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On conformally flat contact metric manifolds

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Abstract. In the present paper we classify the conformally flat contact metric manifolds of dimension $2n+1$ ($n > 1$) satisfying $R(\cdot,\xi)\xi = -k\phi^2$. We prove that these manifolds are Sasakian of constant curvature 1.

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1. Introduction

Let M^{2n+1} be a $(2n+1)$-dimensional contact metric manifold and (ϕ, ξ, η, g) be its contact metric structure. If the characteristic vector field ξ is Killing, then M^{2n+1} is called a K-contact Riemannian manifold. Moreover if the structure is normal, M^{2n+1} is said to be Sasakian. Every Sasakian manifold is K-contact but the opposite is true only for the dimension 3. We denote by ∇, R and Q the Levi-Civita connection, the Riemannian curvature and the Ricci operator on M^{2n+1} respectively. J. Davidov and O. Mushkarov [5] and S.R. Deng [6] gave results which show that there exist K-contact manifolds with $Q\phi = \phi Q$ which are not Sasakian. S. Tanno [11], generalizing the corresponding result of M. Okumura [8] for Sasakian manifolds, proved that every conformally flat K-contact manifold is a space form. Z. Olszak in [9] proved that any contact metric manifold of constant sectional curvature and of dimension ≥ 5 has the sectional curvature equal to 1 and is a Sasakian manifold. D.E. Blair and Th. Koufogiorgos proved in [4] that a conformally flat contact metric manifold M^{2n+1} with $Q\phi = \phi Q$ is of constant curvature 1 if $n > 1$ and 0 or 1 if $n = 1$. K. Bang in [1] proved that in dimension greater than 3, there are no conformally flat contact metric manifolds with $\ell = 0$, ($\ell := R(\cdot,\xi)\xi$). The first author and Ph.J. Xenos proved in [7] that every 3-dimensional conformally flat contact metric manifold satisfying $\nabla_\xi \tau = 0$ ($\tau = L_\xi g$) is either flat or a Sasakian space form. This result extends Bang's result.

An open problem in Riemannian Geometry is the classification of the Riemannian manifolds (M, g) satisfying the conditions: a) M admits a non-vanishing vector field ξ b) The Jacobi operator associated to ξ is proportional to the identity on the complementary subspaces. Motivating the above mentioned problem we study a subclass of these manifolds, i.e., contact metric manifolds of dimension $2n+1$ ($n > 1$) satisfying $\ell = -k\phi^2$, where k is a smooth function. The condition $\ell = -k\phi^2$ (which generalizes the K-contact condition) implies the condition $\nabla_\xi \tau = 0$ for dimensions ≥ 5 and is equivalent to the condition $\nabla_\xi \tau = 0$ for the dimension 3. We have to mention that D. Perrone in [10] gave results which show that there exist contact Riemannian manifolds with $\ell = -k\phi^2$ where k is not constant and D.E. Blair in [3] gave examples of 3-dimensional contact metric manifolds with $\ell = -k\phi^2$, where k is not constant.

The main result of the present paper is the following: Let M^{2n+1} $(n > 1)$ be a conformal1y flat contact metric manifold satisfying $\ell = -k\phi^2$ for some smooth function k on M^{2n+1}. Then M^{2n+1} is Sasakian of constant curvature 1. The above result generalizes the corresponding results of S. Tanno [11] for K-contact manifolds and of the first author and Ph. J. Xenos for the dimension 3.

2. Preliminaries

A *contact metric manifold* is a $(2n+1)$-dimensional differentiable manifold M^{2n+1} which carries a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. Every contact metric manifold has an underlying *contact Riemannian structure* (ϕ, ξ, η, g), where ξ is a global vector field (called the *characteristic vector field*), ϕ a global tensor field of type (1,1) and g a Riemannian metric (called *associated metric*). We note that the set of all contact Riemannian structures associated to a fixed contact form is infinite dimensional. The tensor fields ϕ, ξ, η, g satisfy:

$$\begin{aligned} &\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(X) = g(\xi, X), \\ &d\eta(X, Y) = g(X, \phi Y), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \end{aligned} \tag{1}$$

From now on we assume M^{2n+1} is a contact metric manifold with contact metric structure (ϕ, ξ, η, g).

