| = d + 2k + 1$. We observe that $|X| \geq d + 5$ and that every $d + 2k$ element subset of X is k-inseparable. Hence, if Theorem A is valid for $n = d + 2k$ then it is valid for $n = d + 2k + 1$.

1.5 With $d = 2m \geq 4$ and $n = d + 2k$, our plan of proof is as follows:

- Verify Theorem A for $k = 2$.
- Show that X has a 2-independent subset X'.
- Show that P is obtained from conv X' by a construction that, at each iteration, yields a cyclic d-polytope.

2. Preliminaries

Let Y be a set of points in $\mathbb{R}^d$. Then conv Y and aff Y denote, respectively, the convex hull and the affine hull of Y. For sets $Y_1, Y_2, \ldots, Y_n$, let

$$[Y_1, Y_2, \ldots, Y_n] = \text{conv}(Y_1 \cup Y_2 \cup \ldots \cup Y_n)$$

and

$$\langle Y_1, Y_2, \ldots, Y_n \rangle = \text{aff}(Y_1 \cup Y_2 \cup \ldots \cup Y_n).$$

For a point $y \in \mathbb{R}^d$, let $[y] = [\{y\}]$ and $\langle y \rangle = \langle \{y\} \rangle$.

Let $Y = \{y_1, y_2, \ldots, y_v\}$ be an ordered set of v points, and denote the ordering by $y_1 < y_2 < \cdots < y_v$. We say that y_i and y_{i+1} are *successive* points, and that y_j *separates* y_i and y_k if $y_i < y_j < y_k$. Let $Y' \subset Y$. Then Y' is an *even* subset if it is the union of mutually disjoint subsets $\{y_i, y_{i+1}\}$; otherwise, Y' is an *odd* subset. Next, Y' is a *Gale* subset if any two points of $Y \backslash Y'$ are separated by an even number of points of Y'. We note that even subsets are Gale, and that an odd Gale subset contains y_1 or y_v.

Let $Q \subset \mathbb{R}^d$ denote a convex d-polytope with $\mathcal{V}(Q)$, $\mathcal{F}(Q)$ and $\mathcal{L}(Q)$ denoting, respectively, the set of vertices, the set of facets and the *face lattice* of Q. We recall that $\mathcal{L}(Q)$ is the collection of all faces of Q, and the two d-polytopes are *combinatorially equivalent* if their face lattices are isomorphic.

We say that Q is *neighbourly* if every $[d/2]$ vertices of Q determine a face of Q, and that Q is *cyclic* if it is combinatorially equivalent to the convex hull of some $v \geq d + 1$ points on $\{(t, t^2, \ldots, t^d) \mid t \in \mathbb{R}\}$ in $\mathbb{R}^d$.

We assume familiarity with the basic properties of convex polytopes (cf. [2]) and, in particular, with neighbourly and cyclic polytopes (cf. [1] and [3]).

Let $Y = \{y_1, \ldots, y_v\} = \mathcal{V}(Q)$ and G be a u-face of Q, $0 \leq u \leq d - 1$. Then the *quotient polytope* Q/G is a $(d - u - 1)$-polytope with face lattice isomorphic to $\{G' \in \mathcal{L}(Q) \mid G \subset G'\}$. We say that G is *universal* if either $G \in \mathcal{F}(Q)$ or Q/G is a neighbourly polytope with $|\mathcal{V}(Q)| - |\mathcal{V}(G)|$ vertices; that is, $[G, Y']$ is a face of Q for every $Y' \subset Y$ with $|Y'| \leq [(d - |\mathcal{V}(G)|)/2]$.

We are particularly interested in Q/G in the case $u = d - 3$. Then $Q^* = Q/G$ is a (planar) convex polygon and, as a simplification, we let

$$\mathcal{V}(Q^*) = \{y_i^* \mid [G, y_i] \text{ is a } (d-2)\text{-face of } Q\}.$$

Then $[y_i^*, y_j^*]$ is an edge of Q^* iff $[G, y_i, y_j] \in \mathcal{F}(Q)$, and $\langle y_i^*, y_j^* \rangle$ separates y_r^* and y_s^* in the plane iff the hyperplane $\langle G, y_i, y_j \rangle$ separates y_r and y_s in $\mathbb{R}^d$.

In the case that Y is ordered, we say that $y_1 < y_2 < \cdots < y_v$ is a *vertex array* of Q. Then Q satisfies GEC if for any $Y' \subset Y : [Y'] \in \mathcal{F}(Q)$ if, and only if, $|Y'| = d$ and Y' is Gale. As already noted, Q is a cyclic d-polytope if, and only if, it satisfies GEC with respect to some vertex array.

