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A Note on Blow-Up Criterion to the 3-D Euler Equations in a Bounded Domain

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Abstract. We show that a smooth solution of the 3-D Euler equations in a bounded domain breaks down, if and only if a certain norm of vorticity blows up at the same time. Here the norm introduced by Yudovich, is weaker than L^∞-norm and generates a Banach space including singularities of $\log\log 1/|x|$. Roughly speaking, when a smooth solution breaks down, the vorticity has stronger singularities than $\log\log 1/|x|$ or has infinite number of singularities.

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1. Introduction

In this note, we consider the Euler equations for ideal incompressible fluids in a 3-dimensional bounded domain. Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain with smooth boundary $\partial\Omega$: We study the break down condition for the Euler equations in Ω:

$$\text{(E)}\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = 0, & t \ge 0,\ x\in\Omega,\\ \operatorname{div} u = 0, & t\ge 0,\ x\in\Omega,\\ u\cdot n = 0, & t\ge 0,\ x\in\partial\Omega,\\ u(x,0)=a(x), & \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $p = p(x,t)$ denote unknown velocity vector field and pressure scalar of the fluid at the point $(x,t)\in\mathbb{R}^3\times(0,\infty)$, $a = (a^1(x), a^2(x), a^3(x))$ is a given initial velocity and $n = n(x) = (n^1(x), n^2(x), n^3(x))$ is the unit outward normal at $x\in\partial\Omega$.

It is proved by Kato–Lai [7] that for every $a \in H^m(\Omega)$ with $\operatorname{div} a = 0$ where $m \ge 3$ is an integer, there exist $T > 0$ and a unique solution u of (E) on $[0,T)$ in the class

$$C_m(0,T) = C([0,T); H^m(\Omega)) \cap C^1([0,T); H^{m-1}(\Omega)), \tag{1.1}$$

where T is depending only on $\|a\|_{H^m(\Omega)}$ and m. In the celebrated paper [1], Beale–Kato–Majda showed that the solution u breaks down at a finite time $t = T$ if and only if the maximum norm of vorticity rot $u(t)$ blows up at $t = T$, when Ω is the 3-D entire space $\mathbb{R}^3$ (see also Kato–Ponce [8]). When Ω is a bounded domain, Ferrari [6] and Shirota–Yanagisawa [13] proved an analogous result of break down as in Beale–Kato–Majda [1]. Recently, these results were improved by Kozono–Taniuchi [9] so that the blow-up phenomenon is controlled by the BMO-norm of vorticity rot $u(t)$ when $\Omega = \mathbb{R}^N$. In [11], we observed a slightly improved condition for the break down in case of $\Omega = \mathbb{R}^N (N \geq 3)$.

In this note, we show that it is also possible to improve the result proved by Ferrari and Shirota–Yanagisawa for 3-dimensional bounded domains. Namely the condition on the vorticity for the break down is relaxed including logarithmic singularities compared with the former condition of the boundedness of vorticity. The main idea is based on the generalization of the critical Sobolev inequality of logarithmic type which originated with Beale–Kato–Majda [1], Brezis–Gallouet [2] and Brezis–Wainger [3]. This was developed by Taylor [14], Engler [5], Ozawa [12], Yudovich [17], and Chemin [4]. See also [9], [10], and [11]. For uniqueness theorem of weak solutions to the Euler equations, Yudovich [17] introduced some classes ((B)-spaces), which include $\log\log(e + 1/|x|)$. We prove that the breakdown of smooth solutions is controlled by the (B)-norm of vorticity. It is notable that Vishik [15] showed the uniqueness and global existence of solutions to the Euler equations in $\mathbb{R}^2$ with unbounded vorticity in spaces of Besov type, which include $\log|x|$, see also [16].

2. Preliminary and Main Results

In this section, we recall some function spaces following Yudovich [17].

$B_{\alpha \log \alpha}$ denotes the set of all functions f in $\bigcap_{k=3,4,5,\cdots} L^k(\Omega)$ satisfying

$$\|f\|_{B_{\alpha \log \alpha}} \equiv \sup_{k=3,4,5,\cdots} \frac{\|f\|_{L^k(\Omega)}}{k \log k} < \infty. \tag{2.1}$$

Similarly $B_{\log \alpha}$, $B_{\log \alpha \cdot \log\log \alpha}$ are introduced by

$$\begin{aligned} \|f\|_{B_{\log \alpha}} &\equiv \sup_{k=3,4,5,\cdots} \frac{\|f\|_{L^k(\Omega)}}{\log k} < \infty, \\ \|f\|_{B_{\log \alpha \cdot \log\log \alpha}} &\equiv \sup_{k \geq e^e} \frac{\|f\|_{L^k(\Omega)}}{\log k \cdot \log\log k} < \infty. \end{aligned} \tag{2.2}$$