We denote by L Lie differentiation and we define the tensor fields h and ℓ by

$$h = \tfrac{1}{2} L_\xi \phi \quad and \quad \ell = R(\cdot, \xi)\xi. \tag{2}$$

The tensor field h is symmetric and satisfies

$$Trh\phi = Trh = 0, \quad h\xi = 0, \quad h\phi = -\phi h. \tag{3}$$

On every contact metric manifold M^{2n+1} the following relations are valid:

$$\nabla_X \xi = -\phi X - \phi h X \quad and \quad hence \quad \nabla_\xi \xi = 0, \tag{4}$$

$$\nabla_\xi \phi = 0, \tag{5}$$

$$Tr\ell = g(Q\xi, \xi) = 2n - Trh^2, \tag{6}$$

$$\phi\ell\phi - \ell = 2(\phi^2 + h^2), \tag{7}$$

$$\nabla_\xi h = \phi - \phi\ell - \phi h^2, \tag{8}$$

M^{2n+1} is said to be K-contact manifold if ξ is a Killing vector field. M^{2n+1} is K-contact iff $h = 0$ or iff $\ell = -\phi^2$ [2]. On $M^{2n+1} \times \mathcal{R}$ ($\mathcal{R}$ is the real line) we can define an almost complex structure J by $J(X, fd/dt) = (\phi X - f\xi, \eta(X)d/dt)$, where f is a real valued function. If J is integrable, the contact structure on M^{2n+1} is called *normal* and M^{2n+1} is said to be *Sasakian.* A necessary and sufficient condition in order to M^{2n+1} be Sasakian is [2]:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{9}$$

A Riemannian manifold (M^n, g) is said to be conformally flat if it is conformally equivalent to a Euclidean space. A necessary and sufficient condition in order that M^n is conformally flat is [4]:

$$\begin{aligned} R(X, Y)Z = &\frac{1}{n-2}(g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y) \\ &- \frac{r}{(n-1)(n-2)}(g(Y, Z)X - g(X, Z)Y), \textit{for } n > 3 \end{aligned} \tag{10}$$

and

$$(\nabla_X P)Y = (\nabla_Y P)X, \quad \textit{for} \quad n = 3,$$

where $r = TrQ$ is the scalar curvature of M^n and $P = -Q + \frac{r}{4}Id$.

3. Auxiliary results

Let M^{2n+1} be a contact metric manifold satisfying

$$\ell = -k\phi^2, \tag{11}$$

where k is a smooth function of M^{2n+1}. We prove the following

PROPOSITION 3.1. *On every contact metric manifold M^{2n+1} satisfying* (11) *we have $k \le 1$. If $k < 1$, then M^{2n+1} admits three mutually orthogonal distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$, defined by the eigenspaces of h, where $\lambda = \sqrt{1-k}$.*

Proof. The operator h is symmetric and hence diagonalizable, i.e., there exists an orthonormal frame of eigenvectors. Furthermore, $h\xi = 0$, and hence ξ is one of these eigenvectors. Now, let $X \perp \xi$ be another eigenvector with associated eigenvalue λ, i.e., $hX = \lambda X$. Then, ϕX is also an eigenvector of h with associated eigenvalue $-\lambda$. Also, on the one hand

$$h^2 X = \lambda^2 X.$$

On the other hand, it follows from

$$2(\phi^2 + h^2) = \phi\ell\phi - \ell = -k\phi^4 + k\phi^2 = 2k\phi^2$$

that

$$h^2 = (k-1)\phi^2,$$

and hence

$$h^2 X = (1-k)X.$$

The result follows immediately. □

PROPOSITION 3.2. *Let M^{2n+1} be a contact metric manifold satisfying $\ell = -k\phi^2$ for some smooth function k on it. Then, the distributions $D(\lambda)$ and $D(-\lambda)$ are parallel along ξ.*

Proof. Suppose X is a unit vector field belonging to $D(\lambda)$, i.e., $hX = \lambda X$. On the one hand, we see that

$$\nabla_\xi (hX) = (\xi \cdot \lambda)X + \lambda \nabla_\xi X$$

while, on the other hand,

$$\nabla_\xi (hX) = (\nabla_\xi h)X + h(\nabla_\xi X) = \phi X - \phi\ell X - \phi h^2 X + h(\nabla_\xi X) = h(\nabla_\xi X),$$

and hence

$$h(\nabla_\xi X) = (\xi \cdot \lambda)X + \lambda \nabla_\xi X.$$

Taking the inner product with X and using the fact that $g(X, \nabla_\xi X) = 0$, we see that $\xi \cdot \lambda = 0$ and that $\nabla_\xi X$ again belongs to $D(\lambda)$. □

Let $\{e_1, \ldots, e_n, e_{n+1} = \phi e_1, \ldots, e_{2n} = \phi e_n, \xi\}$ be an orthonormal frame, formed from the unit eigenvectors of h: $e_i \in D(\lambda)$, $\phi e_i \in D(-\lambda)$ and ξ.