Henceforth, we assume that $d = 2m \geq 4$ and that $Q = [y_1, \ldots, y_v]$ is neighbourly. Then a *universal tower* in Q is a strictly increasing sequence $\mathcal{T} = \{G_j\}_{j=1}^m$ of universal faces of Q, with $|\mathcal{V}(G_j)| = 2j$ for $1 \leq j \leq m$.

Let $\mathcal{T} = \{G_j\}_{j=1}^m$ be a universal tower in Q, $\mathcal{F}_j = \{\mathcal{F} \in \mathcal{F}(Q) \mid G_j \in F\}$, $\mathcal{F}_0 = \mathcal{F}(Q)\backslash\mathcal{F}_1$ and

$$\mathcal{C} = \mathcal{C}(Q, \mathcal{T}) = \mathcal{F}_1\backslash(\mathcal{F}_2(\ldots\backslash\mathcal{F}_m)\ldots).$$

Then $\mathcal{C} = (\mathcal{F}_1\backslash\mathcal{F}_2) \cup (\mathcal{F}_3\backslash\mathcal{F}_4) \cup \cdots \cup (\mathcal{F}_{m-1}\backslash\mathcal{F}_m)$ for even m, $\mathcal{C} = (\mathcal{F}_1\backslash\mathcal{F}_2) \cup \cdots \cup \mathcal{F}_m$ for odd m, and there is a point $y \in \mathbb{R}^d$ that *lies exactly beyond* $\mathcal{C}$; that is, y is beyond each $F \in \mathcal{C}$ and beneath each $F \in \mathcal{F}(Q)\backslash\mathcal{C}$. If y is exactly beyond $\mathcal{C}$ then $[Q, y]$ is said to be *obtained from Q by serving y through* $\mathcal{C}$; cf. [3].

Finally, we let $\mathcal{U}(Q)$ denote the set of universal edges of Q and note that $|\mathcal{U}(Q)| \leq v$ if $v \geq d + 3$.

2.1. ([5]) *If Q is a cyclic $2m$-polytope then the vertices of Q lie on a d-order curve.*
2.2. ([4]) $E = [y_i, y_j] \in \mathcal{U}(Q)$ *if, and only if, $H \cap E = \emptyset$ for every hyperplane spanned by points from $\mathcal{V}(Q)\backslash\{y_i, y_j\}$.*
2.3. ([3]) *Q is cyclic with $y_1 < y_2 < \cdots < y_v$ if, and only if,* $\mathcal{U}(Q) = \{[y_i, y_{i+1}] \mid i = 1, \ldots, v$ and $y_{v+1} = y_1\}$, $v \geq 2m + 3$.
2.4. ([4]) *If Q is not cyclic and $v \geq 2m + 5$ then Q has at most $2m + 1$ cyclic $2m$-subpolytopes with $v - 1$ vertices.*
2.5. ([5]) *Let Q be cyclic with $y_1 < y_2 < \cdots < y_v$. Then $\mathcal{T} = \{G_j\}_{j=1}^n$ is a universal tower with $G_j = [y_1, \ldots, y_j, y_{v-j+1}, \ldots, y_v]$; moreover, if $Q' = [Q, y_{v+1}]$ is obtained from Q by sewing y_{v+1} through $\mathcal{C}(Q, T)$ then Q' is cyclic with $y_1 < y_2 < \cdots < y_v < y_{v+1}$.*

We note that although 2.5 is not explicitly stated in the cited article, it is readily deduced from the sewing construction and the description of $\mathcal{U}(Q')$.

As the last entry in this section, we present the manner in which we apply the sewing construction.

LEMMA C. *With the notation of 2.5; let $Q' = [Q, y_{v+1}]$ such that $\{[y_1, \ldots, y_{d-1}, y_d], [y_1, \ldots, y_{d-1}, y_{v+1}]\} \subset \mathcal{F}(Q')$ and $Q'/[y_i, y_{i+1}]$ is a cyclic $(d-2)$-polytope with $v - 1$ vertices and the ordering induced by $y_1 < y_2 \cdots < y_v < y_{v+1}$, $i = v - 3, v - 2, v - 1$. Then Q' is obtained by sewing y_{v+1} through $\mathcal{C}(Q, \mathcal{T})$.*

Proof. We need to show that y_{v+1} lies exactly beyond

$$\mathcal{C}(Q, \mathcal{T}) = (\mathcal{F}_1 \backslash \mathcal{F}_2) \cup (\mathcal{F}_3 \backslash \mathcal{F}_4) \cup \cdots$$

with $\mathcal{F}_j = \{[y_1, \ldots, y_j, S_{d-2j}, y_{v-j+1}, \ldots, y_v] \mid S_{d-2j} \subset \{y_{j+1}, \ldots, y_{v-j}\}\}$,

$$\mathcal{F}_j \backslash \mathcal{F}_{j+1} = \{F \in \mathcal{F}_j \mid \{y_{j+1}, y_{v-j}\} \not\subset F\}$$

and S_u, an u element even subset of $Y = \{y_1, \ldots, y_v\}$.

