J. Math. Fluid Mech. (2024) 26:51

1422-6928/24/030001-21
https://doi.org/10.1007/s00021-024-00887-y

Journal of Mathematical Fluid Mechanics

Blowup Criterion for Viscous Non-baratropic Flows with Zero Heat Conduction Involving Velocity Divergence

Yongfu Wang

Communicated by Gui-Qiang G. Chen

Abstract. In this paper, we prove that the maximum norm of velocity divergence controls the breakdown of smooth (strong) solutions to the two-dimensional (2D) Cauchy problem of the full compressible Navier–Stokes equations with zero heat conduction. The results indicate that the nature of the blowup for the full compressible Navier–Stokes equations with zero heat conduction of viscous flow is similar to the barotropic compressible Navier–Stokes equations and does not depend on the temperature field. The main ingredient of the proof is a priori estimate to the pressure field instead of the temperature field and weighted energy estimates under the assumption that velocity divergence remains bounded. Furthermore, the initial vacuum states are allowed, and the viscosity coefficients are only restricted by the physical conditions.

Mathematics Subject Classification. 35Q35, 35B65, 76N10.

Keywords. Full compressible equations, Strong solutions, Cauchy problem, Zero heat conduction.

1. Introduction and Main Results

The motion of a compressible viscous, heat-conducting fluid is governed by the following compressible Navier–Stokes system ([35]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u + \nabla P = 0, \\ c_v[\partial_t(\rho\theta) + \operatorname{div}(\rho u \theta)] + P \operatorname{div} u - \kappa \Delta \theta = 2\mu |\mathfrak{D}(u)|^2 + \lambda (\operatorname{div} u)^2, \end{cases} \tag{1.1}$$

where $u = (u^1, u^2)$, ρ, θ and $P = R\rho\theta$ $(R > 0)$ denote the velocity, density, absolute temperature and pressure, respectively. The constant viscosity coefficient $\mu > 0$. The positive constants c_v and κ as the heat capacity and the coefficient of heat conduction, respectively.

The deformation tensor $\mathfrak{D}(u)$ denotes as

$$\mathfrak{D}(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^{tr}\right).$$

The constant viscosity coefficient μ and λ satisfy the physical restrictions

$$\mu > 0, \ \mu + \lambda \geq 0. \tag{1.2}$$

The compressible Navier–Stokes system (1.1) consists of the conservation of mass, momentum and energy. There is a large amount of literature on the wellposedness and illposedness to compressible viscous flows in multi-dimensional, see [3,4,9–11,13,14,18,19,23,28,29,32,33,35,39,40]) and the references

This research is partially supported by NSFC Grant No. 12171394, the Sichuan Science and Technology program Grant No. 2022ZYD0002, and Guanghua Talent Project of Southwestern University of Finance and Economics.

cited therein. The local existence and uniqueness of classical solutions were established in [40,43] in the absence of vacuum. When initial data close to a non-vacuum equilibrium in some Sobolev space, Matsumura-Nishida first obtained [39] the global classical solutions. Later, Hoff [14] studied the global weak solutions with strictly positive initial density and temperature for discontinuous initial data. The major breakthrough is due to Lions [35], he obtained a global weak solution to 3D compressible barotropic Navier–Stokes equations, when solutions have finite energy and the pressure $P(\rho) = \rho^\gamma$ with $\gamma \geq \frac{9}{5}$, and Feireisl [13] extended the results with $\gamma > \frac{3}{2}$. Based on the frameworks, Jiang–Zhang [28] constructed the global existence of weak solution for $\gamma > 1$ with the symmetric initial data. Hu-Wang [15,16] extended the Lion's weak solution to the full compressible Navier–Stokes equations and magnetohydrodynamic equations with general large initial data. Moreover, Cho-Kim [3] obtained the local existence and uniqueness of strong solutions with nonnegative initial density in three-dimensional space. Recently, Huang–Li–Xin [18,23,33] and Li–Xu–Zhang [31] established the global existence of the classical solution to multi-dimensional compressible Navier–Stokes equations and magnetohydrodynamic equations when the initial total energy is small but possibly large oscillations, respectively.

In particular, Xin [50] and Xin-Yan [51] showed that the smooth solutions to the compressible viscous flows without heat-conductive will blow up in finite time when the initial density has compact support or contains an isolated mass group. Therefore, it should be noted that one would not expect too much regularity of Lions's weak solutions in general. It is thus important to study the mechanism of blowup and structure of possible singularities of strong (or smooth) solutions to the compressible Navier–Stokes system. In the recent years, there has been some progress along this lines to the multi-dimensional compressible Navier–Stokes equation. The pioneering work can be traced to the incompressible Euler equations, Beale-Kato-Majda (BKM-type) in [1] established a well-known blowup criterion, roughly speaking, that if $T^* < \infty$ is the life span of a strong (or classical) solution, then

$$\lim_{T\to T^*} \|\nabla \times u\|_{L^1(0,T;L^\infty)} = \infty, \qquad \text{(BKM-type)}. \tag{1.3}$$

Later, Ponce [42] rephrased the BKM-type criterion in terms of deformation tensor $\mathfrak{D}(u)$ for 3D incompressible Euler equations, namely,

$$\lim_{T\to T^*} \|\mathfrak{D}(u)\|_{L^1(0,T;L^\infty)} = \infty. \tag{1.4}$$

Recently, Huang and Xin [24] established the following blow-up criterion in a 3-D smooth bounded domain for compressible Navier-Stokes equations, similar to the Beale-Kato-Majda criterion for ideal incompressible flows, namely,

$$\lim_{T\to T^*} \|\nabla u\|_{L^1(0,T;L^\infty)} = \infty, \tag{1.5}$$

provided

$$7\mu > \lambda. \tag{1.6}$$

Very recently, Huang–Li–Xin [22] extended the BKM-type blowup criterion (1.4) to the viscous baratropic Navier–Stokes equations in $\mathbb{R}^3$ or a bounded domain of $\mathbb{R}^3$.

For the non-isentropic compressible Navier–Stokes equations, Jiang–Ou [27] obtained a BKM-type blowup criterion for full Navier-Stokes equations (1.1) over a periodic domain or unit square domain of $\mathbb{R}^2$.

Furthermore, Fan–Jiang–Ou [12] proved that if the strong solution blows up at T^* to (1.1) in a bounded domain of $\mathbb{R}^3$, then

$$\lim_{T\to T^*} \left(\|\theta\|_{L^\infty(0,T;L^\infty)} + \|\nabla u\|_{L^1(0,T;L^\infty)}\right) = \infty,$$

under the assumption (1.6). More recently, in the absence of a vacuum and $7\mu > \lambda$, Sun–Wang–Zhang [46] showed that

$$\lim_{T\to T^*} \left(\|\theta\|_{L^\infty(0,T;L^\infty)} + \|(\rho^{-1}, \rho)\|_{L^\infty(0,T;L^\infty)}\right) = \infty.$$

On the other side, Wang [48] established a blowup criterion only in term of the divergence of the velocity field in two dimension bounded domain, namely,

$$\lim_{T\to T^*}\left(\|\mathrm{div}u\|_{L^1(0,T;L^\infty)}\right)=\infty.$$

Indeed, due to the significant works of Xin [50] and Xin–Yan [51] to compressible viscous flows without heat-conductive, the aim of this paper is to investigate the further blowup mechanism to the 2D full compressible Navier–Stokes equations without heat-conductivity. We will assume that $\kappa=0$, and without loss of generality, take $c_v=R=1$, the system (1.1) is reduced to

$$\begin{cases}\rho_t+\mathrm{div}(\rho u)=0,\\ (\rho u)_t+\mathrm{div}(\rho u\otimes u)-\mu\Delta u-(\mu+\lambda)\nabla\mathrm{div}u+\nabla P=0,\\ P_t+\mathrm{div}(Pu)+P\mathrm{div}u=2\mu|\mathfrak{D}(u)|^2+\lambda(\mathrm{div}u)^2.\end{cases}\tag{1.7}$$

We consider the Cauchy problem to (1.7) with initial conditions

$$(\rho,u,P)(x,0)=(\rho_0,u_0,P_0)(x),\ x\in\mathbb{R}^2,\tag{1.8}$$

and the far-field conditions

$$(\rho,u,P)(x,t)\to(0,0,0)\ as\ |x|\to\infty.\tag{1.9}$$

Before stating the main result, we will use the following notations and conventions. For $1\le r\le\infty$ and integer $k\ge 1$, we denote the standard Lebesgue and Sobolev spaces as follows,

$$\int f dx=\int_{\mathbb{R}^2}f dx,$$

and

$$L^r=L^r(\mathbb{R}^2),\quad W^{k,r}=W^{k,r}(\mathbb{R}^2),\quad H^k=W^{k,2}(\mathbb{R}^2),\quad D^{k,r}=\left\{u\in L^1_{loc}\mid\nabla^k u\in L^r\right\}.$$

We denote the material derivative of f by

$$\dot f:=f_t+u\cdot\nabla f.$$

We begin with the local existence of strong (or classical) solutions. The existence and uniqueness of local strong (or classical) solutions for the system (1.7)–(1.9) are proved recently in [34,36], which the initial vacuum states are allowed. The strong solutions to the Cauchy problem (1.7)–(1.9) are defined as follows.

