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Sets expressible as finite unions of starshaped sets

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Abstract. Several papers have appeared with descriptions of sets that are expressible as unions of 2 or 3 starshaped sets. The general problem is relevant since it is closely related to the classical "*Art Gallery Problem*". Some new solutions, for generic values of the parameters, are presented here. These solutions are adaptations of known characterizations of starshaped sets.

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1. The main problem

In the recent literature on Convexity and Combinatorial Geometry have appeared several partial solutions of the following problem:

PROBLEM 1. Describe geometrical conditions that assure that a certain set $S \subset E^d$ is expressible as the union of k starshaped subsets.

This problem is a generalization of the classical "*Art Gallery Problem*" stated by Victor Klee more than a score of years ago. The statement of the present problem depends on two parameters, the dimension d of the space and the fixed number k of subsets. We remark that the solutions already known are conceived for particular values of these parameters. For instance, [6] gives a solution for $k = 2$ and arbitrary d. The same happens with [4]. On the other hand, [1], [2] and [3] deal with the case $d = 2$ and $k = 2$. Finally, [5] gives a solution for $d = 2$ and $k = 3$. Our aim in this paper is to provide solutions to the general problem, where the parameters d and k are arbitrary fixed positive integers. These new solutions are adaptations of known characterizations of starshaped sets.

2. Notations and first definitions

The environment space is a locally convex topological vector space E, that eventually will be explicitly endowed of dimensional and/or topological properties. The Euclidean d-dimensional space will be denoted E^d. Given two distinct points a and b of E, the *open segment* determined by them is denoted (a, b) and the substitution of one or both parenthesis by square brackets indicates the adjunction of the corresponding extremes. The *ray* (or *halfline*) issuing from a and going through b is $R(a \to b)$. The complement, closure, interior, boundary, convex hull and affine hull of a set S will be denoted S^C, $cl\ S$, $int\ S$, ∂S, $conv\ S$, $aff\ S$, respectively. If S is a subset of E and $a \in S$, $b \in S$ we say that *a sees b via S* if $[a, b] \subset S$. It is clear that the relation of visibility is symmetric. If $x \in S$ we define the *star of x in S* as the set $st(x, S) = \{y \in S \mid x \text{ sees } y \text{ via } S\}$. A *star point* of S is a point $x \in S$ such that $st(x, S) = S$. The *mirador* of S is the set $mir\ S$ of all star points of S. Other words have been used in the literature to describe the mirador: *kernel, convex kernel, hub*. We prefer the word "*mirador*" since "*kernel*" is used in Algebra and in Real Analysis with different meanings. S is a *convex set* if $mir\ S = S$, or, equivalently, if every point of S is a star point. S is *starshaped* if $mir\ S \neq \emptyset$, i.e. if S admits at least one star point. We say that a point $p \in S$ is a *point of local convexity* of S if there exists a neigborhood $\mathcal{U}_p$ of p such that the set $(\mathcal{U}_p \cap S)$ is convex. Otherwise, we say that p is a *point of local nonconvexity* of S. The set of all points of local convexity of S is denoted $lc\ S$, and the set of all points of local nonconvexity is $lnc\ S$. It is clear that for each set S it holds $lnc\ S \subset \partial S$.

We will denote

$$J(x, K) = \bigcup_{y \in K} [x, y]$$

If A and B are two sets, we denote

$$d(A, B) = \inf \{\|x - y\| \mid x \in A,\ y \in B\} \quad \textit{and} \quad d(x, S) = d(\{x\}, S)$$

Finally, we have

$$B(A, \delta) = \{y \in E \mid d(y, A) \leq \delta\}$$

3. Convex components

A *convex component* K of the set S is a maximal convex subset of S. This idea has been used fruitfully to describe the geometry of nonconvex sets. The name was coined in [8] where it was proved the following theorem:

THEOREM 2. *Let S be a nonconvex set and* $\mathcal{F} = \{K_\lambda \mid \lambda \in L\}$ *a covering family of convex components of S. Then* $mir\ S = \bigcap_{\lambda \in L} K_\lambda$.

A *bunch of sets* is a family of sets $\mathcal{F} = \{M_\lambda \mid \lambda \in L\}$ such that $\bigcap_{\lambda \in L} M_\lambda$ is not empty. We say that a family of sets is *k-bunched* if it is partitioned into precisely k subfamilies, each of them a bunch. If d is a positive integer, we say that a family of sets $\mathcal{F}$ is *d-Helly* if every subfamily of $\mathcal{F}$ having $d + 1$ (or less) members is a bunch. From Theorem 2 it follows easily that a set S is starshaped if and only if it is the union of a bunched family of convex sets.