A consequence of Proposition (3.2) is that

$$\nabla_\xi e_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} e_j, \tag{12}$$

$$\nabla_\xi \phi e_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} \phi e_j, \tag{13}$$

where

$$a_{ij} = -a_{ji} \tag{14}$$

If we differentiate the inner products $g(e_i, e_j)$, $g(e_i, \xi)$, $i, j = 1, \ldots, 2n$ with respect to e_k, $k = 1, \ldots, 2n$ we obtain

$$
\begin{aligned}
\nabla_{e_i} e_i &= \sum_{\substack{k=1\\k\neq i}}^{n} A_{ik} e_k + \sum_{\substack{k=1\\k\neq i}}^{n} \overline{A}_{ik} \phi e_k + A_i \phi e_i,\\
\nabla_{\phi e_i} \phi e_i &= \sum_{\substack{k=1\\k\neq i}}^{n} B_{ik} e_k + \sum_{\substack{k=1\\k\neq i}}^{n} \overline{B}_{ik} \phi e_k + B_i e_i,\\
\nabla_{e_i} e_j &= -A_{ij} e_i + \sum_{\substack{k=1\\i\neq k\neq j}}^{n} C^k_{ij} e_k + \sum_{k=1}^{n} \overline{C}^k_{ij} \phi e_k, \quad i \neq j,\\
\nabla_{\phi e_i} \phi e_j &= -\overline{B}_{ij} \phi e_i + \sum_{k=1}^{n} D^k_{ij} e_k + \sum_{\substack{k=1\\i\neq k\neq j}}^{n} \overline{D}^k_{ij} \phi e_k, \quad i \neq j,\\
\nabla_{e_i} \phi e_j &= -\overline{A}_{ij} e_i - \sum_{\substack{k=1\\i\neq k\neq j}}^{n} \overline{C}^j_{ik} e_k - \overline{C}^j_{ij} e_j\\
&\quad - Z_{ij} \phi e_i + \sum_{\substack{k=1\\i\neq k\neq j}}^{n} N^k_{ij} \phi e_k, \quad i \neq j,\\
\nabla_{\phi e_i} e_j &= -E_{ij} e_i - B_{ij} \phi e_i - D^j_{ij} \phi e_j\\
&\quad - \sum_{\substack{k=1\\i\neq k\neq j}}^{n} D^j_{ik} \phi e_k + \sum_{\substack{k=1\\i\neq k\neq j}}^{n} F^k_{ij} e_k, \quad i \neq j,\\
\nabla_{e_i} \phi e_i &= -A_i e_i - \sum_{\substack{k=1\\k\neq i}}^{n} \overline{C}^i_{ik} e_k + \sum_{\substack{k=1\\k\neq i}}^{n} Z_{ik} \phi e_k + (1+\lambda)\xi,\\
\nabla_{\phi e_i} e_i &= -B_i \phi e_i - \sum_{\substack{k=1\\k\neq i}}^{n} D^i_{ik} \phi e_k + \sum_{\substack{k=1\\k\neq i}}^{n} E_{ik} e_k - (1-\lambda)\xi,
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
&N^k_{ij} = -N^j_{ik}, \quad C^k_{ij} = -C^j_{ik}, \quad F^k_{ij} = -F^j_{ik}, \quad \overline{D}^k_{ij} = -\overline{D}^j_{ik},\\
&i, j, k \in \{1, 2, \ldots, n\}, \quad i \neq k \neq j.
\end{aligned}
\tag{16}
$$

4. Conformally flat contact metric manifolds with $\ell = -k\phi^2$

Let M^{2n+1} be a conformally flat contact metric manifold satisfying $\ell = -k\phi^2$. If $n = 1$ the manifold is either flat or a Sasakian manifold of constant curvature 1 [7]. We will investigate the case that $n > 1$.