Definition 1.1 (*Strong solutions*). (ρ,u,P) is called a strong solution to (1.7) in $\mathbb{R}^2\times(0,T)$, if for some $r_0>2$ and $a>1$,

$$\begin{cases}\rho\ge0,\ \rho\bar x^a\in C([0,T];L^1\cap H^1\cap W^{1,r_0}),\\ u\in C([0,T];D^{1,2}\cap D^{2,2})\cap L^2\left(0,T;D^{2,r_0}\right),\\ \sqrt{\rho}\dot u\in L^\infty(0,T;L^2),\ \dot u\in L^2(0,T;D^{1,2}),\\ P\ge0,\ P\in C([0,T];L^1\cap H^1\cap W^{1,r_0}),\end{cases}\tag{1.10}$$

where (ρ,u,P) satisfies (1.7) almost everywhere in $\mathbb{R}^2\times(0,T)$. Here

$$\bar x\triangleq\left(e+|x|^2\right)^{\frac12}\ln^{1+\eta_0}\left(e+|x|^2\right),\tag{1.11}$$

where η_0 is a positive constant.

Without loss of generality, we assume that the initial density ρ_0 satisfies

$$\int \rho_0 dx = 1. \tag{1.12}$$

Our main result of this paper can be stated as follows.

Theorem 1.1. *For given positive constants η_0, $\tilde{q} > 2$ and $a > 1$, the initial data satisfies (1.12) and*

$$\begin{aligned} &\rho_0 \geq 0, \ P_0 \geq 0, \ \rho_0 \bar{x}^a \in L^1 \cap H^1 \cap W^{1,\tilde{q}}, \\ &P_0 \in L^1 \cap H^1 \cap W^{1,\tilde{q}}, \ \nabla u_0 \in H^1, \ \sqrt{\rho} u_0 \in L^2, \end{aligned} \tag{1.13}$$

and the compatibility conditions

$$-\mu \Delta u_0 - (\mu + \lambda) \nabla \mathrm{div} u_0 + \nabla P_0 = \sqrt{\rho_0} g, \tag{1.14}$$

for some $g \in L^2$. Suppose (ρ, u, P) be a strong solution to the Cauchy problem (1.7)–(1.9). If $T^* < \infty$ is the maximal existence time of the strong solution, then

$$\lim_{T \to T^*} \int_0^T \|\mathrm{div} u\|_{L^\infty}^s dt = \infty, \tag{1.15}$$

for all $s \geq 1$.

Remark 1.1. Under the conditions of Theorem 1.1, the local existence of a strong solution was established in [34,36]. Hence, the maximal time T^* is well-defined.

Remark 1.2. In view of the standard Hölder inequality and $\|\mathrm{div} u\|_{L^{q_1}(0,T;L^\infty)} \leq C$ for $q_1 > 1$, we obtain the bound of $\|\mathrm{div} u\|_{L^{q_2}(0,T;L^\infty)} \leq C$ for any $q_2 \in [1, q_1)$. Hence, it suffices to prove the main Theorem 1.1 holds for $s = 1$.

Remark 1.3. Very recently, the authors [26] establish a blowup criterion in terms of the integrability of the density for strong solutions to the compressible viscous baratropic Navier–Stokes equations in $\mathbb{R}^2$ with vacuum. It would be interesting to study whether the integrability of the density can guarantee the global regularity of the system (1.7)–(1.9).

Remark 1.4. Recently, Zhong [54,55] obtained a blowup criterion to system (1.7)–(1.9), if $T^* < \infty$ is the maximal time for the existence of a strong (or classical) solution, then

$$\lim_{T \to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) = \infty.$$

For the 3D Cauchy problem of non-isentropic Navier–Stokes equations with zero heat conduction, Huang–Xin [25] proved that

$$\lim_{T \to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\theta\|_{L^\infty(0,T;L^\infty)} \right) = \infty,$$

under the assumption $\mu > 4\lambda$. In particular, Duan [8], Zhong [53] and Wang [47] established a some blowup criteria for the 3D compressible non-isentropic magnetohydrodynamic (MHD) equations with zero heat-conductivity. It would be interesting to study whether the maximum norm of velocity divergence can guarantee the global regularity of the 3D system (1.7)–(1.9).

Remark 1.5. We would like to mention the recent works on the well-known Serrin-type blowup criteria for compressible flows in [17,20,21,52] for more details. The pioneering work can be traced to Serrin's criterion was first established by J. Serrin in [44] for three-dimensional incompressible Navier–Stokes flows. Some similar blowup criteria for the compressible flows and liquid–gas two-phase flow model have been established in the recent papers, see [5–7,45,49] and references therein.

We now comment on the analysis of this paper.

Compared with [27,48] for 2D the full compressible Navier–Stokes equations in bounded domain, some new difficulties arise for the Cauchy problem of the compressible non-isentropic Navier–Stokes equations with zero heat conduction. First, it seems difficult to bound the norm $\|u\|_{L^q(\mathbb{R}^2)}$ for any $q > 1$ just in terms of $\|\sqrt{\rho}u\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla u\|_{L^2(\mathbb{R}^2)}$, since the Brezis-Waigner's inequality [2] fails for unbounded domain $\mathbb{R}^2$. Inspired by [37,38], one way to overcome this difficulty is to estimate the momentum ρu instead of the velocity u provided that ρ decays for large x. Moreover, in order to estimate $L_t^\infty L_x^\infty$-norm of P, we first show a priori estimate $L_t^\infty L_x^q$-norm of P for any $1 < q < \infty$. Next, we use logarithmic type Gronwall's inequality and a finer estimate for ρ, u and P, and obtain the estimates $L_t^\infty L_x^2$, $L_t^\infty L_x^{\tilde{q}}$-norm of $(\nabla\rho, \nabla P)$ ($\tilde{q} > 2$). Finally, we can obtain the spatial weighted estimate of density. Furthermore, the initial density vacuum is allowed in this paper. bound the norm $\|u\|_{L^q(\mathbb{R}^2)}$ for any $q > 1$ just in terms of $\|\sqrt{\rho}u\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla u\|_{L^2(\mathbb{R}^2)}$, since the Brezis-Waigner's inequality [2] fails for unbounded domain $\mathbb{R}^2$. Inspired by [37,38], one way to overcome this difficulty is to estimate the momentum ρu instead of the velocity u provided that ρ decays for large x.

The remain of this paper is organized as follows. First, some important inequalities and auxiliary lemmas will be given in Sect. 2. Moreover, we prove the main result Theorem 1.1 in Sect. 3.

2. Preliminaries

In this section, we will recall some elementary lemmas and inequalities that will be used later.

First, we will give the result on the local existence of the strong solution to the full Navier–Stokes system (1.7)–(1.9), and the proof can be found in [34].

Proposition 2.1 (Local existence of strong solution). *Assume that the initial data (ρ_0, u_0, P_0) satisfy (1.13) and (1.14). Then there exists a positive constant T_0 and unique strong solution (ρ, u, P) to the Cauchy problem (1.7)–(1.9) on $\mathbb{R}^2 \times (0, T_0)$.*

Next, the following well-known Gagliardo-Nirenberg inequalities, which will be used later frequently (see [41] for the detailed proof).

Lemma 2.2. *For $p \in [2, \infty)$, $m \in (1, \infty)$, and $n \in (2, \infty)$, there exists a generic positive constant C which may depend on m, n such that for any $f \in H^1$, $g \in L^m \cap D^{1,n}$, there holds*

$$\|f\|_{L^p}^p \leq C\,\|f\|_{L^2}^2\,\|\nabla f\|_{L^2}^{p-2}, \tag{2.1}$$

and

$$\|g\|_{L^\infty} \leq C\|g\|_{L^m} + C\|\nabla g\|_{L^n}. \tag{2.2}$$

The effective viscous flux G, and vorticity ω are defined as follows.

$$G = (2\mu + \lambda)\,\mathrm{div} u - P, \quad \omega = \nabla \times u = \partial_{x_1} u^2 - \partial_{x_2} u^1. \tag{2.3}$$

It follows from the momentum equations $(1.7)_2$ that we can obtain two key elliptic system of G and ω,

$$\Delta G = \mathrm{div}\,(\rho\dot{u})\,, \qquad \mu\Delta\omega = \nabla \times (\rho\dot{u})\,. \tag{2.4}$$

In view of the standard L^p-estimate of elliptic system (2.4), we obtain the following estimate.

Lemma 2.3. *Let (ρ, u, P) be a smooth solution of (2.4) and $p \geq 2$. Then there exists a generic positive constant C depending only on μ, λ and p, such that*

$$\|\nabla G\|_{L^p} + \|\nabla\omega\|_{L^p} \leq C\|\rho\dot{u}\|_{L^p}, \tag{2.5}$$

$$\|G\|_{L^p} + \|\omega\|_{L^p} \leq C\|\rho\dot{u}\|_{L^2}^{1-\frac{2}{p}}\left(\|\nabla u\|_{L^2} + \|P\|_{L^2}\right)^{\frac{2}{p}}, \tag{2.6}$$

and

$$\|\nabla u\|_{L^p} \le C\|\rho\dot{u}\|_{L^2}^{1-\frac{2}{p}} \left(\|\nabla u\|_{L^2} + \|P\|_{L^2}\right)^{\frac{2}{p}} + C\|P\|_{L^p}. \tag{2.7}$$

Proof. The standard L^p-estimate for elliptic system (2.4) yields (2.5).