If $F \in \mathcal{F}_j$ and $j \geq 2$ is even then $[y_{v-1}, y_v] \subset F$, $\mathcal{V}(F)$ is an even subset of $Y \cup \{y_{v+1}\}$, and the cyclic property of $Q'/[y_{v-1}, y_v]$ implies that $F \in \mathcal{F}(Q')$; that is, y_{v+1} is beneath F. Similarly, we obtain that y_{v+1} is beneath each $F \in \mathcal{F}(Q) \backslash \mathcal{F}_1$ with $[y_{v-3}, y_{v-2}]$ or $[y_{v-2}, y_{v-1}]$ in F, and beyond each F such that $F \in \mathcal{F}_j \backslash \mathcal{F}_{j+1}$ for odd $j \geq 3$ or $F \in \mathcal{F}_1 \backslash \{[y_1, S_{d-2}, y_v] \mid S_{d-2} \subset \{y_2, \ldots, y_{v-3}\}\}$.

In order to prove that y_{v+1} is beyond $[y_1, S_{d-2}, y_v]$ for $S_{d-2} \subset \{y_2, \ldots, y_{v-3}\}$ and beneath $[S_d]$ for $S_d \subset \{y_1, \ldots, y_{v-3}\}$, we observe that

a) $\{[y_1, \ldots, y_{d-1}, y_d], [y_1, \ldots, y_{d-1}, y_{v+1}]\} \subset \mathcal{F}(Q')$ implies that y_{v+1} is beyond $[y_1, \ldots, y_{d-1}, y_v] \in \mathcal{F}_1$, and
b) if G is a $(d-3)$-face of Q and $\{F_1, F_2\}$ and $\{F_2, F_3\}$ are adjacent pairs of facets of Q such that $G = F_1 \cap F_2 \cap F_3$, y_{v+1} is beyond F_2 and beneath F_1 and F_3 then y_{v+1} is beneath each $F \in \mathcal{F}(Q)$ such that $G \subset F \neq F_2$.

We note that in b): $F_1 \cap F_2$ and $F_2 \cap F_3$ are $(d-2)$-faces of Q, and "beyond" and "beneath" are interchangeable.

Let $F = [y_1, S_{d-2}, y_v]$, $S_{d-2} \subset \{y_2, \ldots, y_{v-3}\}$. By a), we may assume that there is a $d \leq t \leq v-3$ such that if $S'_{d-2} \subset \{y_2, \ldots, y_{t-1}\}$ then y_{v+1} is beyond $[y_1, S'_{d-2}, y_v]$. Let $y_t \in S_{d-2} \subset \{y_2, \ldots, y_t\}$, $S_{d-2} = S_{d-4} \cup \{y_{t-1}, y_t\}$, $G = [y_1, S_{d-4}, y_v]$ and $F_1 = [G, y_{v-2}, y_{v-1}]$. Then $F_1 \in \mathcal{F}_j \backslash \mathcal{F}_{j+1}$ for some odd $j \geq 3$ and y_{v+1} is beyond F_1. Next, there exist $2 \leq r < s \leq t-1$ such that $S_{d-4} \cap \{y_r, y_s\} = \emptyset$ and, $\{y_1\} \cup S_{d-4} \cup \{y_r\}$ and $S'_{d-2} = \{y_r\} \cup S_{d-4} \cup \{y_s\}$ are even sets. Then y_{v+1} is beneath $F_2 = [G, y_r, y_{v-1}] \in \mathcal{F}_2$ and beyond $F_3 = [G, y_r, y_s] = [y_1, S'_{d-2}, y_v]$.

