J. math. fluid mech. 1 (1999) 309–325
1422-6928/99/040309-17 \$ 1.50+0.20/0

Journal of Mathematical Fluid Mechanics

Partial Regularity of Weak Solutions to the Navier–Stokes Equations in the Class $L^\infty(0,T;\, L^3(\Omega)^3)$

Jiří Neustupa

Communicated by G. P. Galdi

Abstract. We show that if v is a weak solution to the Navier–Stokes equations in the class $L^\infty(0,T;\, L^3(\Omega)^3)$ then the set of all possible singular points of v in Ω, at every time $t_0 \in (0,T)$, is at most finite and we also give the estimate of the number of the singular points.

Mathematics Subject Classification (1991). 35Q10, 35B65.

Keywords. Navier–Stokes equations, weak solutions, regularity.

1. Introduction

Suppose that Ω is either $\mathbb{R}^3$ or a bounded domain in $\mathbb{R}^3$ with its boundary $\partial\Omega$ of the class $C^{2+\mu}$ for some $\mu > 0$. Let T be a positive number. Denote $Q_T = \Omega \times (0,T)$. We shall deal with the Navier–Stokes initial-boundary value problem for viscous incompressible fluids

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = f - \nabla p + \nu\, \Delta v \qquad \text{in } Q_T, \tag{1}$$

$$\operatorname{div} v = 0 \qquad \text{in } Q_T, \tag{2}$$

$$v = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{3}$$

$$v|_{t=0} = v_0 \tag{4}$$

where $v = (v_1,\, v_2,\, v_3)$ and p denote the unknown velocity and pressure. f is the external body force and $\nu > 0$ is the viscosity coefficient.

There exists an extensive literature dealing with the qualitative properties of the problem (1)–(4) and particularly with its weak solutions. The detailed surveys of the most of the known results on the existence and regularity of weak solutions

The research was supported by the University of Pittsburgh, USA, and by the Grant Agency of the Czech Republic (grant No. 201/96/0313).

can be found e.g. in the recent works of H. Kozono [13] and G. P. Galdi [6]. The main facts say that the existence of the weak solutions, satisfying the energy inequality, is known already for a long time (see J. Leray [15] and E. Hopf [9]), however their uniqueness and regularity (if all the input data are regular) still remain an open problem. The uniqueness is known to hold only for the weak solutions that find themselves in the class $L^r(0,T;\, L^s(\Omega)^3)$, where $r \in [2,+\infty]$, $s \in [3,+\infty]$ and $2/r+3/s \leq 1$ (see e.g. G. Prodi [16], H. Sohr & W. von Wahl [20], H. Kozono & H. Sohr [12], H. Kozono [13], G. P. Galdi [6]). As to regularity, it is known that a weak solution v of the problem (1)–(4) is regular either if v_0 and f are "smooth" and T is "small enough" (K. K. Kiselev & O. A. Ladyzhenskaya [11], G. P. Galdi [6]) or v_0 and f are "small enough" in certain norms (O. A. Ladyzhenskaya [14], J. G. Heywood [8], G. P. Galdi [6]) or if $v \in L^r(0,T;\, L^s(\Omega)^3)$ where $r \in [2,+\infty]$, $s \in (3,+\infty]$ and $2/r + 3/s \leq 1$ (S. Kaniel & M. Shinbrot [10], Y. Giga [7], G. P. Galdi [6]). There also exist many partial regularity results which give estimates of the measure or the dimension of the set of possible singular points of all weak solutions or only of so called suitable weak solutions even if they do not belong to the class $L^r(0,T;\, L^s(\Omega)^3)$ mentioned above (V. Scheffer [17], [18], [19], C. Foias & R. Temam [4], L. Caffarelli, R. Kohn & L. Nirenberg [2], M. Struwe [21]). The full regularity of the weak solution v that belongs to the class $L^\infty(0,T;\, L^3(\Omega)^3)$ is still an open question. There are reasons why it seems that the possible positive answer could be a fundamental step on the way to the proof of the full regularity of all weak solutions (with the "smooth" input data v_0 and f). Thus, our result on the partial regularity of the weak solution $v \in L^\infty(0,T;\, L^3(\Omega)^3)$ is the contribution to the solution of the problem of full regularity of this solution.

The norm in $L^r(\Omega)^3$ (for $r \in (1,+\infty)$) will be denoted by $\|\,.\,\|_r$. $C_0^\infty(\Omega)^3$ denotes the set of all infinitely differentiable vector-functions defined in Ω, with a compact support in Ω. $C_{0,\sigma}^\infty(\Omega)^3$ is a subset of $C_0^\infty(\Omega)^3$ which contains only the divergence-free vector functions. We shall denote by $H_0^{m,r}(\Omega)^3$ (for $m \in \mathbb{N}$ and $r \in (1,+\infty)$) the completion of $C_0^\infty(\Omega)^3$ in the norm

$$\|v\|_{m,r} = \sum_{|\alpha| \leq m} \|D^\alpha v\|_r.$$

$L_\sigma^r(\Omega)^3$ will be the closure of $C_{0,\sigma}^\infty(\Omega)^3$ in $L^r(\Omega)^3$.

$U_\rho^k(A)$ (for $k = 3$ or $k = 4$) will denote a k-dimensional ρ-neighbourhood of set $A \subset \mathbb{R}^k$. $B_\rho^k(P)$ will be a k-dimensional open ball with the center $P \in \mathbb{R}^k$ and radius ρ.

Let $v_0 \in L_\sigma^2(\Omega)^3$ and $f \in L^2(Q_T)^3$. A measurable vector-function v on Q_T is said to be a *weak solution* of the problem (1)–(4) if

1. $v \in L^2(0,T;\, H_0^{1,2}(\Omega)^3) \cap L^\infty(0,T;\, L_\sigma^2(\Omega)^3)$,
2. v satisfies

$$\int_0^T \int_\Omega \left[v \cdot \varphi_t - \nu\, \nabla v \cdot \nabla\varphi - (v \cdot \nabla v) \cdot \varphi + f \cdot \varphi \right] dx\, dt = -\int_\Omega v_0 \cdot \varphi(\,.\,,0)\, dx$$

for all infinitely differentiable divergence-free vector functions φ on $\overline{Q_T}$ that vanish on $\{[x,t] \in \overline{Q_T};\ x \in \partial\Omega \text{ or } t = T\}$.

A point $[x,t] \in \Omega \times (0,T)$ is called a *regular point* of the weak solution v if there exists a ball $B^4_\rho(x,t)$ in Q_T such that v is essentially bounded on this ball. Points which are not regular are called *singular*. Let us denote by $S(v)$ the set of all singular points of v. If $t_0 \in (0,T)$ then we put $S_{t_0}(v) = \{x \in \Omega;\ [x,t_0] \in S(v)\}$. It is obvious that the sets $S(v)$ and $S_{t_0}(v)$ are closed in Q_T, respectively in Ω.

The main result of this paper is the following:

Theorem 1. *Suppose that $v \in L^\infty(0,T;\, L^3(\Omega)^3)$ is a weak solution of the problem (1)–(4) and the external force f satisfies*

$f \in L^2(Q_T)^3 \cap L^q_{loc}(Q_T)^3$ *for some* $q > \frac{5}{2}$ *and* $\operatorname{div} f = 0$.

Then the set $S_{t_0}(v)$ of all singular points of v developed at any time $t_0 \in (0,T)$ contains no more than K^3/ϵ_5^3 points, where

$$K = \sup_{t\in(0,T)} \text{ess} \left(\int_\Omega |v(\,.\,,t)|^3\, dx \right)^{1/3}$$

and ϵ_5 is the number given by Lemma 9.

2. Auxiliary results and proof of Theorem 1

We shall use the notion of a 1-dimensional Hausdorff measure. The Hausdorff measure is explained in detail e.g. in H. Federer [3] or in L. Caffarelli, R. Kohn & L. Nirenberg [2]. We shall denote the 1-dimensional Hausdorff measure of a set $X \subset \mathbb{R}^k$ (where $k \in \mathbb{N}$) by $\mathcal{H}^1(X)$. The 3-dimensional Lebesgue measure of a set $X \in \mathbb{R}^3$ will be denoted by $m_3(X)$.