In view of (2.1), (2.3) and (2.5), one has (2.6).

Using the similar arguments as Lemma 2.5 in [33], we obtain the desired estimate (2.7). □

The following Hardy-type inequality plays a crucial role in the estimate, the detailed proofs can be found in [32].

Lemma 2.4. *Let $\bar{x}$ and η_0 be as in (1.11) and $B_{N_1} = \{x \mid |x| < N_1\} \subset \mathbb{R}^2$ with $N_1 \ge 1$. Assume that $\rho \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is a non-negative function such that*

$$\|\rho\|_{L^1(B_{N_1})} \ge M_1, \quad \|\rho\|_{L^\infty(\mathbb{R}^2)} \le M_2,$$

for positive constants M_1 and M_2. Then there exists a positive constant C depending on M_1, M_2, γ, N_1 and η_0 such that

$$\left\|v\bar{x}^{-1}\right\|_{L^2} \le C\left(\|\sqrt{\rho}v\|_{L^2} + \|\nabla v\|_{L^2}\right), \tag{2.8}$$

for any $v \in \tilde{D}^{1,2} \triangleq \{v \in H^1_{loc}(\mathbb{R}^2) \mid \nabla v \in L^2(\mathbb{R}^2)\}$. Furthermore, for $\varepsilon > 0$ and $\eta > 0$, there exists a positive constant C depending on ε, η, M_1, M_2, γ, N_1 and η_0 such that every function $v \in \tilde{D}^{1,2}$ satisfies

$$\left\|v\bar{x}^{-\eta}\right\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}} \le C\left(\|\sqrt{\rho}v\|_{L^2} + \|\nabla v\|_{L^2}\right), \tag{2.9}$$

with $\tilde{\eta} = \min\{1, \eta\}$.

In addition, in order to estimate $\|\nabla u\|_{L^p}$, we introduce the following inequality, which is crucial to the velocity field for 2D Cauchy problem (see [30] for the detailed proof).

Lemma 2.5. *Suppose that $u \in C_0^\infty(\mathbb{R}^2)$ for any $p \in (1, \infty)$. There exists a constant C depending only on p, such that*

$$\|\nabla u\|_{L^p} \le C\left(\|\mathrm{div} u\|_{L^p} + \|\omega\|_{L^p}\right). \tag{2.10}$$

In order to improve the regularity of the velocity field, we introduce some regularity estimates to the following so-called Lamé system,

$$\begin{cases} \mu\Delta U + (\mu + \lambda)\nabla \mathrm{div} U = F, & x \in \mathbb{R}^2, \\ U(x) \to 0, & |x| \to \infty, \end{cases} \tag{2.11}$$

where $U = (U_1, U_2)$, $F = (F_1, F_2)$ and μ, λ satisfy the condition (1.2). The following Lemma can be found in [17,45].

Lemma 2.6. *Let $r \in (1, \infty)$, $q \in (2, \infty)$. Suppose $U \in D^{1,r} \cap D^{2,q}$ is a weak solution to the system (2.11). Then there exists some constant C depending only on λ, μ, q and r such that the following estimates hold:*

(1) If $F \in L^r$, then

$$\|\nabla^2 U\|_{L^r} \le C\|F\|_{L^r}. \tag{2.12}$$

(2) If $F = \mathrm{div} f$ with $f = (f_{ij})_{2\times 2} \in L^r \cap D^{1,q}$, then

$$\|\nabla U\|_{L^r} \le C\|f\|_{L^r}, \tag{2.13}$$

and

$$\|\nabla U\|_{L^\infty} \le C\left(1 + \ln(e + \|\nabla f\|_{L^q})\|f\|_{L^\infty} + \|f\|_{L^r}\right). \tag{2.14}$$

3. Proof of The Main Results

Let (ρ, u, P) be a strong solution of (1.7)–(1.9) on $\mathbb{R}^2 \times [0, T^*)$. Suppose that (1.15) was false, for $s = 1$, namely, there exists a constant $M > 0$ such that

$$\lim_{T \to T^*} \int_0^T \|\mathrm{div} u\|_{L^\infty} dt \le M < \infty. \tag{3.1}$$

First, we have the following upper uniform estimates of the density.

Lemma 3.1. *Under the assumption (3.1), it holds that for $0 \le T < T^*$,*

$$\rho \ge 0, \quad P \ge 0, \tag{3.2}$$

and

$$\sup_{0 \le T \le T^*} \|\rho\|_{L^1 \cap L^\infty} \le C. \tag{3.3}$$

Here and after, c, C and C_i $(i = 0, 1)$ denote generic positive constants depending only on M, μ, λ, T^* and the initial data.

Proof. The particle path can be defined before blowup time as follows:

$$\begin{cases} \frac{d}{dt} X(x,t) = u(X(x,t),t), \\ X(x,0) = x. \end{cases} \tag{3.4}$$

It follows from the continuity equation and along the particle path that the density can be expressed by

$$\frac{d}{dt}\rho(X(x,t),t) = -\rho(X(x,t),t)\mathrm{div} u(X(x,t),t),$$

implies

$$\rho(X(x,t),t) = \rho_0(x) \exp\left\{-\int_0^t \mathrm{div} u(X(x,s),s) ds\right\},$$

which together with $\rho_0 \ge 0$, (3.1) and Lemma 2.1 in [22] implies $\rho \ge 0$ and (3.3).

Indeed, in view of the continuity equation, we obtain $\|\rho\|_{L^1} = \|\rho_0\|_{L^1}$.

Furthermore, one has

$$P_t + u \cdot \nabla P = -2P \mathrm{div} u + F, \tag{3.5}$$

where $F = 2\mu|\mathfrak{D}(u)|^2 + \lambda(\mathrm{div} u)^2$.

Thus, along particle path (3.4) and (3.5), we obtain

$$\frac{d}{dt} P(X(x,t),t) = -2P \mathrm{div} u + F,$$

which implies

$$P(X(x,t),t) = \left(P_0 + \int_0^t \exp\left(2\int_0^s \mathrm{div} u d\tau\right) F ds\right) \exp\left\{-2\int_0^t \mathrm{div} u ds\right\} \ge 0. \tag{3.6}$$

Hence, we finish the proof of Lemma 3.1.

□

Next, we will give the standard energy estimates as follows.

Lemma 3.2. *Under the assumption* (3.1)*, there exists a positive constant* N_1 *such that*

$$\sup_{0\le t\le T}\left(\|P\|_{L^1}+\|\sqrt{\rho}u\|_{L^2}^2\right)+\int_0^T\|\nabla u\|_{L^2}^2dt\le C, \tag{3.7}$$

and

$$\inf_{0\le t\le T}\int_{B_{N_1}}\rho dx\ge\frac{1}{4}, \tag{3.8}$$

for any $0\le T<T^*$.

Proof. Indeed, multiplying $(1.7)_2$ by u, we obtain after integrating over $\mathbb{R}^2$ and integrating by parts that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\rho|u|^2dx+\int\left(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2\right)dx&=\int P\mathrm{div}udx\\&\le C\|\mathrm{div}u\|_{L^\infty}\int Pdx.\end{aligned} \tag{3.9}$$

Integrating the pressure equation in $(1.7)_3$ over $\mathbb{R}^2$ yields to

$$\begin{aligned}\frac{d}{dt}\int Pdx&=-\int P\mathrm{div}udx+\int\left(2\mu|\mathfrak{D}(u)|^2+\lambda(\mathrm{div}u)^2\right)dx\\&\le C\|\mathrm{div}u\|_{L^\infty}\int Pdx+C_0\int|\nabla u|^2dx.\end{aligned} \tag{3.10}$$

Set $C_1=\frac{C_0+1}{\mu}$, adding (3.9) multiplied by C_1 to (3.10), one has

$$\begin{aligned}&\frac{d}{dt}\int\left(P+\frac{C_1}{2}\rho|u|^2\right)dx+\int|\nabla u|^2dx\\\le&C\|\mathrm{div}u\|_{L^\infty}\int Pdx,\end{aligned}$$

which together with (3.1) and Gronwall's inequality gives

$$\sup_{0\le t\le T}\int\left(P+\frac{C_1}{2}\rho|u|^2\right)dx+\int_0^T\int|\nabla u|^2dxdt\le C. \tag{3.11}$$

Next, for $N_1>1$, a cutoff function $\eta_{N_1}(x)\in C_0^\infty(\mathbb{R}^2)$ is defined by

$$0\le\eta_{N_1}(x)\le 1,\ \eta_{N_1}(x)=\begin{cases}1, & if\ |x|\le N_1,\\ 0, & if\ |x|\ge 2N_1,\end{cases}\ |\nabla\eta_{N_1}|\le\frac{2}{N_1}.$$

Multiplying $(1.7)_1$ by η_{N_1} and integrating by parts give that

$$\begin{aligned}\frac{d}{dt}\int\rho\eta_{N_1}dx&=\int\rho u\cdot\nabla\eta_{N_1}dx\\&\ge-2{N_1}^{-1}\left(\int\rho dx\right)^{\frac{1}{2}}\left(\int\rho|u|^2dx\right)^{\frac{1}{2}}\ge-CN_1^{-1},\end{aligned}$$

due to Hölder's inequality and (3.11). This gives

$$\inf_{0\le t\le T}\int\rho\eta_{N_1}dx\ge\int\rho_0\eta_{N_1}dx-C{N_1}^{-1}T,$$

which together with (1.12) imply that

$$\inf_{0\le t\le T}\int_{B_{N_1}}\rho dx\ge\frac{1}{4}, \tag{3.12}$$

for N_1 suitably large.

