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Simplices with congruent k-faces

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Abstract. Let S be a non-degenerate simplex in $\mathbb{R}^n$. We prove that S is regular if, for some $k \in \{1, \ldots, n-2\}$, all its k-dimensional faces are congruent. On the other hand, there are non-regular simplices with the property that all their $(n-1)$-dimensional faces are congruent.

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1. Introduction

Due to M. Zacharias [10], E. Genty and E. Lemoine observed about 1880 that the faces of a tetrahedron $T \subset \mathbb{R}^3$ are congruent if they have the same areas. These special three-dimensional simplices are usually called “isosceles tetrahedra” and have many interesting properties, cf. [5, §9] and [6, §1.4] for various references and [7] for a recent contribution. Inspired by this implication, H. Lenz asked in 1987 for analogous statements in $\mathbb{R}^n$, $n \geq 4$, see the book “Mathematical Problems” at the Oberwolfach institute.

P. Frankl and H. Maehara [2] showed that, for $n \geq 4$, an n-dimensional simplex all whose two-dimensional faces have equal areas is necessarily regular (this even holds in spaces of constant curvature, cf. [4]), and B. Weißbach [9] proved that this is no longer true for any $k \in \{3, \ldots, n-1\}$: there are non-regular n-dimensional simplices all whose k-dimensional faces have equal k-volumes. P. McMullen [8] found various further properties of these so-called *k-equiareal simplices* in $\mathbb{R}^n$.

Based on these investigations it is natural to replace the criterion of k-equiareality by the stronger assumption that all k-dimensional faces of an n-dimensional simplex be even *congruent*.

Our main result holds for $n \geq 3$ and any $k \in \{1, \ldots, n-2\}$: An n-dimensional simplex $S \subset \mathbb{R}^n$ is *regular* if all its k-dimensional faces are congruent. (It should be noticed that J. Horváth [3] proved this already for $k \in \{3; 4\}$. He also posed the respective question for higher dimensions.) Since the analogous assumption for the case $n = 2, k = 1$ yields equilateral triangles, and for $n = 3, k = 2$ the above mentioned isosceles tetrahedra are obtained, it remains to look at the cases $n \geq 4, k = n-1$. It turns out that, since an

n-dimensional simplex remains realizable in $\mathbb{R}^n$ if its edge lengths are slightly disturbed, for any of these cases *non-regular* n-dimensional simplices with congruent $(n-1)$-dimensional faces exist. The latter observations were also obtained by other authors, see §3 below.

2. A characterization of regular n-simplices

In this section we want to show that a given n-dimensional simplex is regular if there exists some k with $1 \leq k \leq n-2$ such that all its faces of dimension k are congruent. First we show the following purely combinatorial result.

PROPOSITION 2.1. *Assume M is a finite set with at least 4 elements, and let $\mathcal{P}_2(M)$ denote the family of all subsets of M with exactly 2 elements. Assume $f : \mathcal{P}_2(M) \to S$ is a map with values in some set S. Suppose that f satisfies the following condition:*

For all pairwise distinct elements $a, b, c \in M$ and $M' = M \backslash \{a, b, c\}$ there exists some bijection $\sigma : M' \to M'$ with $f(\{a, x\}) = f(\{b, \sigma(x)\})$ for all $x \in M'$.

Then, with the abbreviation $f_{ab} = f_{ba} := f(\{a, b\})$ for $a, b \in M$ with $a \neq b$, the following holds:

i) *If $a, b, c, d \in M$ are pairwise distinct and $f_{ac} \neq f_{ad}$, then one has*

$$f_{ac} = f_{bc} \text{ and analogously } f_{ad} = f_{bd}.$$

ii) *For all pairwise distinct $a, c, d \in M$ one has* $f_{ac} = f_{ad}$.

iii) *The map f is constant; that is, one has $f_{ab} = f_{cd}$ whenever $a, b, c, d \in M$ satisfy $a \neq b$ and $c \neq d$.*

Proof. i) Put $M'' := M \backslash \{a, b, c, d\}$. By the condition on f we obtain

$$n_1 := \sharp\{x \in M'' \cup \{d\} | f_{ax} = f_{ac}\} = \sharp\{y \in M'' \cup \{d\} | f_{by} = f_{ac}\}$$

as well as

$$n_2 := \sharp\{x \in M'' \cup \{c\} \mid f_{ax} = f_{ac}\} = \sharp\{y \in M'' \cup \{c\} \mid f_{by} = f_{ac}\}.$$

Now the assumption $f_{ac} \neq f_{ad}$ yields $n_2 = n_1 + 1$ and thus $f_{bc} = f_{ac}$.

ii) Assume ii) fails for certain pairwise distinct $a, c, d \in M$, and put $M' := M \backslash \{a, c, d\}$. Then i) yields for all $x \in M'$: $f_{ac} = f_{xc}$ and $f_{ad} = f_{xd}$. On the other hand, the assumption of our proposition implies that there exists some bijection $\sigma : M' \to M'$ with $f_{cx} = f_{d\sigma(x)}$ for all $x \in M'$. Thus we get for all $x \in M'$: $f_{ac} = f_{xc} = f_{cx} = f_{d\sigma(x)} = f_{\sigma(x)d} = f_{ad}$, a contradiction!

iii) is now a direct consequence of ii), because we may assume that a, b, c, d are pairwise distinct, and then we get $f_{ab} = f_{ac} = f_{cd}$.