Let $F = [S_d]$, $S_d \subset \{y_1, \ldots, y_{v-3}\}$. By a), we may assume that there is a $d+1 \leq t \leq v-3$ such that if $S'_d \subset \{y_1, \ldots, y_{t-1}\}$ then y_{v+1} is beneath $[S'_d]$. Let $y_t \in S_d \subset \{y_1, \ldots, y_t\}$, $S_{d-2} \cap \{y_{t-1}, y_t\} = \emptyset$ and $G = [S_{d-2}]$. As already noted, y_{v+1} is beneath $F_1 = [G, y_{v-1}, y_v]$. Next, there exist $1 \leq r < s \leq t-1$ such that $S_{d-2} \cap \{y_r, y_s\} = \emptyset$, $S'_d = S_{d-2} \cup \{y_r, y_s\}$ is even and, either $y_r = y_1$ or $\{y_1, \ldots, y_{r-1}\} \subset S_{d-2}$. Thus, y_{v+1} is beneath $F_3 = [G, y_r, y_s] = [S'_d]$ and beyond $F_2 = [G, y_r, y_v] \in \mathcal{F}_1 \backslash \mathcal{F}_2$. □

3. Proof of Theorem A

In light of the observations 1.1 to 1.5, let $X = \{x_1, x_2, \dots, x_n\}$ be a k-inseparable set of n points in general position in $\mathbb{R}^d$; $n = d + 2k$, $d = 2m \geq 4$ and $k \geq 2$. Let $P = [X]$ and assume that the ordering is $x_1 < \cdots < x_n$. We show that

3.1. $F_i = [x_i, x_{i+1}, \dots, x_{i+d-1}] \in \mathcal{F}(P)$ *for* $i = 1, 2, \dots, n-d+1$,
3.2. P *is cyclic if* $k = 2$, *and*
3.3. X *has a* 2*-inseparable subset* X', *and* P *is obtained from* $[X']$ *by* $2k-4$ *sewings of the type described in* 2.5.

Proof [Proof of 3.1]. Since $\{F_1, F_{n-d+1}\} \subset \mathcal{F}(P)$, we assume that $2 \leq i \leq n-d$. Let $O_i = [x_i, x_{i+1}, \dots, x_{i+d-3}]$ and recall that $P_i^* = P/O_i$ is a convex polygon. Since $P/[x_{n-1}, x_n]$, $P/[x_n, x_1]$ and $P/[x_1, x_2]$ are cyclic $(d-2)$-polytopes with $n-2$ vertices and the induced ordering, it follows that $\{[x_{n-1}, x_n, O_i], [x_n, x_1, O_i]\} \subset \mathcal{F}(P)$ for each i and $[x_1, x_2, O_i] \in \mathcal{F}(P)$ for $i \neq 2$.

Let $i = 2$. Then the preceding and $F_1 = [x_1, O_2, x_d]$ yield that $[x_{n-1}^*, x_n^*]$, $[x_n^*, x_1^*]$ and $[x_1^*, x_d^*]$ are edges of P_2^*, and $\langle x_n^*, x_d^* \rangle$ separates x_1^* and x_{n-1}^*. Since X is k-inseparable, we have that

$$\langle O_2, x_d, x_n \rangle \cap [x_{d+1}, \dots, x_{n-1}] = \emptyset = \langle O_2, x_d, x_{d+1} \rangle \cap [x_{d+2}, \dots, x_n],$$

and from this it readily follows that $[x_d^*, x_{d+1}^*]$ is an edge of P_2^*. Thus, $F_2 \in \mathcal{F}(P)$ and $\langle x_1, O_2, x_{d+1} \rangle$ separates x_d and $[x_{n-1}, x_n]$.

It is now easy to check that iterations of the above argument yield $F_i \in \mathcal{F}(P)$ for $i \leq k$, and that $\langle x_1, O_i, x_{i+d-2} \rangle = \langle x_1, x_i, O_{i+1} \rangle$ separates $[x_2, \dots, x_{i-1}]$ and $[x_{i+d-1}, \dots, x_n]$ for $3 \leq i \leq k+1$.

In summary: $\{F_1, \dots, F_k\} \subset \mathcal{F}(P)$ and $\langle x_1, x_{k+1}, O_{k+2} \rangle = \langle x_1, x_{k+1}, \dots, x_{d+k-1} \rangle$ separates $[x_2, \dots, x_k]$ and $[x_{k+d}, \dots, x_n]$. Then $n = d + 2k$ and reverse labelling yield $\{F_{k+2}, \dots, F_n\} \subset \mathcal{F}(P)$ and $\langle x_{k+2}, \dots, x_{d+k}, x_n \rangle = \langle O_{k+2}, x_{d+k}, x_n \rangle$ separates $[x_{d+k+1}, \dots, x_{n-1}]$ and $[x_1, \dots, x_{k+1}]$.

From P_{k+2}^*, we now readily obtain that $[x_{k+1}^*, x_{d+k}^*]$ is necessarily an edge, and hence, $[x_{k+1}, O_{k+2}, x_{d+k}] = F_{k+1} \in \mathcal{F}(P)$. □

Proof [Proof of 3.2]. We show first that $X = \{x_1, x_2, \dots, x_{d+4}\}$ is neighbourly and then apply 2.2 and 2.3.