Therefore, in view of (3.11) and (3.12), we complete the proof of the Lemma 3.2. □

We can prove the following key estimate on ∇u, which is crucial for deriving the higher order estimates to the solution.

Lemma 3.3. *Under condition (3.1), it holds that for* $0 \le T < T^*$,

$$\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2+\|P\|_{L^2}^2+\|\rho\bar{x}^a\|_{L^1}\right)+\int_0^T\|\sqrt{\rho}\dot{u}\|_{L^2}^2dt\le C. \tag{3.13}$$

Proof. Multiplying the momentum equation $(1.7)_2$ by $\dot{u}$, integrating the resulting equation over $\mathbb{R}^2$ and integrating by parts yield to

$$\begin{aligned}\int\rho|\dot{u}|^2dx&=\int\left(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u\right)\dot{u}dx-\int\nabla P\cdot\dot{u}dx\\&=\sum_{i=1}^2I_i.\end{aligned} \tag{3.14}$$

Indeed, for $\nabla^\perp\triangleq(-\partial_2,\partial_1)$, and $-\Delta u=-\nabla\mathrm{div}u-\nabla^\perp\omega$, integrating by parts, we have the following estimates to I_1.

$$\begin{aligned}I_1&=\int\left(u_t+u\cdot\nabla u\right)\left(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u\right)dx\\&=-\frac{1}{2}\frac{d}{dt}\int\left(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2\right)dx\\&\quad+\mu\int u\cdot\nabla u\cdot\nabla^\perp\omega dx+(2\mu+\lambda)\int u\cdot\nabla u\cdot\nabla\mathrm{div}udx\\&\le-\frac{1}{2}\frac{d}{dt}\int\left(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2\right)dx+C\|\mathrm{div}u\|_{L^\infty}\int|\nabla u|^2dx,\end{aligned} \tag{3.15}$$

where we have used the following facts.

$$\begin{aligned}\int u\cdot\nabla u\cdot\nabla^\perp\omega dx&=\int\left(\partial_2(u^i\partial_iu^1)-\partial_1(u^i\partial_iu^2)\right)\omega dx\\&=-\frac{1}{2}\int\omega^2\mathrm{div}udx,\end{aligned}$$

and

$$\begin{aligned}&\int u\cdot\nabla u\cdot\nabla\mathrm{div}udx\\&=-\int\partial_iu^j\partial_ju^i\mathrm{div}udx+\frac{1}{2}\int(\mathrm{div}u)^3dx.\end{aligned}$$

On the other hand, using Young's inequality and integrating by parts, one has

$$\begin{aligned}I_2&=\int\left(P\mathrm{div}u_t-(u\cdot\nabla u)\cdot\nabla P\right)dx\\&=\frac{d}{dt}\int P\mathrm{div}udx-\int\left(P_t\mathrm{div}u+(u\cdot\nabla u)\cdot\nabla P\right)dx\\&=\frac{d}{dt}\int P\mathrm{div}udx+\int\left(P(\mathrm{div}u)^2-2\mu|\mathfrak{D}(u)|^2\mathrm{div}u-\lambda(\mathrm{div}u)^3+P\partial_ju^i\partial_iu^j\right)dx\\&\le\frac{d}{dt}\int P\mathrm{div}udx+C\int\left(P|\nabla u|+|\nabla u|^2\right)|\mathrm{div}u|dx+C\int|\nabla u|^2Pdx.\end{aligned} \tag{3.16}$$

Furthermore, using Young's inequality, Lemmas 2.2, 2.3 and 2.5 lead to

$$\begin{aligned}\int P|\nabla u|^2dx \le & C\|P\|_{L^2}\|\nabla u\|_{L^4}^2\\ \le & C\|P\|_{L^2}\left(\|\mathrm{div}u\|_{L^4}^2+\|\omega\|_{L^4}^2\right)\\ \le & C\|P\|_{L^2}\left(\|\mathrm{div}u\|_{L^\infty}\|\mathrm{div}u\|_{L^2}+\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}\right)\\ \le & C\|P\|_{L^2}\left(\|\mathrm{div}u\|_{L^\infty}\|\nabla u\|_{L^2}+\|\nabla u\|_{L^2}\|\nabla\omega\|_{L^2}\right)\\ \le & C\|P\|_{L^2}\left(\|\mathrm{div}u\|_{L^\infty}\|\nabla u\|_{L^2}+\|\nabla u\|_{L^2}\|\sqrt{\rho}\dot{u}\|_{L^2}\right)\\ \le & C\left(\|\mathrm{div}u\|_{L^\infty}+\|\nabla u\|_{L^2}^2\right)\left(\|P\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)+C\varepsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2,\end{aligned} \tag{3.17}$$

which together with (3.16) and choosing ε suitable small yields to

$$I_2 \le \frac{d}{dt}\int P\mathrm{div}u dx + C\left(\|\mathrm{div}u\|_{L^\infty}+\|\nabla u\|_{L^2}^2\right)\left(\int|\nabla u|^2dx+\int P^2dx\right)+\frac{1}{2}\|\sqrt{\rho}\dot{u}\|_{L^2}^2. \tag{3.18}$$

Substituting (3.15) and (3.18) into (3.14), we obtain

$$\begin{aligned}&\frac{d}{dt}\int\left(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2-2P\mathrm{div}u\right)dx+\int\rho|\dot{u}|^2dx\\ \le & C\left(\|\mathrm{div}u\|_{L^\infty}+\|\nabla u\|_{L^2}^2\right)\left(\int|\nabla u|^2dx+\int P^2dx\right).\end{aligned} \tag{3.19}$$

Next, we will estimate the term $\|P\|_{L^2}^2$. Multiplying $(1.7)_3$ by P, and integrating it over $\mathbb{R}^2$, after integration by parts together with (3.17) show that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int P^2dx =& \int\left(2\mu|\mathfrak{D}(u)|^2+\lambda(\mathrm{div}u)^2-\mathrm{div}(Pu)-P\mathrm{div}u\right)Pdx\\ =&\int\left(2\mu|\mathfrak{D}(u)|^2+\lambda(\mathrm{div}u)^2-\frac{3}{2}P\mathrm{div}u\right)Pdx\\ \le & C\left(\|\mathrm{div}u\|_{L^\infty}+\|\nabla u\|_{L^2}^2\right)\left(\int|\nabla u|^2dx+\int P^2dx\right)+C\varepsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2,\end{aligned} \tag{3.20}$$

due to Hölder's inequality.

Noting that

$$\left|\int P\mathrm{div}u dx\right| \le \frac{\mu}{4}\int|\nabla u|^2dx+\frac{C_1}{2}\int P^2dx. \tag{3.21}$$

Then, adding (3.20) multiplied by $2(C_1+1)$ to (3.19) and choosing ε suitable small, we obtain that

$$\begin{aligned}&\frac{d}{dt}\int\left(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2-2P\mathrm{div}u+(C_1+1)P^2\right)dx+\frac{1}{2}\int\rho|\dot{u}|^2dx\\ \le & C\left(\|\mathrm{div}u\|_{L^\infty}+\|\nabla u\|_{L^2}^2\right)\left(\int|\nabla u|^2dx+\int P^2dx\right),\end{aligned}$$

with together with (3.1), (3.21) and Gronwall's inequality yields

$$\sup_{0\le t\le T}\int\left(|\nabla u|^2+P^2\right)dx+\int_0^T\int\rho|\dot{u}|^2dxdt \le C. \tag{3.22}$$

Finally, multiplying (1.7)$_1$ by $\bar{x}^a$ and integrating the resulting equation over $\mathbb{R}^2$ and integrating by parts yield that

$$\begin{aligned}\frac{d}{dt}\int \rho\bar{x}^a dx &\le C\int \rho|u|\bar{x}^{a-1}\ln^{1+\eta_0}\left(e+|x|^2\right)dx\\ &\le C\int \rho^{\frac{7+a}{8+a}}\bar{x}^{\frac{a^2+7a}{8+a}}\bar{x}^{-\frac{4}{8+a}}|u|\bar{x}^{-\frac{4}{8+a}}\ln^{1+\eta_0}(e+|x|^2)dx\\ &\le C\left\|\bar{x}^{-\frac{4}{8+a}}\ln^{1+\eta_0}(e+|x|^2)\right\|_{L^\infty}\left\|u\bar{x}^{-\frac{4}{8+a}}\right\|_{L^{8+a}}\left(\int\rho\bar{x}^a dx\right)^{\frac{7+a}{8+a}}\\ &\le C\left(\|\sqrt{\rho}u\|_{L^2}+\|\nabla u\|_{L^2}\right)\left(\int\rho\bar{x}^a dx\right)^{\frac{7+a}{8+a}}\\ &\le C\left(1+\|\nabla u\|_{L^2}^2\right)\left(\int\rho\bar{x}^a dx+1\right),\end{aligned}$$

due to (2.9). This together with Gronwall's inequality and (3.7) gives

$$\sup_{0\le t\le T}\int\rho\bar{x}^a dx\le C.$$

This completes the proof of Lemma 3.3. □

Next, we have the following estimates on the material derivatives of the velocity.