LEMMA 4.1. *Let M^{2n+1} $(n > 1)$ be a conformally flat contact metric manifold. A necessary and sufficient condition in order that the conditions $\ell = -k\phi^2$ and $Q\phi = \phi Q$ are equivalent is that $Q\xi$ is collinear with ξ. In this case $k = 1$.*

Proof. We suppose that $\ell = -k\phi^2$. The relation (10) for $Y = Z = \xi$ takes the form

$$\ell X = \frac{1}{2n-1}(QX - \eta(X)Q\xi + Tr\ell X - \eta(QX)\xi) - \frac{r}{2n(2n-1)}(X - \eta(X)\xi). \tag{17}$$

Furthermore, from the relations (6) and (11) we have

$$\ell X = -\frac{Tr\ell}{2n}\phi^2 X. \tag{18}$$

Comparing the relations (17) and (18) we obtain:

$$QX = \frac{r - Tr\ell}{2n}X + \eta(X)Q\xi + \eta(QX)\xi - \frac{(2n-1)Tr\ell + r}{2n}\eta(X)\xi. \tag{19}$$

Suppose that $Q\xi$ is collinear to ξ. Then for every $Z \in \ker\eta$ we have that $\eta(QZ) = g(Z, Q\xi) = 0$. This relation and (19) yield

$$QZ = \frac{r - Tr\ell}{2n}Z, \quad \forall Z \in \ker\eta.$$

The above relation and $Q\xi = Tr\ell\xi$ imply $Q\phi = \phi Q$. Hence $k = 1$ [3]. On the other hand it has been proved in [4] that on every conformally flat contact metric manifold M^{2n+1} $(n > 1)$ the relation $Q\phi = \phi Q$ implies the relations $Q\xi = Tr\ell\xi$ and $\ell = -\phi^2$. □

From now on we suppose that $k < 1$.

If we calculate the curvature tensors $R(e_i, \phi e_j)\xi$, $R(e_i, e_j)\xi$, $R(\phi e_i, \phi e_j)\xi$ and $R(e_i, \phi e_i)\xi$, $i \neq j$, $i, j = 1, \ldots, n$, once using (10) and (19) and secondly by direct calculation using (11), (13), (14), (15) and the relation $\lambda = \sqrt{1-k}$ we obtain:

$$e_i \cdot \lambda = (1+\lambda)D^j_{ji} - (1-\lambda)B_{ji}, \tag{20}$$

$$\phi e_j \cdot \lambda = (1+\lambda)\overline{A}_{ij} - (1-\lambda)\overline{C}^i_{ij}, \tag{21}$$

$$\frac{1}{2n-1}\eta(Qe_i) = (1+\lambda)(\overline{B}_{ji} - E_{ji}) + 2\lambda\overline{C}_{ij}^j, \tag{22}$$

$$\frac{1}{2n-1}\eta(Q\phi e_j) = (1-\lambda)(Z_{ij} - A_{ij}) + 2\lambda D_{ji}^i, \tag{23}$$

$$(1+\lambda)D_{ji}^k - (1-\lambda)D_{jk}^i + (1-\lambda)C_{ij}^k - (1-\lambda)N_{ij}^k = 0, \quad i \neq k \neq j, \tag{24}$$

$$(1-\lambda)\overline{C}_{ij}^k - (1+\lambda)\overline{C}_{ik}^j + (1+\lambda)\overline{D}_{ji}^k - (1+\lambda)F_{ji}^k = 0, \quad i \neq k \neq j, \tag{25}$$

$$e_i \cdot \lambda = (1+\lambda)(A_{ji} - Z_{ji}), \tag{26}$$

$$e_j \cdot \lambda = (1+\lambda)(A_{ij} - Z_{ij}), \tag{27}$$

$$\frac{1}{2n-1}\eta(Qe_j) = (1+\lambda)\overline{A}_{ij} - (1-\lambda)\overline{C}_{ij}^i - 2\lambda\overline{C}_{ji}^i, \tag{28}$$

$$C_{ij}^k - C_{ji}^k - N_{ij}^k + N_{ji}^k = 0, \quad i \neq k \neq j, \tag{29}$$

$$(1+\lambda)(\overline{C}_{ik}^j - \overline{C}_{jk}^i) + (1-\lambda)(\overline{C}_{ji}^k - \overline{C}_{ij}^k) = 0, \quad i \neq k \neq j, \tag{30}$$