In a similar way, we may introduce the norm $\|f\|_{B_{\log \alpha \log^2 \alpha \log^3 \alpha \cdots}}$. It is worth to note that

$$\log^+ \log^+ \frac{1}{|x|} \in B_{\log\alpha}, \quad \log^+ \frac{1}{|x|} \cdot \log^+ \log^+ \frac{1}{|x|} \in B_{\alpha\log\alpha},$$

(see [17]). Since Ω is a bounded domain, we see

$$\|f\|_{B_{\alpha\log\alpha}} \le C\|f\|_{B_{\log\alpha}} \le C'\|f\|_{L^\infty(\Omega)}, \quad L^\infty(\Omega) \subset B_{\log\alpha} \subset B_{\alpha\log\alpha}.$$

These norm induce Banach spaces and it holds that

$$\|f\|_{B_{\alpha\log\alpha}} \cong \sup_{p\in\mathbb{R}_+, p\ge 3} \frac{\|f\|_{L^p(\Omega)}}{p\log p}.$$

The following lemma obtained by a similar argument found in Engler [5] and Ozawa [12].

Lemma 2.1. *Let $s > N/q$, $1 < q < \infty$ and let $\Omega \subset \mathbb{R}^N$ be a bounded, simply connected domain with smooth boundary $\partial\Omega$. Then there exists a constant $C = C(N, q, s) > 0$ such that for all $f \in B_{\alpha\log\alpha}$*

$$\|f\|_{L^\infty(\Omega)} \le C(1 + \|f\|_{B_{\alpha\log\alpha}} \log(e + \|f\|_{W^{s,q}(\Omega)}) \cdot \log\log(e^e + \|f\|_{W^{s,q}(\Omega)})). \tag{2.3}$$

Remark. We may also obtain a general version of the above type inequality:

$$\|f\|_{L^\infty(\Omega)} \le C(1 + \|f\|_{B_{\Phi(\alpha)}} \Phi(\log(e^M + \|f\|_{W^{s,q}(\Omega)}))), \tag{2.4}$$

where $\Phi(\alpha) \ge 1$ is a nondecreasing function on $[M, \infty)$ for some large integer M. The proof of (2.4) is parallel to that of (2.3).

Thanks to Lemma 2.1, we have the following criterion:

Theorem 2.2. *Let Ω be a bounded, simply connected domain with $\partial\Omega \in C^\infty$ and let u be a solution to the Euler equations in the class $C_m(0, T)$ for some integer $m \ge 3$. Assume that T is maximal, i.e., u cannot be continued to the solution in the class $C_m(0, T')$ for any $T' > T$. Then*

$$\int_0^T \|\mathrm{rot}\ u(\tau)\|_{B_{\log\alpha}} d\tau = \infty$$

holds. In particular, we have

$$\limsup_{t\uparrow T} \|\mathrm{rot}\ u(t)\|_{B_{\log\alpha}} = \infty.$$

Remarks. (i) Ferrari [6] and Shirota–Yanagisawa [13] proved that

$$\int_0^T \|\mathrm{rot}\ u(\tau)\|_{L^\infty(\Omega)} d\tau = \infty$$

under the same assumptions of Theorem 2.2. Since $\|\mathrm{rot}\ u\|_{B_{\log\alpha}} \le C\|\mathrm{rot}\ u\|_{L^\infty(\Omega)}$, Theorem 2.2 covers the result of Ferrari and Shirota–Yanagisawa if Ω is a simply connected bounded domain. We should notice that in [13], Shirota–Yanagisawa dealt with more general domains.

(ii) Making use of (2.4) instead of (2.3), one can replace the consequence of Theorem 2.2 with $B_{\log\alpha}$ by $B_{\Phi(\alpha)}$, where

$$\Phi(\alpha) = \underbrace{\log\alpha \cdot \log\log\alpha \cdot \log\log\log\alpha \cdots}_{\text{finite times iterated}}.$$

3. Proof of Lemma 2.1

In this section, we give the proof of Lemma 2.1, using the similar argument as in Engler [5] and Ozawa [12].