THEOREM 3. *Let S be a nonvoid set. The following statements are equivalent:*

1. *S is expressible as the union of precisely k starshaped sets.*
2. *S admits a covering family of convex components that is k-bunched.*

Proof. (1. $\Longrightarrow$ 2.) Assume that $S = \bigcup_{i=1}^{k} S_i$ where each S_i is nonvoid and starshaped. As we have just remarked, for each i between 1 and k there exists a family $\mathcal{K}_i = \{K_\lambda^i \subset \mid \lambda \in L_i\}$ of convex components of S_i such that $\bigcup_{\lambda \in L_i} K_\lambda^i = S_i$ and $\bigcap_{\lambda \in L_i} K_\lambda^i = mir\, S_i$ that is not empty. Furthermore, $\forall\, i\ \forall\, \lambda$ there exists $\widehat{K}_\lambda^i$ convex component of S such that $K_\lambda^i \subset \widehat{K}_\lambda^i$. Hence, the family

$$\widehat{\mathcal{K}} = \bigcup_{i=1}^{k} \{\widehat{K}_\lambda^i \mid \lambda \in L_i\}$$

is a covering family of convex components of S that is clearly k-bunched.

(2. $\Longrightarrow$ 1.) Assume that $\mathcal{F}$ is a covering family of convex components of S such that $\mathcal{F} = \bigcup_{i=1}^{k} \mathcal{F}_i$ where each $\mathcal{F}_i$ is a bunch. Hence, for each i between 1 and k the set $S_i = \bigcup_{K \in \mathcal{F}_i} K$ is starshaped, and $S = \bigcup_{i=1}^{k} S_i$. □

It is important to remark that Theorem 3 is valid without any topological and/or dimensional restriction.

COROLLARY 4. *Let S be a compact subset of* $\mathbf{E}^d$. *The following statements are equivalent:*

1. *S is expressible as union of precisely k starshaped sets.*
2. *S admits a covering family of convex components and a partition of this family into k d-Helly subfamilies.*

Proof. The well known theorem of Helly on intersections of convex sets imply that a family of compact convex subsets of $\mathbf{E}^d$ that is d-Helly is a bunch. Let us remark that the compactness of S can be dispensed if we ask that the family appearing in statement 2 be finite. □

4. Points having better visibility

Let $x \in S$ and $y \in S$. We say that x has *better visibility via S* than y if $st(y, S) \subset st(x, S)$. Let $M \subset S$, we say that p is an S-boss *of M via S* if p has better visibility via S than any point of M. A *platoon* is a subset P of S that admits an S-boss.

THEOREM 5. *Let S be a nonconvex hunk. If lnc S admits a partition into k platoons, then S is expressible as union of at most k starshaped sets.*

Proof. Assume that $lnc\, S = \bigcup_{i=1}^{k} P_i$ where each set P_i admits an S-boss b_i. For each i between 1 and k denote $S_i = st(b_i, S)$. These sets are clearly starshaped. We claim that they cover S. Let x be a generic point of S. From a result of Valentine [9] there exists $y \in lnc\, S$ such that $x \in st(y, S)$. But according to the hypothesis there is an index j between 1 and k such that $y \in P_j$. Hence, $x \in st(y, S) \subset st(b_j, S) = S_j$. □

The converse implication is false, as the following counterexample shows.

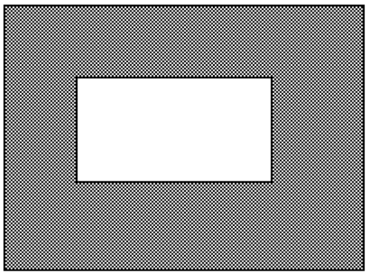

Figure 1

EXAMPLE 6. The shaded set in Figure 1 can be easily partitioned into two starshaped sets but its set of local non-convexity points (the four inner corners) admits no partition into two platoons.

5. Clear visibility

We say that x *sees clearly* y *via* S if y admits a neighborhood $\mathcal{U}_y$ such that x sees the whole $(\mathcal{U}_y \cap S)$ via S. We remark that, contrary to what happens with simple visibility, the relation of clear visibility is not symmetrical. The idea of *clear visibility* was introduced by N. Stavrakas [7] in 1972. If $M \subset S$ and $x \in S$, we say that x is an S *-gazebo of* M if x sees clearly via S every point of M. Let $M \subset S$, then we will denote $\mathcal{N}_M = M \cap lnc\, S$.