$$\phi e_i \cdot \lambda = (1-\lambda)(E_{ji} - \overline{B}_{ji}), \tag{31}$$

$$\phi e_j \cdot \lambda = (1-\lambda)(E_{ij} - \overline{B}_{ij}), \tag{32}$$

$$\frac{1}{2n-1}\eta(Q\phi e_i) = (1+\lambda)D_{ji}^j - (1-\lambda)B_{ji} - 2\lambda D_{ij}^j, \tag{33}$$

$$F_{ij}^k - F_{ji}^k - \overline{D}_{ij}^k + \overline{D}_{ji}^k = 0, \quad i \neq k \neq j, \tag{34}$$

$$(1-\lambda)(D_{jk}^i - D_{ik}^j) + (1+\lambda)(D_{ij}^k - D_{ji}^k) = 0, \quad i \neq k \neq j, \tag{35}$$

$$(1-\lambda)\overline{A}_{ij} - (1+\lambda)(\overline{C}_{ij}^i - \overline{B}_{ij} + E_{ij}) = 0, \tag{36}$$

$$(1+\lambda)B_{ij} - (1-\lambda)(Z_{ij} - A_{ij} + D_{ij}^i) = 0, \tag{37}$$

$$\frac{1}{2n-1}\eta(Qe_i) = 2\lambda A_i - \phi e_i \cdot \lambda, \tag{38}$$

$$\frac{1}{2n-1}\eta(Q\phi e_i) = 2\lambda B_i - e_i \cdot \lambda. \tag{39}$$

If we calculate the curvature tensors $R(e_i,\xi)e_j$, $R(\phi e_j,\xi)\phi e_i$, $R(\phi e_i,\xi)e_i$, $R(\phi e_i,\xi)\phi e_i$ and $R(e_i,\xi)\phi e_i$, $i \neq j$, $i, j = 1,\ldots,n$, once using (10) and (19) and secondly by direct calculation using (11), (13), (14) and (15) we obtain:

$$\begin{aligned} \xi\cdot A_{ij} &= \frac{1}{2n-1}\eta(Qe_j) - e_i\cdot a_{ji} + (1+\lambda)E_{ij} - a_{ij}A_{ji} + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{jk}A_{ik} \\ &\quad + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{ki}C^k_{ij} - \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{ik}C^i_{kj}, \end{aligned} \tag{40}$$

$$\begin{aligned} \xi\cdot D^j_{ji} &= a_{ij}B_j + (1-\lambda)\overline{A}_{ji} + a_{ji}B_{ij} + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{ik}D^j_{jk} \\ &\quad - \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{kj}D^k_{ji} - a_{ij}D^i_{ji} + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{jk}D^j_{ki}, \end{aligned} \tag{41}$$

$$\begin{aligned} \xi\cdot E_{ij} &= \phi e_i\cdot a_{ij} + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{ik}F^j_{ik} - \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{kj}E_{ik} - (1-\lambda)A_{ij} \\ &\quad - a_{ij}E_{ji} + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{ik}F^j_{ki}, \end{aligned} \tag{42}$$

$$\begin{aligned} \xi\cdot \overline{B}_{ij} &= \frac{1}{2n-1}\eta(Q\phi e_j) + \phi e_i\cdot a_{ij} + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{ik}\overline{D}^j_{ik} - \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{kj}\overline{B}_{ik} \\ &\quad - (1-\lambda)Z_{ij} - a_{ij}\overline{B}_{ji} + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{ik}\overline{D}^j_{ki}, \end{aligned} \tag{43}$$

$$\begin{aligned} \xi\cdot Z_{ij} &= e_i\cdot a_{ij} + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{ik}N^j_{ik} - \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{kj}Z_{ik} + (1+\lambda)\overline{B}_{ij} \\ &\quad - a_{ij}Z_{ji} + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} a_{ik}N^j_{ki}. \end{aligned} \tag{44}$$

LEMMA 4.2. *There are no conformally flat contact metric manifolds of dimension ≥ 5 satisfying $\ell = -k\phi^2$ for some constant $k < 1$.*