Proof of Lemma 2.1. Since the boundary of Ω is smooth, it satisfies the interior corn property. Namely there are $\delta > 0$ and $\pi/2 < \theta < \pi$ depending only on Ω with the following property: For any point $x \in \Omega$, there exists a spherical sector $C_\delta^\theta(x) = \{x + \xi \in \mathbb{R}^n : 0 < |\xi| \le \delta, \quad -|\xi| \le \kappa(x) \cdot \xi \le |\xi| \cos\theta\}$ which has a vertex at x and $C_\delta^\theta(x) \subset \Omega$, where $\kappa(x)$ is appropriate unit vector from x. We note that for each $x \in \Omega$, $C_\delta^\theta(x)$ is congruent to $C_\delta^\theta = \{x + \xi \in \mathbb{R}^n : 0 < |\xi| \le \delta, \quad -|\xi| \le \xi_n \le |\xi|\cos\theta\}$. In particular, for any boundary point $x \in \partial\Omega$, $C_\delta^\theta(x)$ can be expressed as $C_\delta^\theta(x) \equiv \{x + \xi \in \Omega; 0 < |\xi| \le \delta, \quad -|\xi| \le \xi \cdot n(x) \le |\xi|\cos\theta\}$, where $n(x)$ denotes the outer normal at x.

Now it suffices to prove (2.3) for the only case $0 < \gamma = s - N/q < 1$.

For any fixed $x \in \Omega$ and $y \in C_\delta^\theta(x) \subset \Omega$, we begin by Morrey's inequality that

$$|f(x) - f(y)| \le C\|f\|_{W^{s,q}(\Omega)}|x - y|^\gamma. \tag{3.1}$$

For $|\xi| \le 1$, $0 < \epsilon < \delta$ and $x + \epsilon\xi \in C_\delta^\theta(x)$, it follows from (3.1) that

$$\begin{aligned} |f(x)| \le& |f(x) - f(x + \epsilon\xi)| + |f(x + \epsilon\xi)| \\ \le& C\epsilon^\gamma \|f\|_{W^{s,q}(\Omega)} + |f(x + \epsilon\xi)|. \end{aligned} \tag{3.2}$$

Integrating both side of (3.2) with respect to ξ over $S \equiv \{\xi; |\xi| \le 1, x + \epsilon\xi \in C_\delta^\theta(x)\}$,

$$\begin{aligned} |f(x)||S| &\le C\epsilon^\gamma \|f\|_{W^{s,q}(\Omega)}|S| + \int_{\xi\in S} |f(x + \epsilon\xi)| d\xi \\ &\le C\epsilon^\gamma \|f\|_{W^{s,q}(\Omega)}|S| + |S|^{1-1/p} \left\{ \int_{\xi\in S} |f(x + \epsilon\xi)|^p d\xi \right\}^{1/p} \\ &\le C\epsilon^\gamma \|f\|_{W^{s,q}(\Omega)}|S| + |S|^{1-1/p} \left\{ \int_M |f(y)|^p dy \right\}^{1/p} \epsilon^{-N/p} \\ &\le C\epsilon^\gamma \|f\|_{W^{s,q}(\Omega)}|S| + |S|^{1-1/p} \|f\|_{L^p(M)} \epsilon^{-N/p}, \end{aligned} \tag{3.3}$$

where $M \equiv C_\epsilon^\theta(x) \subset \Omega \cap \{y; |y-x| < \epsilon\}$. Since $\|f\|_{L^p(\Omega)} \le p \log p \cdot \|f\|_{B_{\alpha \log \alpha}}$ for all $p \ge 3$ and since $|S|$ is only depending on θ and N, we have

$$|f(x)| \le C(\epsilon^\gamma \|f\|_{W^{s,q}(\Omega)} + \epsilon^{-N/p} p \log p \cdot \|f\|_{B_{\alpha \log \alpha}}) \tag{3.4}$$

for all $0 < \epsilon < \delta$ and all $p \ge 3$. Setting $p = \log \frac{1}{\epsilon}$ so that $\epsilon^{-N/p} = e^N$ (under $\epsilon \le e^{-3}$), we have

$$|f(x)| \le C\Big(\epsilon^\gamma \|f\|_{W^{s,q}(\Omega)} + \log \frac{1}{\epsilon} \cdot \log\log \frac{1}{\epsilon} \cdot \|f\|_{B_{\alpha \log \alpha}}\Big) \tag{3.5}$$

for all $0 < \epsilon \le \mathrm{Min}\{\delta, e^{-3}\} \equiv \kappa$. Then we optimize ϵ by letting $\epsilon = (1/\|f\|_{W^{s,q}(\Omega)})^{1/\gamma}$ if $\|f\|_{W^{s,q}(\Omega)} \ge \kappa^{-\gamma}$ and letting $\epsilon = \kappa$ if $\|f\|_{W^{s,q}(\Omega)} \le \kappa^{-\gamma}$ to obtain (2.3). □

4. Proof of Theorem 2.2

Proof of Theorem 2.2. According to Yudovich [17, Lemma 4.1], we have

$$\|\nabla u\|_{B_{\alpha \log \alpha}} \le C \|\mathrm{rot}\ u\|_{B_{\log \alpha}}. \tag{4.1}$$

It follows from Lemma 2.1 (2.3) and (4.1) that

$$\|\nabla u\|_{L^\infty(\Omega)} \le C(1 + \|\mathrm{rot}\ u\|_{B_{\log \alpha}}) \log(e + \|u\|_{H^m(\Omega)}) \cdot \log\log(e^e + \|u\|_{H^m(\Omega)}). \tag{4.2}$$

(In Section 5, we shall give an alternative proof for (4.2).)