Lemma 3.4. *Under the condition (3.1), it holds that for* $0\le T<T^*$,

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|P\|_{L^4}^4\right)+\int_0^T\|\nabla\dot{u}\|_{L^2}^2dt\le C. \tag{3.23}$$

Proof. Applying $\dot{u}^j(\partial_t+\mathrm{div}(u\cdot))$ to the j-th equation in the momentum equations ($j=1,2$), integrating the resulting equation over $\mathbb{R}^2$ and integration by parts, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\rho|\dot{u}|^2dx =&-\int\dot{u}^j(\partial_jP_t+\mathrm{div}(u\partial_jP))dx+\mu\int\dot{u}^j\left(\Delta u_t^j+\mathrm{div}(u\Delta u^j)\right)dx\\ &+(\mu+\lambda)\int\dot{u}^j\left(\partial_j\mathrm{div}u_t+\mathrm{div}(u\partial_j\mathrm{div}u)\right)dx\\ =&\sum_{j=1}^3 J_i.\end{aligned}\tag{3.24}$$

For the first term on the right-hand side of (3.24), using (1.7)$_3$, Young's inequality, and integration by parts yields to

$$\begin{aligned}J_1=&-\int\dot{u}^j\left(\partial_jP_t+\mathrm{div}(u\partial_jP)\right)dx\\ =&\int\partial_j\dot{u}^j\left(2\mu|\mathfrak{D}(u)|^2+\lambda(\mathrm{div}u)^2-\mathrm{div}(Pu)-P\mathrm{div}u\right)dx+\int\partial_k\dot{u}^j\partial_jPu^kdx\\ =&\int\partial_j\dot{u}^j\left(2\mu|\mathfrak{D}(u)|^2+\lambda(\mathrm{div}u)^2-P\mathrm{div}u\right)dx-\int P\partial_k\dot{u}^j\partial_ju^kdx\\ \le& C\int|\nabla\dot{u}||\nabla u|(|\nabla u|+P)\,dx\\ \le&\frac{\mu}{8}\int|\nabla\dot{u}|^2dx+C\int|\nabla u|^4dx+C\int P^4dx.\end{aligned}\tag{3.25}$$

Next, after integrating by parts, it follows from Young's inequality that

$$
\begin{aligned}
J_2 =&\mu \int \dot{u}^j(\Delta u_t^j + \operatorname{div}(u\Delta u^j))dx \\
=&-\mu \int \left(\partial_i \dot{u}^j \partial_i u_t^j + \Delta u^j u\cdot\nabla \dot{u}^j\right)dx \\
=&-\mu \int \left(|\nabla \dot{u}|^2 - \partial_i\dot{u}^j u^k\partial_k\partial_i u^j - \partial_i\dot{u}^j\partial_i u^k\partial_k u^j + \partial_i\partial_i u^j u^k \partial_k \dot{u}^j\right)dx \\
=&-\mu \int \left(|\nabla \dot{u}|^2 + \partial_i\dot{u}^j\partial_i u^j \mathrm{div}u - \partial_i\dot{u}^j\partial_i u^k\partial_k u^j - \partial_k\dot{u}^j\partial_i u^k \partial_i u^j\right)dx \\
\le&-\frac{5\mu}{8}\int |\nabla\dot{u}|^2dx + C\int |\nabla u|^4 dx.
\end{aligned}
\tag{3.26}
$$

Similarly,

$$
J_3 \le -\frac{5}{8}(\mu+\lambda)\int (\operatorname{div}\dot{u})^2dx + C\int|\nabla u|^4dx \le C\int |\nabla u|^4dx. \tag{3.27}
$$

Putting (3.25)–(3.27) into (3.24), we obtain

$$
\frac{d}{dt}\int \rho|\dot{u}|^2dx + \mu\int|\nabla\dot{u}|^2dx \le C\int|\nabla u|^4dx + C\int P^4dx. \tag{3.28}
$$

Furthermore, we multiply $(1.7)_3$ by $4P^3$, and integrating it over $\mathbb{R}^2$, after integration by parts, Hölder's inequality, (3.13), Lemmas 2.2, 2.3 and 2.5 yield to

$$
\begin{aligned}
\frac{d}{dt}\int P^4dx =&4\int \left(2\mu|\mathfrak{D}(u)|^2 + \lambda(\operatorname{div}u)^2 - \nabla P\cdot u - 2P\operatorname{div}u\right)P^3dx \\
=&4\int\left(2\mu|\mathfrak{D}(u)|^2 + \lambda(\operatorname{div}u)^2 - \frac{7}{4}P\operatorname{div}u\right)P^3dx \\
\le& C\int \left((\operatorname{div}u)^2P^3 + |\operatorname{div}u|P^4 + |\nabla u|^2P^3\right)dx \\
\le& C\|\operatorname{div}u\|_{L^\infty}\left(\int|\nabla u|^4dx + \int P^4dx\right) + C\|\nabla u\|_{L^8}^2\|P\|_{L^4}^3 \\
\le& C\|\operatorname{div}u\|_{L^\infty}\left(\int|\nabla u|^4dx + \int P^4dx\right) + C\left(\|\operatorname{div}u\|_{L^8}^2 + \|\omega\|_{L^8}^2\right)\|P\|_{L^4}^3 \\
\le& C\|\operatorname{div}u\|_{L^\infty}\left(\int|\nabla u|^4dx + \int P^4dx\right) + C\left(\|\operatorname{div}u\|_{L^8}^2 + \|\omega\|_{L^8}^2\right)\|P\|_{L^4}^3 \\
\le& C\|\operatorname{div}u\|_{L^\infty}\left(\int|\nabla u|^4dx + \int P^4dx\right) \\
&+ C\left(\|\operatorname{div}u\|_{L^\infty}\|\operatorname{div}u\|_{L^4} + \|\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{3}{2}}\right)\|P\|_{L^4}^3 \\
\le& C\|\operatorname{div}u\|_{L^\infty}\left(\int|\nabla u|^4dx + \int P^4dx\right) \\
&+ C\left(\|\operatorname{div}u\|_{L^\infty}\|\nabla u\|_{L^4} + \|\nabla u\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{3}{2}}\right)\|P\|_{L^4}^3 \\
\le& C\|\operatorname{div}u\|_{L^\infty}\left(\int|\nabla u|^4dx + \int P^4dx\right) + C\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+1\right)\left(\|P\|_{L^4}^4+1\right) \\
\le& C\left(\|\operatorname{div}u\|_{L^\infty} + \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + 1\right)\left(\int|\nabla u|^4dx + \int P^4dx + 1\right).
\end{aligned}
$$

This together with (3.28) gives that

$$\begin{aligned}&\frac{d}{dt}\int\left(\rho|\dot{u}|^2+P^4\right)dx+\mu\int|\nabla\dot{u}|^2dx\\ \le& C\left(\|\mathrm{div}u\|_{L^\infty}+\|\sqrt{\rho}\dot{u}\|_{L^2}^2+1\right)\left(\int|\nabla u|^4dx+\int P^4dx+1\right).\end{aligned} \tag{3.29}$$

To deal with the right-hand side of (3.29), using (2.7) and (3.13), one has

$$\begin{aligned}\int|\nabla u|^4dx&\le C\|\rho\dot{u}\|_{L^2}^2\left(\|\nabla u\|_{L^2}+\|P\|_{L^2}\right)^2+C\|P\|_{L^4}^4\\ &\le C\int\rho|\dot{u}|^2dx+C\int P^4dx.\end{aligned} \tag{3.30}$$

This together with (3.29), we obtain

$$\begin{aligned}&\frac{d}{dt}\int\left(\rho|\dot{u}|^2+P^4\right)dx+\mu\int|\nabla\dot{u}|^2dx\\ \le& C\left(\|\mathrm{div}u\|_{L^\infty}+\|\sqrt{\rho}\dot{u}\|_{L^2}^2+1\right)\left(\int\rho|\dot{u}|^2dx+\int P^4dx+1\right),\end{aligned}$$

which together with (3.1) and (3.13), we immediately obtain (3.23) by applying Gronwall's inequality. The proof of Lemma 3.4 is finished.

□

Next, in view of Lemmas 3.1–3.4, we will prove the higher order integrability of P.