Suppose in the following that $v \in L^\infty(0,T;\, L^3(\Omega)^3)$ is a weak solution of the problem (1)–(4) whose external force f satisfies the assumptions of Theorem 1 and $t_0 \in (0,T)$. The reason why we assume function f to be in $L^2(Q_T)^3 \cap L^q_{loc}(Q_T)^3$ for some $q > \frac{5}{2}$ is that we wish to apply results from paper [2] and this kind of integrability of f is needed in [2].

Lemma 1. *The 1-dimensional Hausdorff measure of the set $S(v) \cap \Big(\Omega \times [\sigma, T)\Big)$ is zero for every $\sigma \in (0,T)$.*

Proof. If $\Omega = \mathbb{R}^3$ then due to L. Caffarelli, R. Kohn & L. Nirenberg [2], pp. 772–773 and 784, there exists a so called "suitable" weak solution u of the problem (1)–(4). Its singular set $S(u)$ satisfies $\mathcal{H}^1(S(u)) = 0$. The "suitable" weak solution u satisfies the energy inequality (even the so called strong energy inequality — see [2], pp. 779–780). Thus, the uniqueness theorem for weak solutions from $L^\infty(0,T;\, L^3(\Omega)^3)$ (see H. Kozono & H. Sohr [12], pp. 258–259) says that $u = v$ and so $S(v) = S(u)$.

The only difference in the case when Ω is a bounded domain is that the proof of existence of a "suitable" weak solution u of (1)–(4) requires fractional derivatives of v_0 of the order $\frac{2}{5}$ to be in $L^{5/4}(\Omega)^3$ (see [2], pp. 772, 773). We do not assume this regularity of our initial data v_0. However, since $v(\,.\,,t^*) \in H_0^{1,2}(\Omega)^3$ at a.a. times $t^* \in (0,\sigma)$, $v(\,.\,,t^*)$ already has this higher regularity and so v is a "suitable" weak solution of (1)–(4) on the time interval (t^*,T). t^* can surely be found such that $[\sigma,T) \subset (t^*,T)$. Then the results of [2], pp. 772–773, 784, give the equality $\mathcal{H}^1\Big[S(v)\cap\Big(\Omega\times[\sigma,T)\Big)\Big]=0$. $\square$

A set $A \subset S_{t_0}(v)$ will be called a *separated* subset of $S_{t_0}(v)$ if either $A = S_{t_0}(v)$ or $U^3_\epsilon(A) \cap U^3_\epsilon\Big(S_{t_0}(v) - A\Big) = \emptyset$ for some $\epsilon > 0$. It is obvious that every separated subset of $S_{t_0}(v)$ is closed in Ω. A nonempty separated subset A_0 of $S_{t_0}(v)$ will be called a *component* of $S_{t_0}(v)$ if A_0 cannot be expressed as a union of two disjoint non-empty separated subsets of $S_{t_0}(v)$.

Lemma 2. *If A_0 is a component of $S_{t_0}(v)$ then A_0 contains just one point.*

Proof. Let $x_0 \in A_0$ and G be an open bounded set such that $x_0 \in G \subset \overline{G} \subset \Omega$. Then $A_0 \cap \overline{G}$ is obviously compact.

1. Suppose that $A_0 \subset \overline{G}$ at first. Assume that A_0 contains point y_0, different from x_0. Let z be a point on the line segment $x_0\,y_0$ and σ_z be a plane perpendicular to this segment and intersecting it at point z. To every $k \in \mathbb{N}$ there exists a sequence of points $x_1, \ldots, x_m$ in A_0 such that $|x_0 - x_1| < 1/k$, $|x_1 - x_2| < 1/k$. $\ldots$, $|x_m - y_0| < 1/k$. The distance of at least one of these points from σ_z is less than $1/k$. Let us denote this point by z_k. The sequence $\{z_k\}_{k=1}^{+\infty}$ is bounded and so it contains a convergent subsequence. Let z' be the limit of this subsequence. Obviously, $z' \in \sigma_z \cap A_0$.

Denote by $(x_0\,y_0)'$ the union of all such points z' over all z from the line segment $x_0\,y_0$. Since $\mathcal{H}^1((x_0\,y_0)') = 0$, there exists a covering $B^3_{\rho_i}(x'_i)$ $(i = 1,2,\ldots)$ of $(x_0\,y_0)'$ by balls with centers x'_i and radii ρ_i such that

$$\sum_{i=1}^{+\infty} \rho_i < \frac{1}{4}\,|x_0 - y_0|. \tag{5}$$

Denote by x_i the orthogonal projection of point x'_i to the line segment $x_0\,y_0$. The orthogonal projection of the set $(x_0\,y_0)'$ to the line segment $x_0\,y_0$ is the whole segment $x_0\,y_0$. The balls $B^3_{\rho_i}(x_i)$ $(i = 1,2,\ldots)$ (i.e. the balls $B^3_{\rho_i}(x'_i)$ "shifted" so that their centers are the points x_i) form a covering of the segment $x_0\,y_0$. However, this is a contradiction with (5). Thus, A_0 cannot have two different points.

2. Suppose now that $A_0 \not\subset \overline{G}$. Let us define $J_\epsilon(x_0)$ (for $\epsilon > 0$) as a set of all points $x \in A_0 \cap \overline{G}$ such that there exist other points $x_1, x_2, \ldots, x_m \in A_0 \cap \overline{G}$ such that $|x_0 - x_1| \le \epsilon$, $|x_1 - x_2| \le \epsilon$, $\ldots$, $|x_m - x| \le \epsilon$. In other words, $J_\epsilon(x_0)$ is the set

of those points $x \in A_0 \cap \overline{G}$ that one can get from x_0 to x, jumping on points of $A_0 \cap \overline{G}$, by jumps whose length does not exceed ϵ. It is clear that $J_\epsilon(x_0)$ is a closed subset of $A_0 \cap \overline{G}$. The set

$$M_n = J_{1/n}(x_0) \cap \overline{U^3_{1/n}(\partial G)}$$

is non-empty for every $n \in \mathbb{N}$. (If it were empty for some $n \in \mathbb{N}$ then it would not be possible to get from x_0 to $y \in A_0 - \overline{G}$ by jumps of the length at most $1/n$. This would be the contradiction with the assumption that A_0 is a component of $S_{t_0}(v)$.) Since $M_{n+1} \subset M_n$, there exists $y_0 \in \cap_{n=1}^{+\infty} M_n$. Then $y_0 \in A_0 \cap \overline{G}$ and one can get from x_0 to y_0, jumping on points of $A_0 \cap \overline{G}$, by arbitrarily short jumps. However, it can be shown that this leads to the contradiction with the equality $\mathcal{H}^1(A_0 \cap \overline{G}) = 0$ in the same way as in part 1 of this proof. □

Lemma 3. *Let $x_0 \in S_{t_0}(v)$ and $a > 0$. Then there exists a separated subset A of $S_{t_0}(v)$ such that $x_0 \in A$, $\operatorname{diam} A < a$ and $\overline{A} \subset \Omega$.*

Proof. Define $K_\epsilon(x_0)$ (for $\epsilon > 0$) as a set of points $x \in S_{t_0}(v)$ such that there exists a sequence $\{x_1, x_2, \dots, x_m\}$ in $S_{t_0}(v)$ satisfying the inequalities $|x_0 - x_1| \leq \epsilon$, $|x_1 - x_2| \leq \epsilon$, …, $|x_m - x| \leq \epsilon$. Obviously, $K_\epsilon(x_0)$ is a separated subset of $S_{t_0}(v)$, closed in Ω.

1. Suppose at first that there exists $n \in \mathbb{N}$ such that $\operatorname{diam} K_{1/n}(x_0) < a$ and $\overline{K_{1/n}(x_0)} \subset \Omega$. Then we can put $A = K_{1/n}(x_0)$.