Proof. K. Bang proved in [1] that there are no conformally flat contact metric manifolds of dimension ≥ 5 satisfying $\ell = 0$. Hence $\lambda^2 \neq 1$. Using the relations (14), (26), (29), (32), (40), (44) and $\lambda^2 \neq 1$ we obtain

$$\xi \cdot A_{ij} - \xi \cdot Z_{ij} = \frac{1}{2n-1}\eta(Qe_j) + \frac{1+\lambda}{1-\lambda}\phi e_j \cdot \lambda - a_{ij}\frac{e_i \cdot \lambda}{1+\lambda} + \sum_{\substack{k=1 \\ i\neq k\neq j}}^{n} a_{jk}\frac{e_k \cdot \lambda}{1+\lambda}.$$

The above relation, using (14), takes the form

$$\xi \cdot A_{ij} - \xi \cdot Z_{ij} = \frac{1}{2n-1}\eta(Qe_j) + \frac{1+\lambda}{1-\lambda}\phi e_j \cdot \lambda + \sum_{\substack{k=1 \\ k\neq j}}^{n} a_{jk}\frac{e_k \cdot \lambda}{1+\lambda}. \tag{45}$$

If we differentiate with respect to ξ the relation (27) we obtain using the relations (4), (12) and $\xi \cdot \lambda = 0$,

$$\begin{aligned}\xi \cdot A_{ij} - \xi \cdot Z_{ij} &= \frac{\xi \cdot e_j \cdot \lambda - e_j \cdot \xi \cdot \lambda}{1+\lambda} = \frac{[\xi, e_j] \cdot \lambda}{1+\lambda} = \frac{(\nabla_\xi e_j - \nabla_{e_j}\xi) \cdot \lambda}{1+\lambda} \\ &= \frac{\sum_{\substack{k=1 \\ k\neq j}}^{n} a_{jk}(e_k \cdot \lambda)}{1+\lambda} + \phi e_j \cdot \lambda.\end{aligned}$$

Hence,

$$\xi \cdot A_{ij} - \xi \cdot Z_{ij} = \frac{\sum_{\substack{k=1 \\ k\neq j}}^{n} a_{jk}(e_k \cdot \lambda)}{1+\lambda} + \phi e_j \cdot \lambda. \tag{46}$$

Comparing the relations (45) and (46) we obtain

$$\eta(Qe_j) = -\frac{2\lambda(2n-1)}{1-\lambda}\phi e_j \cdot \lambda. \tag{47}$$

Similarly, using the relations (4), (13), (14), (27), (31), (34), (42), (43) and $\xi \cdot \lambda = 0$, we obtain

$$\eta(Q\phi e_j) = \frac{2\lambda(2n-1)}{1+\lambda} e_j \cdot \lambda. \tag{48}$$

If k is a constant, then λ is constant. Hence $\eta(Qe_j) = 0$, $\eta(Q\phi e_j) = 0$, $\forall j \in \{1, 2, \ldots, n\}$, i.e., $Q\xi$ is collinear with ξ. Using Lemma 4.1 we obtain $Q\phi = \phi Q$, which means that M^{2n+1} is a Sasakian space form [4]. Hence $k = 1$. A contradiction. □

LEMMA 4.3. *Let M^{2n+1} $(n > 1)$ be a conformally flat contact metric manifold satisfying $\ell = -k\phi^2$ for some smooth function $k < 1$ on M^{2n+1}. If $\{e_1, \ldots, e_n, \phi e_1, \ldots, \phi e_n, \xi\}$ is an orthonormal base formed from the unit eigenvectors $e_i \in D(\lambda)$, $\phi e_i \in D(-\lambda)$ and ξ, then*

$$\begin{aligned}
\nabla_{e_i} e_i &= \sum_{\substack{k=1\\k\neq i}}^{n} A_{ik} e_k + \frac{1-3\lambda}{2\lambda(1-\lambda)}(\phi e_i \cdot \lambda)\phi e_i,\\
\nabla_{\phi e_i} \phi e_i &= \sum_{\substack{k=1\\k\neq i}}^{n} \overline{B}_{ik} \phi e_k + \frac{1+3\lambda}{2\lambda(1+\lambda)}(e_i \cdot \lambda) e_i,\\
\nabla_{e_i} e_j &= -A_{ij} e_i + \sum_{\substack{k=1\\i\neq k\neq j}}^{n} C_{ij}^k e_k + \sum_{\substack{k=1\\i\neq k\neq j}}^{n} \overline{C}_{ij}^k \phi e_k\\
&\quad + \frac{1}{2\lambda}(\phi e_i \cdot \lambda)\phi e_j - \frac{1}{1-\lambda}(\phi e_j \cdot \lambda)\phi e_i,
\end{aligned}$$