Lemma 3.5. *Under the condition (3.1), for any $0\le T<T^*$ and $q\in[2,\infty)$, it holds that*

$$\sup_{0\le t\le T}\|P(\cdot,t)\|_{L^q}\le C, \tag{3.31}$$

and

$$\int_0^T\|P(\cdot,t)\|_{L^\infty}dt\le C. \tag{3.32}$$

Proof. For any $q\in[2,\infty)$, multiplying $(1.7)_3$ by qP^{q-1} and integrating by parts over $\mathbb{R}^2$ lead to

$$\begin{aligned}\frac{d}{dt}\int P^qdx=&\int\left(2\mu q|\mathfrak{D}(u)|^2P^{q-1}+\lambda q(\mathrm{div}u)^2P^{q-1}-u\cdot\nabla P^q-2qP^q\mathrm{div}u\right)dx\\ =&(1-2q)\int P^q\mathrm{div}udx+q\int\left(2\mu|\mathfrak{D}(u)|^2+\lambda(\mathrm{div}u)^2\right)P^{q-1}dx\\ \le& C\int P^q|\mathrm{div}u|dx+Cq\int\left((\mathrm{div}u)^2+|\nabla u|^2\right)P^{q-1}dx\\ \le& C\|\mathrm{div}u\|_{L^\infty}\left(\int P^qdx+\int|\nabla u|^qdx\right)+C\|\nabla u\|_{L^{2q}}^2\|P\|_{L^q}^{q-1}\\ \le& C\|\mathrm{div}u\|_{L^\infty}\left(\int P^qdx+\int|\nabla u|^qdx\right)+C\left(\|\mathrm{div}u\|_{L^{2q}}^2+\|\omega\|_{L^{2q}}^2\right)\|P\|_{L^q}^{q-1}\\ \le& C\|\mathrm{div}u\|_{L^\infty}\left(\int P^qdx+\int|\nabla u|^qdx\right)\\ &+C\left(\|\mathrm{div}u\|_{L^\infty}\|\mathrm{div}u\|_{L^q}+\|\omega\|_{L^2}^{\frac{2}{q}}\|\nabla\omega\|_{L^2}^{2-\frac{2}{q}}\right)\|P\|_{L^q}^{q-1}\\ \le& C\|\mathrm{div}u\|_{L^\infty}\left(\int P^qdx+\int|\nabla u|^qdx\right)+C\|\sqrt{\rho}\dot{u}\|_{L^2}^{2-\frac{2}{q}}\|P\|_{L^q}^{q-1}\\ \le& C\left(\|\mathrm{div}u\|_{L^\infty}+1\right)\left(\int P^qdx+\int|\nabla u|^qdx+1\right),\end{aligned} \tag{3.33}$$

where have used (2.1), (2.5), (2.10), (3.13), (3.23), Hölder's inequality and the following facts.

$$\begin{aligned}\|\nabla u\|_{L^q} &\le C\|\rho\dot{u}\|_{L^2}^{1-\frac{2}{q}}\left(\|\nabla u\|_{L^2}+\|P\|_{L^2}\right)^{\frac{2}{q}}+C\|P\|_{L^q}\\ &\le C+C\|P\|_{L^q},\end{aligned}$$

due to (2.7) and (3.23). This together with (3.33) yield that

$$\frac{d}{dt}\|P\|_{L^q} \le C\left(\|\mathrm{div}u\|_{L^\infty}+1\right)\left(\|P\|_{L^q}+1\right), \tag{3.34}$$

with constant C only depending on μ, λ and q.

Combining (3.34) with (3.1) and using Gronwall's inequality, one has

$$\sup_{0\le t\le T}\|P(\cdot,t)\|_{L^q}\le C.$$

Moreover, for some $\tilde{q}>2$, in view of (2.1), (2.5), (2.9), (3.3), (3.13), (3.23) and Hölder's inequality, we have

$$\begin{aligned}\|G\|_{L^\infty} &\le C\|G\|_{L^2}+C\|\nabla G\|_{L^{\tilde{q}}}\\ &\le C\|\nabla G\|_{L^{\tilde{q}}}+C\\ &\le C\|\rho\dot{u}\|_{L^{\tilde{q}}}+C\\ &\le C\left\|\sqrt{\rho}\dot{u}\right\|_{L^2}^{\frac{2(\tilde{q}-1)}{\tilde{q}^2-2}}\left\|\sqrt{\rho}\dot{u}\right\|_{L^{\tilde{q}^2}}^{\frac{\tilde{q}(\tilde{q}-2)}{\tilde{q}^2-2}}+C\\ &\le C\left\|\rho^{\tilde{q}^2}\bar{x}^a\right\|_{L^1}^{\frac{\tilde{q}-2}{2\tilde{q}(\tilde{q}^2-2)}}\left\|\dot{u}\bar{x}^{-\frac{a}{2\tilde{q}^2}}\right\|_{L^{2\tilde{q}^2}}^{\frac{\tilde{q}(\tilde{q}-2)}{\tilde{q}^2-2}}+C\\ &\le C\left\|\rho\bar{x}^a\right\|_{L^1}^{\frac{\tilde{q}-2}{2\tilde{q}(\tilde{q}^2-2)}}\left\|\dot{u}\bar{x}^{-\frac{a}{2\tilde{q}^2}}\right\|_{L^{2\tilde{q}^2}}^{\frac{\tilde{q}(\tilde{q}-2)}{\tilde{q}^2-2}}+C\\ &\le C\left(1+\|\nabla\dot{u}\|_{L^2}^{\frac{\tilde{q}(\tilde{q}-2)}{\tilde{q}^2-2}}\right)\\ &\le C\left(1+\|\nabla\dot{u}\|_{L^2}\right),\end{aligned} \tag{3.35}$$

due to $\frac{\tilde{q}(\tilde{q}-2)}{\tilde{q}^2--2}\le 1$. This together with $G=(2\mu+\lambda)\,\mathrm{div}u-P$ and (3.23) yield to

$$\int_0^T\|P\|_{L^\infty}dt\le C\int_0^T\left(\|\mathrm{div}u\|_{L^\infty}+\|G\|_{L^\infty}\right)dt\le C.$$

Hence, the proof of Lemma 3.5 is completed. □

The following lemma is to prove the higher-order derivatives of strong solution (ρ,u,P). In particular, we obtain the second spatial derivatives of u and the $L^2\cap L^{\tilde{q}}$-norm ($\tilde{q}>2$) of the first spatial derivative of ρ and P.

Lemma 3.6. *Suppose that the conditions (3.1) holds, and let $\tilde{q}>2$ be defined in Theorem 1.1, we have that*

$$\sup_{0\le t\le T}\left(\|\nabla u\|_{H^1}+\|\rho\|_{H^1\cap W^{1,\tilde{q}}}+\|P\|_{H^1\cap W^{1,\tilde{q}}}\right)+\int_0^T\|\nabla^2u\|_{L^{\tilde{q}}}^2dt\le C, \tag{3.36}$$

and

$$\sup_{0\le t\le T}\|P\|_{L^\infty}\le C, \tag{3.37}$$

for any $0\le T<T^*$.

Proof. Indeed, $|\nabla\rho|^r$ $(2\le r\le\tilde q)$ satisfies

$$\begin{aligned}&(|\nabla\rho|^r)_t+\mathrm{div}(|\nabla\rho|^r u)+(r-1)|\nabla\rho|^r\mathrm{div}u\\&\quad+r|\nabla\rho|^{r-2}(\nabla\rho)^{tr}\nabla u(\nabla\rho)+r\rho|\nabla\rho|^{r-2}\nabla\rho\cdot\nabla\mathrm{div}u=0,\end{aligned}$$

which together with integrating the resulting equation over $\mathbb{R}^2$ and Hölder's inequality gives that

$$\frac{d}{dt}\|\nabla\rho\|_{L^r}\le C\left(1+\|\nabla u\|_{L^\infty}\right)\|\nabla\rho\|_{L^r}+C\left\|\nabla^2u\right\|_{L^r}. \tag{3.38}$$

Furthermore, in view of (1.7)$_3$, $|\nabla P|^r$ satisfies

$$\begin{aligned}&(|\nabla P|^r)_t+\mathrm{div}(|\nabla P|^r u)+(2r-1)|\nabla P|^r\mathrm{div}u\\&\quad+r|\nabla P|^{r-2}(\nabla P)^{tr}\nabla u(\nabla P)+2rP|\nabla P|^{r-2}\nabla P\cdot\nabla\mathrm{div}u\\&\quad-2r\mu|\nabla P|^{r-2}\nabla P\cdot\nabla|\mathfrak{D}(u)|^2-\lambda r|\nabla P|^{r-2}\nabla P\cdot\nabla(\mathrm{div}u)^2=0,\end{aligned}$$

which together with effective viscous flux $G=(2\mu+\lambda)\,\mathrm{div}u-P$ yields

$$\begin{aligned}&(|\nabla P|^r)_t+\mathrm{div}(|\nabla P|^r u)+(2r-1)|\nabla P|^r\mathrm{div}u+r|\nabla P|^{r-2}(\nabla P)^{tr}\nabla u(\nabla P)\\&+\frac{2r}{2\mu+\lambda}P|\nabla P|^{r-2}\nabla P\cdot\nabla G+\frac{2r}{2\mu+\lambda}P|\nabla P|^r\\&-2r\mu|\nabla P|^{r-2}\nabla P\cdot\nabla|\mathfrak{D}(u)|^2-\lambda r|\nabla P|^{r-2}\nabla P\cdot\nabla(\mathrm{div}u)^2=0.\end{aligned}$$

In view of (2.5) and (3.31), one has

$$\begin{aligned}\frac{d}{dt}\|\nabla P\|_{L^r}^r\le&C\|\nabla u\|_{L^\infty}\|\nabla P\|_{L^r}^r+C\|P\|_{L^{2r}}\|\nabla G\|_{L^{2r}}\|\nabla P\|_{L^r}^{r-1}\\&+C\|\nabla u\|_{L^\infty}\|\nabla^2u\|_{L^r}\|\nabla P\|_{L^r}^{r-1}\\\le&C\|\nabla u\|_{L^\infty}\|\nabla P\|_{L^r}^r+C\|\rho\dot u\|_{L^{2r}}\|\nabla P\|_{L^r}^{r-1}+C\|\nabla u\|_{L^\infty}\|\nabla^2u\|_{L^r}\|\nabla P\|_{L^r}^{r-1}\\\le&C\|\nabla u\|_{L^\infty}\left(\|\nabla^2u\|_{L^r}+\|\nabla P\|_{L^r}\right)\|\nabla P\|_{L^r}^{r-1}+C\|\rho\dot u\|_{L^{2r}}\|\nabla P\|_{L^r}^{r-1}.\end{aligned}\tag{3.39}$$

where have used the fact

$$\int P|\nabla P|^r dx\ge 0.$$

It follows from (3.39) that

$$\frac{d}{dt}\|\nabla P\|_{L^r}\le C\|\nabla u\|_{L^\infty}\left(\|\nabla^2u\|_{L^r}+\|\nabla P\|_{L^r}\right)+C\|\rho\dot u\|_{L^{2r}}. \tag{3.40}$$