2. Suppose now the opposite, i.e. for every $n \in \mathbb{N}$, $\operatorname{diam} K_{1/n}(x_0) \geq a$ or $\overline{K_{1/n}(x_0)} \not\subset \Omega$. Let G be an open subset of Ω such that $x_0 \in G \subset \overline{G} \subset \Omega$ and $\operatorname{diam} G < a$. The sets $K_{1/n}(x_0) \cap \overline{G}$ are compact. (This follows from the closedness of $K_{1/n}(x_0)$ in Ω.) The sets

$$M_n = K_{1/n}(x_0) \cap \overline{G} \cap \overline{U^3_{1/n}(\partial G)}$$

are non-empty, compact and $M_{n+1} \subset M_n$. Let $y_0 \in \cap_{n=1}^{+\infty} M_n$. y_0 is a point on $\partial G \cap S_{t_0}(v)$ such that one can get from x_0 to y_0, jumping on points of $\overline{G} \cap S_{t_0}(v)$, by arbitrarily short jumps. However, using the same argument as in part 1 of the proof of Lemma 2, we can show that this is the contradiction with the fact that $\mathcal{H}^1(S_{t_0}(v)) = 0$. Thus, the assumption of part 2 of this proof cannot be true. □

Lemma 4. *Let A be a bounded separated subset of $S_{t_0}(v)$ such that $\overline{A} \subset \Omega$. Then to any given $\epsilon_1 > 0$, $r > 0$ and $\sigma_0 > 0$ (such that $[t_0 - \sigma_0, t_0 + \sigma_0] \subset (0, T)$) there exist $\xi > 0$ and an infinitely differentiable function η on $\mathbb{R}^3$ such that*

1. $m_3(U^3_\xi(A)) < \epsilon_1$,
2. $0 \leq \eta(x) \leq 1$ *for all* $x \in \mathbb{R}^3$,
3. $\eta(x) = 1$ *for all* $x \in U^3_\xi(A)$,
4. $\operatorname{supp} \eta \subset U^3_r(A)$,
5. *if we put* $E = \{x \in \Omega;\ 0 < \eta(x) < 1\}$ *then* $\left(\overline{E} \times [t_0 - \sigma_0, t_0 + \sigma_0]\right) \cap S(v) = \emptyset$.

Proof. The boundedness of A, the closedness of A in Ω and the assumption $\overline{A} \subset \Omega$ imply that A is compact. Let $r_0 \in (0, r)$ be such a number that $U^3_{r_0}(A) \cap U^3_{r_0}(S_{t_0}(v) - A) = \emptyset$ and $U^3_{r_0}(A) \cap \partial\Omega = \emptyset$.

Put $\delta = \frac{1}{2} r_0$. A can be covered by a finite number of balls $B^3_{a_i}(x_i)$ $(i = 1, \dots, n)$ such that their centers x_i are in A, their radii a_i satisfy $a_i < \delta$ and if we put

$$M = \bigcup_{i=1}^{n} B^3_{a_i}(x_i)$$

then $m_3(M) < \epsilon_1$. Denote $d = \mathrm{dist}\,(A, \partial M)$. We claim that

$$\exists\, \rho \in (0, \delta) : \Big(\partial B^3_{a_i+\rho}(x_i) \times [t_0 - \sigma_0,\, t_0 + \sigma_0]\Big) \cap S(v) = \emptyset \quad (i = 1, \dots, n). \tag{6}$$

To prove (6), we show that $\mathcal{H}^1(\mathcal{J}_i) = 0$ $(i = 1, \dots, n)$ where

$$\mathcal{J}_i = \Big\{\rho \in (0, \delta);\ \Big(\partial B^3_{a_i+\rho}(x_i) \times [t_0 - \sigma_0,\, t_0 + \sigma_0]\Big) \cap S(v) \neq \emptyset\Big\}.$$

Due to Lemma 1, the 1-dimensional Hausdorff measure of the set

$$\bigcup_{\rho \in (0,\delta)} \Big(\partial B^3_{a_i+\rho}(x_i) \times [t_0 - \sigma_0,\, t_0 + \sigma_0]\Big) \cap S(v) \tag{7}$$

is zero. Thus, to any given $\alpha > 0$ there exists a covering of set (7) by balls $B^4_{r_j}(y_j,\, t_j)$ $(j = 1, 2, \dots)$ with the centers $[y_j,\, t_j] \in \mathbb{R}^4$ and radii r_j such that $\sum_j r_j < \alpha$. The intervals $\mathcal{I}_{ij} = \Big(|y_j - x_i| - a_i - r_j,\ |y_j - x_i| - a_i + r_j\Big)$ $(j = 1, 2, \dots)$ form a covering of $\mathcal{J}_i$. (Indeed, if $\rho' \in \mathcal{J}_i$ then there exists $[y', t'] \in \Big(\partial B^3_{a_i+\rho'}(x_i) \times [t_0 - \sigma_0,\, t_0 + \sigma_0]\Big) \cap S(v)$. The point $[y', t']$ belongs to $B^4_{r_j}(y_j, t_j)$ for some $j \in \{1; 2; \dots\}$ and so we have $\Big|\rho' - (|y_j - x_i| - a_i)\Big| = \Big||y' - x_i| - |y_j - x_i|\Big| \leq |y' - y_j| < r_j$. This implies that $\rho' \in \mathcal{I}_{ij}$.) The total length of the intervals $\mathcal{I}_{ij}$ $(j = 1, 2, \dots)$ is $\sum_j 2r_j < 2\alpha$. $\alpha > 0$ was chosen arbitrarily, hence $\mathcal{H}^1(\mathcal{J}_i) = 0$ $(i = 1, \dots, n)$ and consequently, $\mathcal{H}^1(\cup_{i=1}^n \mathcal{J}_i) = 0$. This proves (6).

Furthermore, each of the intervals $\mathcal{J}_i$ is closed in $(0, \delta)$. Hence its complement in $(0, \delta)$ is open and statement (6) can be extended: There exist $\rho_1,\ \rho_2 \in (0, \delta)$ such that $\rho_1 < \rho_2$ and

$$\bigcup_{\rho \in [\rho_1,\, \rho_2]} \Big(\partial B^3_{a_i+\rho}(x_i) \times [t_0 - \sigma_0,\, t_0 + \sigma_0]\Big) \cap S(v) =$$

$$= \Big(\Big[\overline{B^3_{a_i+\rho_2}(x_i)} - B^3_{a_i+\rho_1}(x_i)\Big] \times [t_0 - \sigma_0,\, t_0 + \sigma_0]\Big) \cap S(v) = \emptyset \tag{8}$$

for all $i = 1, \dots, n$.

Put $\rho_0 = (\rho_1 + \rho_2)/2$ and $M_0 = \cup_{i=1}^n B^3_{a_i+\rho_0}(x_i)$. Let χ be the characteristic function of set M_0. Choose $\xi > 0$ such that $\xi < (\rho_2 - \rho_1)/2$ and $\xi < \frac{1}{2} d$. (The last condition guarantees the validity of statement 1 of the lemma.) Put $\eta = R_\xi \chi$

where R_ξ is the mollifier with the kernel different from zero on a ball with the radius ξ. The validity of statement 2 is now obvious. Since $\operatorname{dist}(A; \partial M_0) > 2\xi$, $\chi(x) = 1$ for all $x \in U^3_{2\xi}(A)$ and so statement 3 is also true. Statement 4 follows from the inclusions

$$\operatorname{supp}\eta = \bigcup_{i=1}^{n} \overline{B^3_{a_i+\rho_0+\xi}(x_i)} \subset \bigcup_{i=1}^{n} B^3_{a_i+\rho_2}(x_i) \subset U^3_{r_0}(A) \subset U^3_{r}(A).$$