Taking a L^2 inner product of (E) and u, we see

$$\frac{d}{dt}\|u(t)\|_{L^2(\Omega)}^2 = 0,$$

which implies

$$\|u(t)\|_{L^2(\Omega)} \le \|a\|_{L^2(\Omega)} \quad \text{for all } 0 < t < T. \tag{4.3}$$

Since $\|u\|_{L^\infty(\Omega)} \le C(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)})$, we obtain from (4.2) and (4.3) that

$$\begin{aligned}\|u(t)\|_{W^{1,\infty}(\Omega)} \le & C(1 + \|a\|_{L^2(\Omega)})(1 + \|\mathrm{rot}\ u(t)\|_{B_{\log \alpha}}) \log(e + \|u(t)\|_{H^m(\Omega)}) \\ & \times \log\log(e^e + \|u(t)\|_{H^m(\Omega)}).\end{aligned} \tag{4.4}$$

On the other hand, it is known that the smooth solution of the Euler equations satisfies that

$$\|u(t)\|_{H^m(\Omega)}^2 \le \|a\|_{H^m(\Omega)}^2 + C\int_0^t \|u(\tau)\|_{H^m(\Omega)}^2 \|u(\tau)\|_{W^{1,\infty}(\Omega)} d\tau \tag{4.5}$$

for all $0 < t < T$ (cf., Ferrari [6] and Shirota–Yanagisawa [13]). By Gronwall's inequality we have that

$$\|u(t)\|_{H^m(\Omega)} \le \|a\|_{H^m(\Omega)} \exp\left(C\int_0^t \|u(\tau)\|_{W^{1,\infty}(\Omega)} d\tau\right), \tag{4.6}$$

which yields

$$\log(\|u(t)\|_{H^m(\Omega)} + e^e) \le \log(\|a\|_{H^m(\Omega)} + e^e) + C\int_0^t \|u(\tau)\|_{W^{1,\infty}(\Omega)} d\tau. \quad (4.7)$$

Letting $z(t) = \log(\|u(t)\|_{H^m(\Omega)} + e^e)$ and using (4.4), we have

$$z(t) \le z(0) + CT + C\int_0^t (1 + \|\mathrm{rot}\ u(\tau)\|_{B_{\log\alpha}}) z(\tau) \log z(\tau) d\tau. \quad (4.8)$$

Again by applying Gronwall's inequality, we obtain

$$z(t) \le (z(0) + CT) \exp\left(C\int_0^t (1 + \|\mathrm{rot}\ u(\tau)\|_{B_{\log\alpha}}) \log z(\tau) d\tau \right). \quad (4.9)$$

Repeating this procedure once more, we get

$$\log\log(\|u(t)\|_{H^m(\Omega)} + e^e) \le \log(z(0) + CT) \exp\left(C\int_0^t (1 + \|\mathrm{rot}\ u(\tau)\|_{B_{\log\alpha}}) d\tau \right),$$

for all $t \in (0, T)$, which competes the proof of Theorem 2.2. □

5. Another proof of (4.2)

Here we give another proof of (4.2), without using (4.1). Ferrari [6] and Shirota–Yanagisawa [13] proved the following inequality:

$$\|\nabla u\|_{L^\infty(\Omega)} \le C(1 + \|\mathrm{rot}\ u\|_{L^\infty(\Omega)} \log(e + \|u\|_{H^3(\Omega)})). \quad (5.1)$$

On the other hand, by (2.4) with $\Phi(\alpha) = \log\alpha$ and $M = e$ we have

$$\|\mathrm{rot}\ u\|_{L^\infty(\Omega)} \le C(1 + \|\mathrm{rot}\ u\|_{B_{\log\alpha}} \log\log(e^e + \|u\|_{H^3(\Omega)})). \quad (5.2)$$

The combination of (5.2) with (5.1) yields

$$\|\nabla u\|_{L^\infty(\Omega)} \le C(1 + \|\mathrm{rot}\ u\|_{B_{\log\alpha}}) \log(e + \|u\|_{H^3(\Omega)}) \cdot \log\log(e^e + \|u\|_{H^3(\Omega)}). \quad (5.3)$$

Then we get the desired estimate (4.2).

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