Combined with (3.38) and (3.40) gives directly that

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla\rho\|_{L^r}+\|\nabla P\|_{L^r}\right)\\&\le C\left(\|\nabla u\|_{L^\infty}+1\right)\left(\|\nabla^2u\|_{L^r}+\|\nabla P\|_{L^r}+\|\nabla\rho\|_{L^r}\right)+C\|\rho\dot u\|_{L^{2r}}.\end{aligned}\tag{3.41}$$

In order to estimate $\|\nabla u\|_{L^\infty}$, we decompose the velocity field into two parts, namely $u=v+w$, which satisfy the following equations with null boundary Dirichlet conditions, respectively,

$$\mu\Delta v+(\mu+\lambda)\nabla\mathrm{div}v=\nabla P, \tag{3.42}$$

and

$$\mu\Delta w+(\mu+\lambda)\nabla\mathrm{div}w=\rho\dot u. \tag{3.43}$$

In view of Lemmas 2.6, 3.3–Lemma 3.5 and (3.35), we obtain

$$\|\nabla^2w\|_{L^r}\le C\|\rho\dot u\|_{L^r}\le C\left(\|\rho\dot u\|_{L^2}+\|\rho\dot u\|_{L^{\tilde q}}\right)\le C\left(1+\|\nabla\dot u\|_{L^2}\right), \tag{3.44}$$

$$\|\nabla v\|_{L^r}\le C\|P\|_{L^r}\le C, \tag{3.45}$$

and

$$\begin{aligned}\|\nabla v\|_{L^\infty} &\le C\left(1+\ln(e+\|\nabla P\|_{L^{\tilde q}})\|P\|_{L^\infty}+\|P\|_{L^r}\right)\\ &\le C\left(1+\ln(e+\|\nabla P\|_{L^{\tilde q}})\|P\|_{L^\infty}\right).\end{aligned}\tag{3.46}$$

It follows from (2.2), (3.35), (3.41), (3.44)–(3.46) that

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla\rho\|_{L^r}+\|\nabla P\|_{L^r}\right)\\ \le&C\left(\|\nabla u\|_{L^\infty}+1\right)\left(\|\nabla^2u\|_{L^r}+\|\nabla P\|_{L^r}+\|\nabla\rho\|_{L^r}\right)+C\|\rho\dot u\|_{L^{2r}}\\ \le&C\left(\|\nabla w\|_{L^\infty}+\|\nabla v\|_{L^\infty}+1\right)\left(\|\nabla^2u\|_{L^r}+\|\nabla P\|_{L^r}+\|\nabla\rho\|_{L^r}\right)+C\left(1+\|\nabla\dot u\|_{L^2}\right)\\ \le&C\left(\|\nabla w\|_{L^2}+\|\nabla^2w\|_{L^{\tilde q}}+\|\nabla v\|_{L^\infty}+\|\nabla\dot u\|_{L^2}+1\right)\left(\|\nabla^2u\|_{L^r}+\|\nabla P\|_{L^r}+\|\nabla\rho\|_{L^r}+1\right)\\ \le&C\left(\|\rho\dot u\|_{L^{\tilde q}}+\|\nabla v\|_{L^\infty}+\|\nabla\dot u\|_{L^2}+1\right)\left(\|\nabla^2u\|_{L^r}+\|\nabla P\|_{L^r}+\|\nabla\rho\|_{L^r}+1\right)\\ \le&C\left(\ln(e+\|\nabla P\|_{L^{\tilde q}})\|P\|_{L^\infty}+\|\nabla\dot u\|_{L^2}+1\right)\left(\|\nabla^2u\|_{L^r}+\|\nabla P\|_{L^r}+\|\nabla\rho\|_{L^r}+1\right).\end{aligned}\tag{3.47}$$

Set $r=\tilde q$, the standard $L^{\tilde q}$-estimate of elliptic system $(1.7)_2$, (2.9), (3.23) and (3.42) give to

$$\begin{aligned}\|\nabla^2u\|_{L^{\tilde q}}&\le C\left(\|\rho\dot u\|_{L^{\tilde q}}+\|\nabla P\|_{L^{\tilde q}}\right)\\ &\le C\left(1+\|\nabla\dot u\|_{L^2}+\|\nabla P\|_{L^{\tilde q}}\right).\end{aligned}\tag{3.48}$$

Inserting (3.48) into (3.47), one has

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla\rho\|_{L^{\tilde q}}+\|\nabla P\|_{L^{\tilde q}}\right)\\ \le&C\left(\ln(e+\|\nabla P\|_{L^{\tilde q}})\|P\|_{L^\infty}+\|\nabla\dot u\|_{L^2}+1\right)\left(\|\nabla P\|_{L^{\tilde q}}+\|\nabla\rho\|_{L^{\tilde q}}+1\right)\\ &+C\left(\ln(e+\|\nabla P\|_{L^{\tilde q}})\|P\|_{L^\infty}+\|\nabla\dot u\|_{L^2}+1\right)\|\nabla\dot u\|_{L^2}\\ \le&C\left(\ln(e+\|\nabla P\|_{L^{\tilde q}})\|P\|_{L^\infty}+\|\nabla\dot u\|_{L^2}+1\right)\left(\|\nabla P\|_{L^{\tilde q}}+\|\nabla\rho\|_{L^{\tilde q}}+1\right)\\ &+C\left(\ln(e+\|\nabla P\|_{L^{\tilde q}})(1+\|\nabla P\|_{L^{\tilde q}})+\|\nabla\dot u\|_{L^2}+1\right)\|\nabla\dot u\|_{L^2}\\ \le&C\left(1+\|P\|_{L^\infty}+\|\nabla\dot u\|_{L^2}^2\right)\left(e+\|\nabla\rho\|_{L^{\tilde q}}+\|\nabla P\|_{L^{\tilde q}}\right)\ln\left(e+\|\nabla\rho\|_{L^{\tilde q}}+\|\nabla P\|_{L^{\tilde q}}\right).\end{aligned}\tag{3.49}$$

Set $f(t)=e+\|\nabla\rho\|_{L^{\tilde q}}+\|\nabla P\|_{L^{\tilde q}}$ and $g(t)=1+\|P\|_{L^\infty}+\|\nabla\dot u\|_{L^2}^2$, together with (3.49), we obtain

$$f'(t)\le Cg(t)f(t)\ln f(t),$$

which implies

$$(\ln f(t))'\le Cg(t)\ln f(t).$$

This together with (3.23), (3.32) and Gronwall's inequality yields to

$$\sup_{0\le t\le T}\left(\|\nabla\rho\|_{L^{\tilde q}}+\|\nabla P\|_{L^{\tilde q}}\right)\le C.\tag{3.50}$$

It follows from (2.2), (3.13), (3.48) and (3.50) that

$$\begin{aligned}\|\nabla u\|_{L^\infty}\le&C\left(\|\nabla u\|_{L^2}+\|\nabla^2u\|_{L^{\tilde q}}\right)\\ \le&C\left(1+\|\nabla\dot u\|_{L^2}+\|\nabla P\|_{L^{\tilde q}}\right)\\ \le&C\left(1+\|\nabla\dot u\|_{L^2}^2\right),\end{aligned}$$

which together with (3.23) yields to

$$\int_0^T\|\nabla u\|_{L^\infty}dt\le C.\tag{3.51}$$

Taking $r=2$, we obtain

$$\frac{d}{dt}\left(\|\nabla\rho\|_{L^2}+\|\nabla P\|_{L^2}\right)\le C\left(1+\|\nabla u\|_{L^\infty}\right)\left(\|\nabla\rho\|_{L^2}+\|\nabla P\|_{L^2}+1\right)+C\|\nabla\dot u\|_{L^2},\tag{3.52}$$

where we have used the following facts.

$$\begin{aligned} \|\nabla^2 u\|_{L^2} &\leq C\left(\|\rho\dot{u}\|_{L^2} + \|\nabla P\|_{L^2}\right) \\ &\leq C\left(1 + \|\nabla P\|_{L^2}\right). \end{aligned} \tag{3.53}$$

Using (3.23), (3.51) and Gronwall's inequality yield to

$$\sup_{0\leq t\leq T}\left(\|\nabla\rho\|_{L^2} + \|\nabla P\|_{L^2}\right) \leq C, \tag{3.54}$$

which together with (3.53) yields to

$$\sup_{0\leq t\leq T}\|\nabla^2 u\|_{L^2} \leq C. \tag{3.55}$$

In view of (3.48), (3.23) and (3.50), we obtain

$$\begin{aligned} \int_0^T \|\nabla^2 u\|_{L^{\tilde q}}^2 dt &\leq C\int_0^T \left(1 + \|\nabla\dot{u}\|_{L^2} + \|\nabla P\|_{L^{\tilde q}}^2\right) dt \\ &\leq C\int_0^T \left(1 + \|\nabla\dot{u}\|_{L^2}^2 + \|\nabla P\|_{L^{\tilde q}}^2\right) dt \\ &\leq C. \end{aligned} \tag{3.56}$$

Finally, due to (2.2), one has

$$\|P\|_{L^\infty} \leq C\|P\|_{L^2} + C\|\nabla P\|_{L^{\tilde q}},$$

which together with (3.50) gives to

$$\sup_{0\leq t\leq T}\|P\|_{L^\infty} \leq C. \tag{3.57}$$

Thus the desired (3.36) and (3.37) follows from (3.50), (3.54), (3.55), (3.56) and (3.57). We complete the proof of Lemma 3.6.