Further, we have

$$\overline{E} \subset \bigcup_{i=1}^{n} \overline{B^3_{a_i+\rho_0+\xi}(x_i)} - \bigcup_{i=1}^{n} B^3_{a_i+\rho_0-\xi}(x_i) \subset \bigcup_{i=1}^{n} \Big[\overline{B^3_{a_i+\rho_2}(x_i)} - B^3_{a_i+\rho_1}(x_i)\Big].$$

Due to (8), the cartesian product of the last set with the time interval $[t_0-\sigma_0,\, t_0+\sigma_0]$ has the empty intersection with $S(v)$. This proves statement 5. $\square$

Lemma 5. *Let D be a bounded Lipschitz domain in $\mathbb{R}^3$. Let further $1 < r < +\infty$, $m \in \mathbb{N}_0$. Then there exists a linear operator R from $H_0^{m,r}(D)$ into $H_0^{m+1,r}(D)^3$ with the following properties:*

1. $\operatorname{div} Rf = f$ *for all $f \in H_0^{m,r}(D)$ with $\int_D f\, dx = 0$,*
2. $\|\nabla^{m+1} Rf\|_r \le c_1\, \|\nabla^m f\|_r$ *for all $f \in H_0^{m,r}(D)$.*

Lemma 5 follows from W. Borchers & H. Sohr [1, Theorem 2.4] and G. P. Galdi [5, Theorem 3.2, Chap. III.3]. All the mentioned norms are the norms in function spaces on D. Moreover, it also follows from G. P. Galdi [5, Theorem 3.3, Chap. III.3] that

$$\|Rf\|_r \le c_2\, \|g\|_r \tag{9}$$

for $f = \operatorname{div} g$ with g having a compact support in D and such that $g \in L^r(D)^3$ and $\operatorname{div} g \in L^r(D)$.

Lemma 6. *There exists a constant $\epsilon_2 > 0$ with this property: If v satisfies*

$$\sup_{t\in(t_0-\sigma,\, t_0+\sigma)} \left(\int_{B^3_r(x_0)} |v(\,.\,,t)|^3\, dx\right)^{1/3} \le \epsilon_2 \tag{10}$$

for some $r > 0$ and $\sigma > 0$ then $[x_0, t_0]$ is a regular point of v.

Lemma 6 is a modified version of Theorem 7 in H. Kozono [13]. In Theorem 7 of [13], f is supposed to be zero and instead of (10), the supremum over $t \in (t_0-\sigma,\, t_0+\sigma)$ of a weak L^3-norm of v on $B^3_r(x_0)$ is supposed to be "small enough". However, since the weak L^3-norm can be dominated by the L^3-norm, our assumption (10) is still stronger. The fact that our f need not be zero represents only a technical difficulty which does not remarkably influence the proof.

Lemma 7. *There exists $\epsilon_3 > 0$ such that if A is a nonempty subset of $S_{t_0}(v)$ then*

$$\lim_{r\to 0+}\ \limsup_{t\to t_0}\ \left(\int_{U_r^3(A)} |v(\,.\,,t)|^3\,dx\right)^{1/3} \geq \epsilon_3. \tag{11}$$

Proof. By contradiction. Suppose that to every $\epsilon_3 > 0$ there exists a nonempty subset A of $S_{t_0}(v)$ such that

$$\lim_{r\to 0+}\ \limsup_{t\to t_0}\ \left(\int_{U_r^3(A)} |v(\,.\,,t)|^3\,dx\right)^{1/3} < \epsilon_3.$$

This means that there exist $r_0 > 0$ and $\sigma > 0$ such that

$$\sup_{t\in(t_0-\sigma,\,t_0+\sigma)} \left(\int_{B_r^3(x_0)} |v(\,.\,,t)|^3\,dx\right)^{1/3} < 2\epsilon_3$$

for all $r \in (0, r_0)$ and for each point $x_0 \in A$. If ϵ_3 is small enough then Lemma 6 says that x_0 is a regular point of v. This is the desired contradiction. □

Let a be a fixed positive number in the rest of this paper.

Lemma 8. *There exists $\delta_0 > 0$ such that IF A is a nonempty separated subset of $S_{t_0}(v)$ such that* diam $A < a$, $\overline{A} \subset \Omega$ *and* $\epsilon_4 > 0$, $r > 0$ *and* $\sigma_0 > 0$ (*such that* $[t_0 - \sigma_0,\, t_0 + \sigma_0] \subset (0, T)$) *are given numbers THEN there exist $c_3 > 0$, $\rho > 0$ and $\tau > 0$ such that for each $t^* \in [t_0 - \sigma_0, t_0 + \sigma_0)$ the inequality*

$$\int_{U_r^3(A)} |v(\,.\,,t^*)|^3\,dx < \delta_0 \tag{12}$$

implies

$$\int_{U_\rho^3(A)} |v(\,.\,,t)|^3\,dx \leq c_3 \int_{U_r^3(A)} |v(\,.\,,t^*)|^3\,dx + \epsilon_4 \tag{13}$$

for all $t \in (t^*, t^* + \tau) \cap (t_0 - \sigma_0,\, t_0 + \sigma_0)$.

Proof. The proof is based on a cut-off function techniques which was also used e.g. by H. Kozono in [13].

There exists a ball $B_a^3(z_0)$ in $\mathbb{R}^3$ such that $\overline{A} \subset B_a^3(z_0)$. Put either $D = \Omega$ (if Ω is bounded) or $D = B_a^3(z_0)$ (if $\Omega = \mathbb{R}^3$). The norms $\|\,.\,\|_r$ and $\|\,.\,\|_{m,r}$ will denote the norms in function spaces on D in this proof. Since $\overline{A} \subset D$, A has a positive distance from ∂D.

Suppose that ϵ_4, $r > 0$ and $\sigma_0 > 0$ are given numbers and σ_0 is so small that $[t_0 - \sigma_0, t_0 + \sigma_0] \subset (0, T)$. We can suppose without loss of generality that r is so small that $U_r^3(A) \cap \partial D = \emptyset$ and if $S_{t_0}(v) - A \neq \emptyset$ then $U_r^3(A) \cap U_r^3(S_{t_0}(v) - A) = \emptyset$. Let ϵ_1 be a positive number. (Its value will be specified later.)

Let ξ, η be the number (respectively the function) given by Lemma 4. We set $V(.\,,t) = R\Big(\nabla\eta \cdot v(.\,,t)\Big)$ where R is the operator from Lemma 5. We have

$$\int_D \nabla\eta \cdot v(.\,,t)\,dx = \int_D \operatorname{div}\,[\eta\, v(.\,,t)]\,dx = \int_{\partial D} \eta\, v(.\,,t)\cdot n\,dS = 0$$

where n is the outer normal vector to ∂D. Thus, $\operatorname{div} V = \nabla\eta\cdot v$ in $D \times (t_0-\sigma_0,\, t_0+\sigma_0)$. Moreover, we have at least

$$V \in L^\infty(t_0-\sigma_0,\, t_0+\sigma_0;\, H_0^{1,3}(D)^3) \cap L^2(t_0-\sigma_0,\, t_0+\sigma_0;\, H_0^{2,2}(D)^3), \tag{14}$$

$$\frac{\partial V}{\partial t} \in L^s(t_0-\sigma_0,\, t_0+\sigma_0;\, H_0^{1,q}(D)^3) \tag{15}$$

for $2/s+3/q=4$ with $1<s\le 2$, $1<q<\frac{3}{2}$. (14) and (15) follow from Lemma 5 and the results of Y. Taniuchi [22, Lemma 5.1] and H. Kozono [13, Lemma 7.2]. Further, it follows from the proof of Lemma 5 (see W. Borchers & H. Sohr [1, Theorem 2.4], pp. 73–76) that since $\nabla\eta$ has a compact support in D, V also has a compact support in D. Put $w=\eta v - V$. It can be verified that w is the weak solution of the following problem in $D\times(t_0-\sigma_0,\, t_0+\sigma_0)$:

$$\frac{\partial w}{\partial t} + w\cdot\nabla w + [(1-\eta)v + V]\cdot\nabla w = -\nabla(\eta p) + \nu\,\Delta w + g, \tag{16}$$

$$\operatorname{div} w = 0, \tag{17}$$

$$w = 0 \quad \text{on } \partial D\times(t_0-\sigma_0,\, t_0+\sigma_0) \tag{18}$$

where

$$g = (v\cdot\nabla\eta)v + p\,\nabla\eta + f\eta - 2\nu\,\nabla\eta\cdot\nabla v - \nu\,\Delta\eta\, v - \frac{\partial V}{\partial t} - v\cdot\nabla V + \nu\,\Delta V.$$

w has a compact support in D and moreover, due to the relation between w and v and (14), we also have

$$w \in L^\infty(t_0-\sigma_0, t_0+\sigma_0;\, L^3_\sigma(D)^3) \cap L^2(t_0-\sigma_0, t_0+\sigma_0;\, H_0^{1,2}(D)^3). \tag{19}$$

p is a pressure associated with the weak solution v. It follows from Y. Taniuchi [22, Lemma 5.1] and H. Kozono [13, Lemma 7.2] that p can be taken so that

$$p \in L^s(t_0-\sigma_0,\, t_0+\sigma_0;\, L^q(D)) \tag{20}$$

for $2/s+3/q=3$ with $1<s<2$, $1<q<3$.

Let us denote by Π the orthogonal projector of $L^2(D)^3$ onto $L^2_\sigma(D)^3$. Its detailed analysis can be found e.g. in G. P. Galdi [5]. We shall especially use the fact that its closure in $L^q(D)^3$ (for $1<q<2$) or restriction to $L^q(D)^3$ (for $q>2$) is bounded in $L^q(D)^3$. Further, we have $\Pi w = w$, $\Pi(\partial w/\partial t) = \partial w/\partial t$ and $\Pi(\Delta w) = \Delta w$.

The interval $[t_0-\sigma_0,\, t_0+\sigma_0)$ can be expressed as $\cup_{\gamma\in\Gamma}\,[a_\gamma,\, b_\gamma)$ where set Γ is at most countable, $[a_\gamma,\, b_\gamma)$ (for $\gamma\in\Gamma$) are disjoint intervals and the restriction of v to $[a_\gamma, b_\gamma)$ is in $BC([a_\gamma,\, b_\gamma);\, L^3(\Omega)^3)$ for every $\gamma\in\Gamma$ (see H. Kozono & H. Sohr

[12]). It follows from H. Kozono [13, Theorem 5] and G. P. Galdi [6, Lemma 5.4] that

$$\frac{\partial v}{\partial t} \in L^2(a_\gamma+\vartheta, b_\gamma-\vartheta;\, L^2_\sigma(D)^3), \quad v \in L^2(a_\gamma+\vartheta, \beta_\gamma-\vartheta;\, W^{2,2}(D)^3) \tag{21}$$

for every $\gamma \in \Gamma$ and $\vartheta > 0$ such that $a_\gamma + \vartheta < b_\gamma - \vartheta$. (In fact, the mentioned results of H. Kozono and G. P. Galdi concern only the case $f \equiv 0$. However, using the same approaches, (21) can also be extended to the situation when $f \not\equiv 0$ and $f \in L^2(Q_T)^3$.) The $L^2(\Omega)^3$-weak continuity of v on $[t_0-\sigma_0,\, t_0+\sigma_0]$ (which follows relatively easily from the definition of the weak solution — see e.g. G. P. Galdi [6]) and the $L^3(\Omega)^3$-essential boundedness of v on $[t_0-\sigma_0,\, t_0+\sigma_0]$ imply the $L^3(\Omega)^3$-weak continuity of v on $[t_0-\sigma_0,\, t_0+\sigma_0]$. Thus, function w is $L^3(D)^3$-right continuous and $L^3(D)^3$-weakly continuous on $[t_0-\sigma_0,\, t_0+\sigma_0]$ and it is also in $BC([a_\gamma, b_\gamma);\, L^3(D)^3)$ for every $\gamma \in \Gamma$. Hence it satisfies except others

$$\lim_{t\to b_\gamma-} \sup \int_D |w(\,.\,,t)|^3\, dx \geq \int_D |w(\,.\,,b_\gamma)|^3\, dx \tag{22}$$

at every point b_γ (for $\gamma \in \Gamma$). It is obvious that function w also satisfies (21).

Suppose now that $t \in (a_\gamma, b_\gamma)$ for some $\gamma \in \Gamma$. For simplicity, we will often write only w instead of $w(\,.\,,t)$. Multiplying equation (16) by $\Pi(w|w|)$ and integrating over D, we obtain

$$\begin{aligned} \frac{1}{3}\,\frac{d}{dt}\int_D |w|^3\, dx + \int_D (w\cdot\nabla w)\cdot\Pi(w|w|)\, dx + \int_D \Big([(1-\eta)v+V]\cdot\nabla w\Big)\cdot\Pi(w|w|)\, dx \; = \\ = -\nu\int_D |\nabla w|^2\, |w|\, dx - \frac{4}{9}\,\nu\int_D \Big|\nabla|w|^{3/2}\Big|^2\, dx + \int_D g\cdot\Pi(w|w|)\, dx. \end{aligned} \tag{23}$$

Since function v is locally essentially bounded on $Q_T - S(v)$, it follows from statement 5 of Lemma 4 that it is essentially bounded on $\overline{E}\times[t_0-\sigma_0,\, t_0+\sigma_0]$. We will now estimate the integrals in (23). c_4 and c_5 will be generic constants, i.e. constants whose values may change from one line to the next and which depend only on D, the essential bound of v on $E\times(t_0-\sigma_0,\, t_0+\sigma_0)$ and $\sup\,\mathrm{ess}_{\,t\in(0,T)}\,\|v(\,.\,,t)\|_3$, $\sup\,\mathrm{ess}_{\,t\in(0,T)}\,\|w(\,.\,,t)\|_3$ and $\sup\,\mathrm{ess}_{\,t\in(0,T)}\,\|V(\,.\,,t)\|_{1,3}$. (The norms mean the norms on D.) Moreover, c_5 will also depend on a certain number δ whose exact value will be specified later. c_4 and c_5 do not depend on t.

$$\begin{aligned} \left|\int_D (w\cdot\nabla w)\cdot\Pi(w|w|)\, dx\right| &\leq \delta\int_D |w|\,|\nabla w|^2\, dx + c_5(\delta)\int_D |w|\,\Big|\Pi(w|w|)\Big|^2\, dx \leq \\ &\leq \delta\int_D |w|\,|\nabla w|^2\, dx + c_5(\delta)\left(\int_D |w|^3\, dx\right)^{2/3}\left(\int_D |w|^9\, dx\right)^{1/3} \leq \\ &\leq \delta\int_D |w|\,|\nabla w|^2\, dx + \left[\delta + c_5(\delta)\int_D |w|^3\, dx\right]\int_D \Big|\nabla|w|^{3/2}\Big|^2\, dx, \end{aligned} \tag{24}$$