LEMMA 7. *Let S be a nonconvex, compact and connected set. Let K be a convex component of S, and $x \in S$ an S-gazebo of $\mathcal{N}_K$. Then $x \in K$.*

Proof. Assume that x is not in K and denote $J = J(x, K)$. Define Q as the boundary of K in the relative topology of J, i. e.

$$Q = \{t \in K \mid \forall \varepsilon > 0 \ \ it\ holds\ \ U(p, \varepsilon) \cap K \neq \emptyset \ \ and\ U(p, \varepsilon) \cap (\sim K) \neq \emptyset\}.$$

For each $p \in \mathcal{N}_K$ denote U_p an open neighborhood of p such that $U_p \cap S$ is totally visible from x, and $C_p = J(x, U_p) \sim \{x\}$. Hence, each C_p is open and so is $C_K = \bigcup_{p \in \mathcal{N}_K} C_p$.

This implies that $\widetilde{Q} = Q \sim C_K$ is compact and free from points of local nonconvexity of S. If we denote $F = (J \sim C_K) \cap lnc\, S$, this is another compact set disjoint from $\widetilde{Q}$. Hence there exists $\delta > 0$ such that $B(\widetilde{Q}, \delta) \cap F = \emptyset$. For each $q \in \widetilde{Q}$ define $\hat{q} \in (q, x)$ such that $d(\hat{q}, \widetilde{Q}) = \delta$ and denote $Q_\delta = \bigcup_{q \in \widetilde{Q}} [q, \hat{q}]$. It is clear that $Q_\delta \subset B(\widetilde{Q}, \delta)$. *We claim that* $Q_\delta \subset S$. Let T be a generic connected component of $\widetilde{Q}$ and denote $T_\delta = \bigcup_{q \in T} [q, \hat{q}]$. From the construction we know that T_δ is free from points of local nonconvexity of S. Moreover, Q_δ is the union of all such sets T_δ generated by its connected components. If all the points of T are S-visible from x, then $T_\delta \subset S$, clearly. Now assume that there are points of T not visible from x. In this case we claim that the interior of T (in the relative topology of Q) is not empty. Otherwise, every point of T would be a limit of points of $C_K \subset st(x, S)$ and, since this star is closed, it would hold $T \subset st(x, S)$, in contradiction with our assumption. Our claim implies that T_δ has nonempty interior. Since S^C is open, the set $T_\delta \cap S^C$ would include an open ball. Denote U the largest open ball such that $U \subset T_\delta \cap S^C$. Then, owing to the maximality of U, $\partial U \cap \partial S$ would not be empty and we would be able to pick $z \in \partial U \cap \partial S$ and (since $z \in lc\, S$) a hyperplane H_z that locally separates U from S. Now pick $w \in S \cap cl\, T_\delta$ and w "above" H_z, i.e. in the same side of H_z as U. For instance, if q is a point in the relative boundary of T, we can pick $w = \hat{q}$ defined as above and with the required property. It is clear that under these circumstances, z and w do not see each other via S. Define Λ as an arc of curve of minimal length included in T_δ and joining z with w. Since Λ is not a segment, it must include a point m of local nonconvexity of S. This is a contradiction since $m \in T_\delta \subset Q_\delta \subset B(\widetilde{Q}, \delta)$ that is free from points of local nonconvexity. Hence, $T_\delta \subset S$ and our claim is proved. Now let us define $\widehat{K} = J(x, K) \cap B(K, \delta)$. It is clear that $\widehat{K}$ would be convex and would include K properly. But our previous construction would yield that $\widehat{K} \subset S$ and this statement contradicts the maximality of K as a convex subset of S. This contradiction originates at the assumption that x does not belong to K. □

THEOREM 8. *Let S be a nonconvex, compact and connected set. Assume that S admits a covering family $\mathcal{F}$ of convex components and a set of precisely k points $B = \{b_1; b_2; \ldots; b_k\} \subset S$ such that $\forall K \in \mathcal{F}\ \exists\, b_i \in B$ that is an S-gazebo of $\mathcal{N}_K$. Then S can be partitioned into at most k starshaped subsets.*

Proof. For each index i from 1 to k denote

$$S_i = \bigcup \{K_\lambda \in \mathcal{F} \mid b_i \in K_\lambda\}$$

Clearly, each of these sets is starshaped and they cover S. □

The converse implication of this result is false. The counterexample is the same set of Figure 1. As we mention above, this set S is the union of two starshaped sets, and it admits a minimal covering by four convex components $\{K_1\,; K_2\,; K_3\,; K_4\}$, each of them a rectangle. Nevertheless, it is impossible to find a pair $B = \{b_1 ; b_2\}$ of points that act as S-gazebos of $\mathcal{N}_{K_i}$ for $i = 1, \ldots, 4$.

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