Let $\tilde{X}$ be an m element subset of X. By 3.1, we may assume that $\tilde{X} \not\subset F_3$ and that, say, $\tilde{X} \cap \{x_1, x_2\} \neq \emptyset$. Then $[x_1, x_2] \in \mathcal{U}(X)$ and $|\tilde{X} \backslash \{x_1, x_2\}| \leq m-1$ yield that $[x_1, x_2, \tilde{X}]$ is a face of P. Since P is simplicial, $[\tilde{X}]$ is also a face of P.

Since P is neighbourly and $\{[x_{d+3}, x_{d+4}], [x_{d+4}, x_1], [x_1, x_2]\} \subset \mathcal{U}(P)$, it follows by 2.2 that each $G \in \{[x_{d+3}, x_{d+4}, x_1], [x_{d+4}, x_1, x_2]\}$ is disjoint from every hyperplane spanned by points from $X \backslash \mathcal{V}(G)$. Then X is 2-inseparable with respect to the cyclic ordering $x_1 < x_2 < \cdots < x_{d+4} < x_1$ and, as noted in 1.2, $P/[x_i, x_{i+1}]$ is a cyclic $(d-2)$-polytope with $d+2$ vertices for $i = 1, \ldots, d+4$ and $x_{d+5} = x_1$. Thus, $|\mathcal{U}(P)| = d+4$ and P is cyclic by 2.3. □

Proof [Proof of 3.3]. Let $k \geq 3$, $X' = \{x_1, x_2, x_{k+1}, \ldots, x_{k+d}, x_{n-1}, x_n\}$, $\tilde{X}$ be a set of three successive points in X' and H be a hyperplane spanned by points from $X' \backslash \tilde{X}$.

Since $H \cap (\tilde{X} \cup \{x_3, \ldots, x_k\} \cup \{x_{k+d+1}, \ldots, x_{k+d+k-2}\}) = \emptyset$ and X is k-inseparable, we may assume that $\tilde{X} \cap \{x_{k+1}, x_{k+d}\} = \emptyset$; that is, $\tilde{X} = \{x_i, x_{i+1}, x_{i+2}\} \subset \{x_{k+2}, \ldots, x_{k+d-1}\}$. Then $|X' \backslash \tilde{X}| = d+1$ implies that, say, $[x_1, x_2] \subset H$. Let $H = \langle x_1, x_2, y_1, \ldots, y_{d-2} \rangle$ with $\{y_1, \ldots, y_{d-2}\} \subset X' \backslash (\tilde{X} \cup \{x_1, x_2\})$. Since $\bar{P} = P/[x_1, x_2]$ is cyclic with $\bar{x}_3 < \cdots < \bar{x}_n$, it follows from 2.1 that $\{\bar{x}_3, \ldots, \bar{x}_n\}$ is 1-inseparable. Then $\langle \bar{y}_1, \ldots, \bar{y}_{d-2} \rangle \cap [\bar{x}_i, \bar{x}_{i+1}, \bar{x}_{i+2}] = \emptyset$ and $H \cap [\tilde{X}] = \emptyset$.

Since X' is 2-inseparable, $Q = [X']$ is a cyclic d-polytope with $x_1 < x_2 < x_{k+1} < \cdots < x_{k+d} < x_{d+2k-1} < x_{d+2k}$. Let $Q_0 = [Q, x_k]$. Then 1.2, 3.1, Lemma C and 2.5 yield that Q_0 is cyclic with $x_1 < x_2 < x_k < x_{k+1} < \cdots < x_{k+d} < x_{d+2k-1} < x_{d+2k}$.

It is now clear that $Q_i = [Q_{i-1}, x_{k-i}]$ is cyclic for $i = 1, \ldots, k-3$, and that with $P_0 = Q_{k-3} = [x_1, x_2, \ldots, x_{d+k}, x_{d+2k-1}, x_{d+2k}]$, $P_i = [P_{i-1}, x_{d+k+i}]$ is cyclic for $i = 1, \ldots, k-2$ and $P = P_{k-2}$. □

As a final comment, we recall that a neighbourly $2m$-polytope Q with v vertices is *almost-cyclic* if $|\mathcal{U}(Q)| = v-2$. In view of the preceding, it is natural to ask for example if almost-cyclic $2m$-polytopes with $n+2k$ or $n+2k+1$ vertices are characterizable in terms of ℓ-inseparability for some $\ell > k$?

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