$$
\begin{aligned}
&\left|\int_D (1-\eta)(v\cdot\nabla w)\cdot\Pi(w|w|)\,dx\right| = \left|\int_E (1-\eta)[v\cdot\nabla(\eta v - V)]\cdot\Pi(w|w|)\,dx\right| \le \\
&\le \left|\int_E (1-\eta)[v\cdot\nabla(\eta v)]\cdot\Pi(w|w|)\,dx\right| + \left|\int_E (1-\eta)[v\cdot\nabla V]\cdot\Pi(w|w|)\,dx\right| \le \\
&\le c_4\int_E |\nabla(\eta v)|\,\Big|\Pi(w|w|)\Big|\,dx + c_4\int_E |\nabla V|\,\Big|\Pi(w|w|)\Big|\,dx \le c_4\int_E |\nabla(\eta v)|^2\,dx\; + \\
&\quad + c_4\int_E \Big|\Pi(w|w|)\Big|^2\,dx \;+\; c_4\left(\int_E |\nabla V|^3\,dx\right)^{1/3}\left(\int_D \Big|\Pi(w|w|)\Big|^{3/2}\,dx\right)^{2/3} \le \\
&\le c_4\int_D |\nabla v|^2\,dx + c_4 + c_4\int_D |w|^4\,dx + c_4\left(\int_D |w|^3\,dx\right)^{2/3} \le \\
&\le c_4\int_D |\nabla v|^2\,dx + c_4 + c_4\left(\int_D |w|^3\,dx\right)^{5/6}\left(\int_D |w|^9\,dx\right)^{1/6} \le \\
&\le c_4\int_D |\nabla v|^2\,dx + \delta\left(\int_D |w|^9\,dx\right)^{1/3} + c_5(\delta) \le \\
&\le c_4\int_D |\nabla v|^2\,dx + \delta\, c_4\int_D \Big|\nabla |w|^{3/2}\Big|^2\,dx + c_5(\delta),
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&\left|\int_D (V\cdot\nabla w)\cdot\Pi(w|w|)\,dx\right| \le \\
&\le c_4\left(\int_D |V|^6 dx\right)^{1/6}\left(\int_D |\nabla w|^2\,dx\right)^{1/2}\left(\int_D |w|^6\,dx\right)^{1/3} \le \\
&\le c_4\int_D |\nabla w|^2\,dx + c_4\left(\int_D |w|^6\,dx\right)^{2/3} \le \\
&\le c_4\int_D |\nabla w|^2\,dx + c_4\left(\int_D |w|^3\,dx\right)^{1/3}\left(\int_D |w|^9\,dx\right)^{1/3} \le \\
&\le c_4\int_D |\nabla w|^2\,dx + \left[\delta + c_5(\delta)\int_D |w|^3\,dx\right]\int_D \Big|\nabla |w|^{3/2}\Big|^2\,dx.
\end{aligned}
\tag{26}
$$

In order to estimate the integral of $g\cdot\Pi(w|w|)$ on D, we shall also need the following inequalities:

$$
\begin{aligned}
&\left|\int_D [(v\cdot\nabla\eta)\cdot v]\cdot\Pi(w|w|)\,dx\right| = \left|\int_E [(v\cdot\nabla\eta)\cdot v]\cdot\Pi(w|w|)\,dx\right| \le \\
&\le c_4\int_E \Big|\Pi(w|w|)\Big|\,dx \le c_4\int_D \Big|\Pi(w|w|)\Big|^{3/2}\,dx + c_4 \le \\
&\le c_4\int_D |w|^3\,dx + c_4 \le c_4,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&\left|\int_D p\nabla\eta\cdot\Pi(w|w|)\,dx\right| = \left|\int_E p\,\nabla\eta\cdot\Pi(w|w|)\,dx\right| \le c_4\int_E |p|\Big|\Pi(w|w|)\Big|\,dx \le \\
&\le c_4\left(\int_D |p|^{13/8}\,dx\right)^{8/13}\left(\int_D \Big|\Pi(w|w|)\Big|^{13/5}\,dx\right)^{5/13} \le \\
&\le c_4\left(\int_D |p|^{13/8}\,dx\right)^{16/15} + c_4\left(\int_D |w|^{26/5}\,dx\right)^{10/11} \le \\
&\le c_4\left(\int_D |p|^{13/8}\,dx\right)^{16/15} + c_4\left(\int_D |w|^{3}\,dx\right)^{19/33}\left(\int_D |w|^{9}\,dx\right)^{1/3} \le \\
&\le c_4\left(\int_D |p|^{13/8}\,dx\right)^{16/15} + \left[\delta + c_5(\delta)\int_D |w|^3\,dx\right]\int_D \Big|\nabla|w|^{3/2}\Big|^2\,dx, \qquad (28)
\end{aligned}
$$

$$
\left|\int_D f\eta\cdot\Pi(w|w|)\,dx\right| \le c_4\int_D |f|^2\,dx + c_4\int_D |w|^4\,dx.
$$

The integral of $|w|^4$ can now be estimated in the same way as in (25). So we get

$$
\left|\int_D f\eta\cdot\Pi(w|w|)\,dx\right| \le c_4\int_D |f|^2\,dx \,+\, \delta\,c_4\int_D \Big|\nabla|w|^{3/2}\Big|^2\,dx + c_5(\delta). \qquad (29)
$$

Further, we have

$$
\begin{aligned}
&\left|2\nu\int_D (\nabla\eta\cdot\nabla v)\cdot\Pi(w|w|)\,dx\right| \le \\
&\le c_4\int_D |\nabla v|\,\Big|\Pi(w|w|)\Big|\,dx \le c_4\int_D |\nabla v|^2\,dx + c_4\int_D |w|^4\,dx \le \\
&\le c_4\int_D |\nabla v|^2\,dx + \delta\,c_4\int_D \Big|\nabla|w|^{3/2}\Big|^2\,dx + c_5(\delta), \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
\left|\nu\int_D \Delta\eta\,v\cdot\Pi(w|w|)\,dx\right| &= \left|\nu\int_E \Delta\eta\,v\cdot\Pi(w|w|)\,dx\right| \le \\
&\le c_4\int_E\Big|\Pi(w|w|)\Big|\,dx \le c_4, \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_D \frac{\partial V}{\partial t}\cdot\Pi(w|w|)\,dx\right| \le \left(\int_D \left|\frac{\partial V}{\partial t}\right|^{9/5}\,dx\right)^{5/9}\left(\int_D \Big|\Pi(w|w|)\Big|^{9/4}\,dx\right)^{4/9} \le \\
&\le c_4\left\|\frac{\partial V}{\partial t}\right\|_{1,\,9/8}\left(\int_D |w|^{9/2}\,dx\right)^{4/9} \le c_4\left\|\frac{\partial V}{\partial t}\right\|_{1,\,9/8}^{3/2} + c_4\left(\int_D |w|^{9/2}\,dx\right)^{4/3} \le \\
&\le c_4\left\|\frac{\partial V}{\partial t}\right\|_{1,\,9/8}^{3/2} + c_4\left(\int_D |w|^3\,dx\right)\left(\int_D |w|^9\,dx\right)^{1/3} \le \\
&\le c_4\left\|\frac{\partial V}{\partial t}\right\|_{1,\,9/8}^{3/2} + c_4\left(\int_D |w|^3\,dx\right)\left(\int_D \Big|\nabla|w|^{3/2}\Big|^2\,dx\right), \qquad (32)
\end{aligned}
$$

$$\begin{aligned}
&\left| \int_D (v\cdot\nabla V)\cdot\Pi(w|w|)\,dx \right| \le \\
&\le \left(\int_D |v|^3\,dx\right)^{1/3} \left(\int_D |\nabla V|^3\,dx\right)^{1/3} \left(\int_D |w|^6\,dx\right)^{1/3} \le \\
&\le c_4 \left(\int_D |w|^3\,dx\right)^{1/6} \left(\int_D |w|^9\,dx\right)^{1/6} \le c_4 \left(\int_D |w|^9\,dx\right)^{1/6} \le \\
&\le \delta \left(\int_D |w|^9\,dx\right)^{1/3} + c_5(\delta) \le \delta\, c_4 \int_D \left|\nabla |w|^{3/2}\right|^2 dx + c_5(\delta), \qquad (33)
\end{aligned}$$

$$\begin{aligned}
&\left| \nu \int_D \Delta V\cdot\Pi(w|w|)\,dx \right| \le \int_D |\Delta V|^2\,dx + c_4 \int_D |w|^4\,dx \le \\
&\le \int_D |\Delta V|^2\,dx + \delta\, c_4 \int_D \left|\nabla |w|^{3/2}\right|^2 dx + c_5(\delta). \qquad (34)
\end{aligned}$$