and for the same couple $\{i, j\}$ we have:

$$\begin{aligned}
\nabla_{e_j} e_i = -A_{ji} e_j + \sum_{\substack{k=1\\i\neq k\neq j}}^{n} C_{ji}^k e_k + \sum_{\substack{k=1\\i\neq k\neq j}}^{n} \overline{C}_{ji}^k \phi e_k - \frac{1}{1-\lambda}(\phi e_i \cdot \lambda)\phi e_j\\
- \frac{1+\lambda}{2\lambda(1-\lambda)}(\phi e_j \cdot \lambda)\phi e_i,\\
\nabla_{\phi e_j} \phi e_i = -B_{ji} \phi e_j + \sum_{\substack{k=1\\i\neq k\neq j}}^{n} D_{ji}^k e_k + \frac{1}{2\lambda}(e_j \cdot \lambda) e_i + \frac{1}{1+\lambda}(e_i \cdot \lambda) e_j\\
+ \sum_{\substack{k=1\\i\neq k\neq j}}^{n} \overline{D}_{ij}^k \phi e_k,
\end{aligned}$$

where $i \neq j$ and $\overline{C}_{ij}^k = -\overline{C}_{ik}^j$, $D_{ik}^j = -D_{ij}^k$, $D_{ij}^k = D_{ji}^k$, $\overline{C}_{ij}^k = \overline{C}_{ji}^k$, $i \neq k \neq j$.

Proof. Since $k < 1$ we have that $\lambda \neq 0$. Hence, from the relation (38) we obtain

$$A_i = \frac{1}{2\lambda(2n-1)}\eta(Qe_i) + \frac{1}{2\lambda}\phi e_i \cdot \lambda.$$

The above equation and (47) yield

$$A_i = \frac{1-3\lambda}{2\lambda(1-\lambda)}\phi e_i \cdot \lambda. \tag{49}$$

Similarly we obtain from the equations (39) and (48)

$$B_i = \frac{1+3\lambda}{2\lambda(1+\lambda)} e_i \cdot \lambda. \tag{50}$$

The relations (32), (36) and $\lambda^2 \neq 1$ imply

$$(1-\lambda)\overline{A}_{ij} - (1+\lambda)\overline{C}_{ij}^i - \frac{1+\lambda}{1-\lambda}\phi e_j \cdot \lambda = 0. \tag{51}$$

Comparing the last relation with (21) we obtain

$$\overline{A}_{ij} = 0, \quad i, j = 1, 2, \ldots, n, \quad i \neq j. \tag{52}$$

The relations (51) and (52) yield

$$\overline{C}^{i}_{ij} = -\frac{1}{1-\lambda}\phi e_j \cdot \lambda. \tag{53}$$

From the last relation and (28), (47) and (52) we obtain

$$\overline{C}^{i}_{ji} = -\frac{1+\lambda}{2\lambda(1-\lambda)}\phi e_j \cdot \lambda. \tag{54}$$

Using the relations (22), (31) and (47) for the same couple $\{i, j\}$ we have

$$\overline{C}^{j}_{ij} = \frac{1}{2\lambda}\phi e_i \cdot \lambda. \tag{55}$$

The relation (37) can be written in the form:

$$(1+\lambda)B_{ji} - (1-\lambda)(Z_{ji} - A_{ji} + D^{j}_{ji}) = 0.$$

Comparing the last relation with (26) we obtain

$$(1+\lambda)B_{ji} - (1-\lambda)D^{j}_{ji} + \frac{1-\lambda}{1+\lambda}e_i \cdot \lambda = 0. \tag{56}$$

The above relation and (20) give

$$B_{ji} = 0, \quad j, i = 1, \ldots, n, \quad i \neq j. \tag{57}$$

If we compare the last two relations we have

$$D^{j}_{ji} = \frac{1}{1+\lambda}e_i \cdot \lambda. \tag{58}$$

Using the relations (23), (27) and (48) we obtain

$$D^{i}_{ji} = \frac{1}{2\lambda}e_j \cdot \lambda. \tag{59}$$

At the end we will prove that $D^{i}_{jk} = -D^{k}_{ji}, \overline{C}^{i}_{jk} = -\overline{C}^{k}_{ji}$ and $D^{k}_{ij} = D^{k}_{ji}, \overline{C}^{k}_{ij} = \overline{C}^{k}_{ji}, i \neq k \neq j$. The relations (24) and (35) yield

$$(1+\lambda)D^{k}_{ij} - (1-\lambda)D^{j}_{ik} = (1-\lambda)(N^{k}_{ij} - C^{k}_{ij}). \tag{60}$$