□

Based on Proposition 2.1, we can now prove

PROPOSITION 2.2. *Assume $n \geq 3$, and $S \subseteq \mathbb{R}^n$ be a non-degenerate simplex such that all faces of dimension $n-2$ are congruent. Then S is regular; that means, all edges of S exhibit the same length.*

Proof. Let $V = V(S) = \{P_0, \ldots, P_n\}$ denote the vertex set of S, put $M := \{0, 1, \ldots, n\}$, and for $i, j \in M$ with $i \neq j$ let l_{ij} denote the Euclidean distance between P_i and P_j. By Proposition 2.1, it suffices to prove the following: For pairwise distinct $i, j, k \in M$ and $M' := M \backslash \{i, j, k\}$ there exists some bijection $\sigma : M' \to M'$ with $l_{i\nu} = l_{j\sigma(\nu)}$ for all $\nu \in M'$.

Consider the $(n-2)$-dimensional faces $S_1 :=$ conv $(\{P_0, \ldots, P_n\} \backslash \{P_j, P_k\})$ and $S_2 :=$ conv $(\{P_0, \ldots, P_n\} \backslash \{P_i, P_k\})$, where "conv" means convex hull.

By our assumption, there exists some congruence transformation $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ which maps S_1 onto S_2. In particular, for every positive number l, the number of edges of S_1 not belonging to S_2 and exhibiting the length l equals the number of edges of this length belonging to S_2 but not to S_1. This implies our assertion, because the edges of S_1 not belonging to S_2 are the line segments $\overline{P_i P_\nu}$ for $\nu \in M'$, while $\overline{P_j P_\mu}$, $\mu \in M'$, are the edges of S_2 not belonging to S_1. □

REMARK. In the last proof, some edge $\overline{P_i P_\nu}$, $\nu \in M'$, is not necessarily mapped to $\overline{P_j P_{\sigma(\nu)}}$ under α.

More generally, we can now also prove

THEOREM 2.3. *Assume $n \geq 3$, and $S \subseteq \mathbb{R}^n$ be a non-degenerate simplex such that for some k with $1 \leq k \leq n-2$ all k-dimensional faces of S are congruent. Then S is regular.*

Proof. By Proposition 2.2, applied to $n' := k+2$, all $(k+2)$-dimensional faces are regular. Thus the assertion follows from the fact that any two edges of S belong to some $(k+2)$-dimensional face. □

3. Simplices with congruent $(n-1)$-faces

In contrast to the last section, for $n \geq 3$ there exist n-dimensional simplices which are not regular but have the property that all their faces of dimension $n-1$ are congruent. (For $n = 2$, such simplices do not exist since each triangle with congruent sides is regular.) Namely, by using the theory of eigenvalues it is already proved in [1, Lemma 1] that a non-degenerate simplex remains realizable in Euclidean space under any slight disturbance of its edge lengths. Thus we can prove

PROPOSITION 3.1. *For any $n \geq 3$, there exists a non-degenerate simplex $S \subseteq \mathbb{R}^n$ which is not regular, but has the property that all its faces of dimension $n-1$ are congruent.*

Proof. Let $P_0, \ldots, P_n$ denote the vertices of some regular simplex S_0 in $\mathbb{R}^n$ all whose edges exhibit length 1, say. By the arguments from [1] there exists some $\delta \in (0,1)$ and some non-degenerate simplex S with vertices $P'_0, \ldots, P'_n$ and edge lengths

$$\begin{aligned} \|P'_i - P'_j\| &= 1 + \delta \text{ for } |i-j| \geq 2 \text{ but } \{i,j\} \neq \{0,n\}, \\ \|P'_i - P'_j\| &= 1 \text{ for } \{i,j\} = \{0,n\} \text{ or } |i-j| = 1. \end{aligned}$$

Thus, the graph with vertex set $\{P'_0, \ldots, P'_n\}$ and edge set $\{\{P'_i, P'_j\} : \|P'_i - P'_j\| = 1\}$ is a cycle of length $n+1$. By symmetry, it is clear that all $(n-1)$-dimensional faces of S are congruent. However, S is not regular since $n > 2$. □

B. Weissbach (private communication) informed the authors to have a further, independent approach to this result, and we also refer to [3].

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