Substituting from (25)–(34) to (23), we obtain

$$\begin{aligned}
&\frac{1}{3}\,\frac{d}{dt}\int_D |w|^3\,dx + (\nu-\delta\, c_4)\int_D |\nabla w|^2\,|w|\,dx + \\
&+ \left[\frac{4}{9}\,\nu - \delta\, c_4 - c_4 \int_D |w|^3\,dx\right]\cdot \int_D \left|\nabla |w|^{3/2}\right|^2 dx \le c_5(\delta) + F \qquad (35)
\end{aligned}$$

where

$$F = c_4 \int_D \left(|\nabla v|^2 + |\nabla w|^2 + |f|^2 + |\Delta V|^2\right) dx + c_4 \left(\int_D |p|^{13/8}\,dx\right)^{16/15} + c_4 \left\| \frac{\partial V}{\partial t}\right\|_{1,\,9/8}^{3/2}.$$

It follows from (14), (15), (19) and (20) that F, as a function of t, is integrable on $(t_0-\sigma_0,\, t_0+\sigma_0)$. Now we choose δ so that $\delta\, c_4 = \frac{2}{9}\,\nu$. Suppose from now that c_4 and c_5 are no more generic, i.e. that their values are fixed. If

$$c_4 \int_D |w(\,.\,,t^*)|^3\,dx < \frac{1}{9}\,\nu \qquad (36)$$

then due to the right continuity of w in $L^3(D)^3$ there exists $\tau > 0$ such that

$$c_4 \int_D |w(\,.\,,t)|^3\,dx < \frac{2}{9}\,\nu \quad \text{for } t\in[t^*, t^*+\tau)\cap[t_0-\sigma_0,\, t_0+\sigma_0]. \qquad (37)$$

We will now show that τ can be chosen independent on t^* and we will derive a more accurate estimate of the integral of $|w(\,.\,,t)|^3$ on D for $t\in(t^*, t^*+\tau)\cap(t_0-\sigma_0,\, t_0+\sigma_0)$. Inequality (35) implies

$$\frac{1}{3}\,\frac{d}{dt}\int_D |w(\,.\,,t)|^3\,dx \le c_5 + F(t) \qquad (38)$$

for a.a. $t \in (t^*, t^*+\tau) \cap (t_0-\sigma_0,\, t_0+\sigma_0)$. Integrating this inequality on the interval (t^*, t) and using (22) and the $L^3(D)^3$-right continuity of w on $[t_0-\sigma_0,\, t_0+\sigma_0)$, we obtain

$$\int_D |w(.\,,t)|^3\, dx \le \int_D |w(.\,,t^*)|^3\, dx + 3\int_{t^*}^t (c_5+F(s))\, ds \le \frac{\nu}{9c_4} + 3\int_{t^*}^t (c_5+F(s))\, ds. \tag{39}$$

Now it is seen that if $\tau > 0$ is chosen so that the right hand side of (39) is less than $2\nu/(9c_4)$ whenever t^*, $t \in [t_0-\sigma_0,\, t_0+\sigma_0]$ and $|t-t^*| < \tau$ then (37) holds and this further implies the validity of (39) on $(t^*, t^*+\tau) \cap (t_0-\sigma_0,\, t_0+\sigma_0)$.

Put $\rho = \xi$. Using the relation between w and v (i.e. $w = \eta v - V$), the fact that $V = R(\operatorname{div}(\eta v))$ and inequality (9), we can derive the estimate

$$\begin{aligned} \int_D |\, w(.\,,t^*)|^3\, dx &\le \int_D |\eta v(.\,,t^*)|^3\, dx + \int_D |V(.\,,t^*)|^3\, dx \le \\ &\le \int_{U_r^3(A)} |v(.\,,t^*)|^3\, dx + c_6 \int_D |\eta\, v(.\,,t^*)|^3\, dx \;\le\; c_7 \int_{U_r^3(A)} |v(.\,,t^*)|^3\, dx. \end{aligned} \tag{40}$$

Thus, there exists $\delta_0 > 0$ such that the inequality (12) implies (36). Further, due to (14), the first part of (39) and (40), we have:

$$\begin{aligned} \int_{U_\rho^3(A)} |\, v(.\,,t)|^3\, dx &= \int_{U_\rho^3(A)} |\eta\, v(.\,,t)|^3\, dx \le \\ &\le c_8 \int_{U_\rho^3(A)} \Big(|w(.\,,t)|^3 + |V(.\,,t)|^3 \Big)\, dx \;\le \\ &\le c_8 \int_D |w(.\,,t)|^3\, dx + c_8 \left(\int_{U_\rho^3(A)} dx \right)^{1/2} \left(\int_{U_\rho^3(A)} |V(.\,,t)|^6\, dx \right)^{1/2} \le \\ &\le c_8 \int_D |w(.\,,t^*)|^3\, dx + 3c_8 \int_{t^*}^t (c_5 + F(s))\, ds + \\ &\qquad + c_8\, m_3(U_\xi^3(A))^{1/2}\, \|V(.\,,t)\|_{1,2}^3 \le \\ &\le c_8\, c_7 \int_{U_r^3(A)} |v(.\,,t^*)|^3\, dx + 3c_8 \int_{t^*}^t (c_5 + F(s))\, ds + c_9\, \epsilon_1^{1/2}. \end{aligned} \tag{41}$$

If ϵ_1 is so small that $c_9\, \epsilon_1^{1/2} < \frac{1}{2}\, \epsilon_4$ and τ is, in addition, chosen so small that

$$\left| 3c_8 \int_{t_1}^{t_2} (c_5 + F(s))\, ds \right| < \frac{1}{2}\, \epsilon_4$$

for all t_1, $t_2 \in [t_0-\sigma_0,\, t_0+\sigma_0]$ such that $|t_2 - t_1| < \tau$ then estimate (41) implies (13). $\square$

Lemma 9. *There exists $\epsilon_5 > 0$ such that if A is a nonempty separated subset of*

$S_{t_0}(v)$ *such that* diam $A < a$ *and* $\overline{A} \subset \Omega$ *then*

$$\lim_{r\to 0+} \lim_{t\to t_0-} \inf \left(\int_{U_r^3(A)} |v(\,.\,,t)|^3 \, dx \right)^{1/3} \geq \epsilon_5. \tag{42}$$

Proof. By contradiction. Suppose that to every $\epsilon_5 > 0$ there exists a nonempty separated subset A of $S_{t_0}(v)$ such that diam $A < a$, $\overline{A} \subset \Omega$ and

$$\lim_{r\to 0+} \lim_{t\to t_0-} \inf \left(\int_{U_r^3(A)} |v(\,.\,,t)|^3 \, dx \right)^{1/3} \leq \epsilon_5.$$

Thus, there exist $r_0 > 0$ and an increasing sequence $t_n \to t_0-$ such that for all $r \in (0, r_0)$ and all $n \in \mathbb{N}$, one has

$$\left(\int_{U_r^3(A)} |v(\,.\,,t_n)|^3 \, dx \right)^{1/3} \leq 2\epsilon_5.$$

Suppose that ϵ_5 is chosen so small that $(2\epsilon_5)^3 < \delta_0$. ($\delta_0$ is the number from Lemma 8.) Then

$$\int_{U_\rho^3(A)} |v(\,.\,,t)|^3 \, dx \leq c_3 \int_{U_r^3(A)} |v(\,.\,,t_n)|^3 \, dx + \epsilon_4 \leq c_3 \, (2\epsilon_5)^3 + \epsilon_4$$

for all $t \in (t_n, t_n + \tau) \cap (t_0 - \sigma_0,\, t_0 + \sigma_0)$. ($\rho$, c_3, ϵ_4 and τ are the numbers given by Lemma 8.) Since $t_n \to t_0-$ and τ does not depend on n, the intervals $(t_n,\, t_n + \tau)$ $(n = 1, 2, \dots)$ cover an interval $(t_0 - \sigma,\, t_0 + \sigma)$ for some $\sigma > 0$. So

$$\lim_{t\to t_0} \sup \left(\int_{U_\rho^3(A)} |v(\,.\,,t)|^3 \, dx \right)^{1/3} \leq \left[c_3 \, (2\epsilon_5)^3 + \epsilon_4 \right]^{1/3}.$$

If ϵ_4 and ϵ_5 are chosen small enough then this is in contradiction with inequality (11) in Lemma 7. $\square$

Proof of Theorem 1. $S_{t_0}(v)$ can be expressed in the form

$$S_{t_0}(v) \;=\; \Big(\bigcup_{i\in I} C_i \Big) \cup P$$

where C_i $(i \in I)$ are all components of $S_{t_0}(v)$ and $(\cup_{i\in I}\, C_i) \cap P = \emptyset$. Set I is at most countable. It follows from Lemma 2 that every component consists of just one point. Thus, $C_i = \{c_i\}$ $(i \in I)$.