Using the relations (16) we obtain from (60)

$$(1+\lambda)D^k_{ij} - (1-\lambda)D^j_{ik} = -(1-\lambda)(N^j_{ik} - C^j_{ik}). \tag{61}$$

The relation (24) yields

$$(1-\lambda)(N^j_{ik} - C^j_{ik}) = (1+\lambda)D^j_{ik} - (1-\lambda)D^k_{ij}. \tag{62}$$

Comparing the relations (61) and (62) we obtain

$$D^j_{ik} = -D^k_{ij}. \tag{63}$$

Similarly, using the relations (16), (25) and (30), we obtain

$$\overline{C}^i_{jk} = -\overline{C}^k_{ji}. \tag{64}$$

Using the relations (35) and (63) we have

$$D^k_{ij} = D^k_{ji}. \tag{65}$$

Similarly using the relations (30) and (64) we have

$$\overline{C}^k_{ij} = \overline{C}^k_{ji}. \tag{66}$$

□

THEOREM 4.1. *Let M^{2n+1} be a conformally flat contact metric manifold satisfying $\ell = -k\phi^2$ for a smooth function k defined on it. Then M^{2n+1} is a Sasakian space form if $n > 1$ and flat or Sasakian space form if $n = 1$.*

Proof. If $n = 1$ then M^3 is either flat or a Sasakian space form [7]. □

Let $n > 1$. If $k = 1$, M^{2n+1} is K-contact. S. Tanno proved in [11] that every conformally flat K-contact manifold is of constant sectional curvature and J. Olzak proved in [9] that any contact metric manifold of constant sectional curvature and of dimension ≥ 5 is Sasakian of constant curvature 1. Hence in the case that $k = 1$, M^{2n+1} is Sasakian space form.

We suppose now that $k < 1$.

Using the relations (41), (52), (57), (63) and (65) we obtain

$$\sum_{\substack{k=1 \\ i\neq k\neq j}}^{n} a_{ik} D^j_{jk} + a_{ij} B_j - \xi \cdot D^j_{ji} + a_{ji} D^i_{ji} = 0.$$

The above relation and (14), (50), (58), (59), $\xi \cdot \lambda = 0$ yield

$$\frac{1}{1+\lambda} \sum_{\substack{k=1 \\ i\neq k\neq j}}^{n} a_{ik}(e_k \cdot \lambda) + \frac{1+3\lambda}{2\lambda(1+\lambda)} a_{ij}(e_j \cdot \lambda)$$
$$-\frac{1}{1+\lambda}[\xi, e_i] \cdot \lambda - \frac{1}{2\lambda} a_{ij}(e_j \cdot \lambda) = 0,$$

or because of (4) and (12)

$$\frac{1}{1+\lambda} \sum_{\substack{k=1 \\ i\neq k\neq j}}^{n} a_{ik}(e_k \cdot \lambda) + \frac{1+3\lambda}{2\lambda(1+\lambda)} a_{ij}(e_j \cdot \lambda)$$
$$-\frac{1}{1+\lambda} \sum_{\substack{k=1 \\ k\neq i}}^{n} a_{ik}(e_k \cdot \lambda) - \phi e_i \cdot \lambda - \frac{1}{2\lambda} a_{ij}(e_j \cdot \lambda) = 0.$$

From the last relation we obtain

$$\phi e_i \cdot \lambda = 0. \tag{67}$$

The above relation and $\xi \cdot \lambda = 0$ imply:

$$\xi \cdot \phi e_i \cdot \lambda - \phi e_i \cdot \xi \cdot \lambda = 0.$$

Hence,

$$[\xi, \phi e_i] \cdot \lambda = 0,$$

or because of (4), (13) and (67)

$$e_i \cdot \lambda = 0. \tag{68}$$

Hence λ is a constant which means that k is a constant. This is a contradiction because of the Lemma 4.2.

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