Since every component is a separated subset of $S_{t_0}(v)$, there exist numbers r_i $(i \in I)$ such that $B^3_{r_i}(c_i) \cap S_{t_0}(v) = \{c_i\}$ $(i \in I)$. The numbers r_i can be chosen so that the balls $B_{r_i}(c_i)$ are disjoint. Suppose that $\{c_1; \dots; c_n\}$ is a finite subset of

$\cup_{i\in I} c_i$. Then we have:

$$K^3 \geq \sup_{t\in(0,T)} \text{ess} \sum_{i=1}^{n} \int_{B^3_{r_i}(c_i)} |v(.\,,t)|^3\,dx \;\geq\; \lim_{t\to t_0-}\inf \sum_{i=1}^{n} \int_{B^3_{r_i}(c_i)} |v(.\,,t)|^3\,dx \geq$$

$$\geq \sum_{i=1}^{n} \lim_{t\to t_0-}\inf \int_{B^3_{r_i}(c_i)} |v(.\,,t)|^3\,dx \;\geq\; \sum_{i=1}^{n} \epsilon_5^3 \;=\; n\,\epsilon_5^3.$$

(The last inequality follows from Lemma 9.) This implies:

$$n \leq K^3/\epsilon_5^3 \tag{43}$$

Thus, the total number of all components C_i of $S_{t_0}(v)$ cannot exceed K^3/ϵ_5^3.

Further, we claim that $P = \emptyset$. Suppose the opposite. Let $x_0 \in P$ and $a > 0$. Due to Lemma 3, there exists a separated subset A of $S_{t_0}(v)$ such that $\operatorname{diam} A < a$, $\overline{A} \subset \Omega$, the intersection of A with all components of $S_{t_0}(v)$ is empty, and $x_0 \in A$. A cannot be a component, so $A = A_1 \cup A_2$ where A_1 and A_2 are disjoint non-empty separated subsets of $S_{t_0}(v)$. The same argument can now be applied to each of the sets A_1 and A_2. After m such steps, we have:

$$A = \bigcup_{i=1}^{2^m} A_i$$

where A_i $(i = 1,\dots,2^m)$ are disjoint non-empty separated subsets of $S_{t_0}(v)$. Thus, there exist numbers $\rho_i > 0$ $(i = 1,\dots,2^m)$ such that $U^3_{\rho_i}(A_i) \cap S_{t_0}(v) = A_i$ $(i = 1,\dots,2^m)$. Using Lemma 9, we obtain:

$$K^3 \geq \sup_{t\in(0,T)} \text{ess} \sum_{i=1}^{2^m} \int_{U^3_{\rho_i}(A_i)} |v(.\,,t)|^3\,dx \;\geq\; \lim_{t\to t_0-}\inf \sum_{i=1}^{2^m} \int_{U^3_{\rho_i}(A_i)} |v(.\,,t)|^3\,dx \geq$$

$$\geq \sum_{i=1}^{2^m} \lim_{t\to t_0-}\inf \int_{U^3_{\rho_i}(A_i)} |v(.\,,t)|^3\,dx \;\geq\; \sum_{i=1}^{2^m} \epsilon_5^3 \;=\; 2^m\,\epsilon_5^3.$$

However, this inequality cannot hold if m is chosen large enough. Thus, $P = \emptyset$.

□

References

[1] W. Borchers and H. Sohr, On the equations $\operatorname{rot} v = g$ and $\operatorname{div} u = f$ with zero boundary conditions, *Hokkaido Math. J.* **19** (1990), 67–87.

[2] L. Caffarelli, R. Kohn and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier–Stokes equations, *Comm. on Pure and Apl. Math.* **35** (1982), 771–831.

[3] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[4] C. Foias and R. Temam, Some analytic and geometric properties of the solutions of the evolution Navier–Stokes equations, *J. Math. Pures et Appl.* **58** (1979), 339–368.

[5] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations, Vol I: Linearized Steady Problems*, Springer Tracts in Natural Philosophy, Vol. 38, Springer-Verlag, New York–Berlin–Heidelberg, 1994.

[6] G. P. Galdi, *An Introduction to the Navier–Stokes Initial-Boundary Value Problem*, to be published.
[7] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes equations, *J. of Diff. Equations* **61** (1986), 186–212.
[8] J. G. Heywood, The Navier–Stokes equations: On the existence, uniqueness and decay of solutions, *Indiana Univ. Math. J.* **29** (1980), 639–681.
[9] E. Hopf, Über die Anfangwertaufgabe für die hydrodynamischen Grundgleichungen, *Math. Nachr.* **4** (1950), 213–231.
[10] S. Kaniel and M. Shinbrot, Smoothness of weak solutions of the Navier–Stokes equations, *Arch. Rat. Mech. Anal.* **24** (1967), 302–324.
[11] K. K. Kiselev and O. A. Ladyzhenskaya, On existence and uniqueness of the solutions of the nonstationary problem for a viscous incompressible fluid, *Izv. Akad. Nauk SSSR* **21** (1957), 655–680 (in Russian).
[12] H. Kozono and H. Sohr, Remark on uniqueness of weak solutions to the Navier–Stokes equations, *Analysis* **16** (1996), 255–271.
[13] H. Kozono, Uniqueness and regularity of weak solutions to the Navier–Stokes equations, *Lecture Notes in Num. and Appl. Anal.* **16** (1998), 161–208.
[14] O. A. Ladyzhenskaya, Uniqueness and smoothness of generalized solutions of the Navier–Stokes equations, *Zap. Nauch. Sem. LOMI* **5** (1967), 169–185 (in Russian).
[15] J. Leray, Sur le mouvements d'un liquide visqueux emplissant l'espace, *Acta Math.* **63** (1934), 193–248.
[16] G. Prodi, Un teorema di unicità per el equazioni di Navier–Stokes, *Ann. Mat. Pura Appl.* **48** (1959), 173–182.
[17] V. Scheffer, Turbulence and Hausdorff dimension, *Lecture Notes in Mathematics* **565**, 94–112, Springer-Verlag, Berlin–Heidelberg–New York, 1976.
[18] V. Scheffer, Partial regularity of solutions to the Navier–Stokes equations, *Pacific J. Math.* **66** (1976), 535–552.
[19] V. Scheffer, Hausdorff measure and the Navier–Stokes equations, *Comm. Math. Phys.* **55** (1977), 97–112.
[20] H. Sohr and W. von Wahl, On the singular set and the uniqueness of weak solutions of the Navier–Stokes equations, *Manuscripta Math.* **49** (1984), 27–59.
[21] M. Struwe, On partial regularity for the Navier–Stokes equations, *Comm. on Pure and Appl. Math.* **41** (1988), 437–458.
[22] Y. Taniuchi, On generalized energy inequality of the Navier–Stokes equations, *Manuscripta Math.* **94** (1997), 365–384.
[23] R. Temam, *Navier–Stokes Equations*, North-Holland, Amsterdam–New York–Oxford, 1977.

J. Neustupa
Czech Technical University
Faculty of Mechanical Engineering
Karlovo nám. 13
121 35 Praha 2
Czech Republic
e-mail: neustupa@marian.fsik.cvut.cz

(accepted: April 29, 1999)