□

We have the following spatial weighted estimate on the density.

Lemma 3.7. *With the assumption* (3.1)*, and let $\tilde q > 2$ be defined in Theorem* 1.1*, it holds*

$$\sup_{0\leq t\leq T}\|\rho\bar{x}^a\|_{H^1\cap W^{1,\tilde q}} \leq C. \tag{3.58}$$

Proof. Indeed, in view of $(1.7)_1$, $\rho\bar{x}^a$ satisfies

$$(\rho\bar{x}^a)_t + u\cdot\nabla\left(\rho\bar{x}^a\right) - a\rho\bar{x}^a u\cdot\nabla\ln\bar{x} + \rho\bar{x}^a \mathrm{div} u = 0.$$

Thus, $|\nabla\left(\rho\bar{x}^a\right)|^p$ $(p\in[2,r])$ satisfies

$$\begin{aligned} &(|\nabla\left(\rho\bar{x}^a\right)|^p)_t + \mathrm{div}(|\nabla\left(\rho\bar{x}^a\right)|^p u) + (p-1)|\nabla\left(\rho\bar{x}^a\right)|^p \mathrm{div} u \\ &\quad + p|\nabla\left(\rho\bar{x}^a\right)|^{p-2}(\nabla\left(\rho\bar{x}^a\right))^{tr}\nabla u(\nabla\left(\rho\bar{x}^a\right)) + p\rho\bar{x}^a|\nabla\left(\rho\bar{x}^a\right)|^{p-2}\nabla\left(\rho\bar{x}^a\right)\cdot\nabla\mathrm{div} u \\ &\quad - ap|\nabla\left(\rho\bar{x}^a\right)|^p u\cdot\nabla\ln\bar{x} - ap\rho\bar{x}^a|\nabla\left(\rho\bar{x}^a\right)|^{p-2}\left(\nabla\left(\rho\bar{x}^a\right)\right)^{tr}\left(\nabla u\nabla\ln\bar{x} + u\nabla^2\ln\bar{x}\right) = 0. \end{aligned}$$

Integrating this equality over $\mathbb{R}^2$, using (2.1), (2.9), (3.13) and (3.36), we obtain

$$\begin{aligned} \frac{d}{dt}\|\nabla\left(\rho\bar{x}^a\right)\|_{L^p} \leq& C\left(1 + \|\nabla u\|_{L^\infty} + \|u\cdot\nabla\ln\bar{x}\|_{L^\infty}\right)\|\nabla\left(\rho\bar{x}^a\right)\|_{L^p} \\ &+ C\|\rho\bar{x}^a\|_{L^\infty}\left(\|\nabla u\nabla\ln\bar{x}\|_{L^p} + \|u\nabla^2\ln\bar{x}\|_{L^p} + \left\|\nabla^2 u\right\|_{L^p}\right) \\ \leq& C\left(1 + \|\nabla u\|_{W^{1,r}}\right)\|\nabla\left(\rho\bar{x}^a\right)\|_{L^p} \\ &+ C\|\rho\bar{x}^a\|_{L^\infty}\left(\|\nabla u\|_{L^p} + \|u\bar{x}^{-\frac{1}{2}}\|_{L^p} + \left\|\nabla^2 u\right\|_{L^p}\right) \\ \leq& C\left(1 + \left\|\nabla^2 u\right\|_{L^p} + \|\nabla u\|_{W^{1,r}}\right)\left(1 + \|\nabla\left(\rho\bar{x}^a\right)\|_{L^p} + \|\nabla\left(\rho\bar{x}^a\right)\|_{L^r}\right), \end{aligned} \tag{3.59}$$

where we have used the following facts. In fact, it follows from (2.1), (2.9) and (3.36) that

$$\begin{aligned}\|u\cdot\nabla\ln\bar{x}\|_{L^\infty} &\le C\|u\bar{x}^{-\frac{1}{2}}\|_{L^\infty}\\ &\le C\left(\|u\bar{x}^{-\frac{1}{2}}\|_{L^6}+\left\|\nabla\left(u\bar{x}^{-\frac{1}{2}}\right)\right\|_{L^6}\right)\\ &\le C\left(\|u\bar{x}^{-\frac{1}{2}}\|_{L^6}+\|\nabla u\|_{L^6}\right)\\ &\le C\left(1+\|\nabla u\|_{H^1}\right)\\ &\le C,\end{aligned}$$

and

$$\begin{aligned}\|\rho\bar{x}^a\|_{L^\infty} \le &C\left(\|\rho\bar{x}^a\|_{L^2}+\|\nabla\left(\rho\bar{x}^a\right)\|_{L^{\tilde{q}}}\right)\\ \le &C\left(\|\rho\bar{x}^a\|_{L^1}^{\frac{1}{2}}\|\rho\bar{x}^a\|_{L^\infty}^{\frac{1}{2}}+\|\nabla\left(\rho\bar{x}^a\right)\|_{L^{\tilde{q}}}\right)\\ \le &C\left(\|\rho\bar{x}^a\|_{L^1}+\varepsilon\|\rho\bar{x}^a\|_{L^\infty}+\|\nabla\left(\rho\bar{x}^a\right)\|_{L^{\tilde{q}}}\right)\\ \le &C\left(1+\varepsilon\|\rho\bar{x}^a\|_{L^\infty}+\|\nabla\left(\rho\bar{x}^a\right)\|_{L^{\tilde{q}}}\right),\end{aligned}$$

choosing ε suitably small yields to

$$\|\rho\bar{x}^a\|_{L^\infty}\le C\left(1+\|\nabla\left(\rho\bar{x}^a\right)\|_{L^{\tilde{q}}}\right).$$

Letting $p=\tilde{q}$ in (3.59), together with Gronwall's inequality and (3.36) gives

$$\sup_{0\le t\le T}\|\nabla\left(\rho\bar{x}^a\right)\|_{L^{\tilde{q}}}\le C. \tag{3.60}$$

Furthermore, taking $p=2$ in (3.57), we deduce from (3.36) and (3.60) that

$$\sup_{0\le t\le T}\|\nabla\left(\rho\bar{x}^a\right)\|_{L^2}\le C.$$

This combined with (3.60) gives (3.58). This finishes the proof of Lemma 3.7. □

Proof of Theorem 1.1 Suppose that (1.15) was false, namely, (3.1) holds.

With the aid of the a priori estimates Lemma 3.1–Lemma 3.7, the functions $(\rho,u,P)(x,T^*)=\lim_{t\to T^*}(\rho,u,P)(x,t)$ satisfy the conditions imposed on the initial data at the time $t=T^*$. Hence, we obtain $\rho\dot{u}\in C([0,T^*];L^2)$, which implies

$$\rho\dot{u}(x,T^*)=\lim_{t\to T^*}\rho\dot{u}(x,t)\in L^2.$$

It follows from (3.36) that

$$\left(-\mu\Delta u-(\mu+\lambda)\nabla\mathrm{div}u+\nabla P\right)|_{t=T^*}=\sqrt{\rho}(x,T^*)g(x,T^*),$$

where

$$g(x)=\begin{cases}\rho^{-\frac{1}{2}}(x,T^*)(\rho\dot{u})(x,T^*), & \text{for } x\in\{x|\rho(x,T^*)>0\},\\ \\ 0, & \text{for } x\in\{x|\rho(x,T^*)=0\},\end{cases}$$

satisfying $g\in L^2$. This implies that $(\rho,u,P)(x,T^*)$ satisfy compatibility conditions (1.14). Therefore, one can take $(\rho,u,P)(x,T^*)$ as the initial data and apply Proposition 2.1 to extend the local strong solution beyond T^*, which contradicts the assumption on T^*. We thus finish the proof of Theorem 1.1.

Acknowledgements. The author would like to thank the anonymous referees for their useful comments and suggestions, which led to the final version of this paper.

Data Availibility Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author states that there is no conflict of interest.

References

[1] Beale, J.T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Commun. Math. Phys. **94**, 61–66 (1984)
[2] Brézis, H., Wainger, S.: A note on limiting cases of Sobolev embeddings and convolution inequalities. Commun. Partial Differ. Equ. **5**, 773–789 (1980)
[3] Cho, Y., Kim, H.: Existence results for viscous polytropic fluids with vacuum. J. Differ. Equ. **228**, 377–411 (2006)
[4] Cho, Y., Kim, H.: On classical solutions of the compressible Navier–Stokes equations with nonnegative initial densities. Manuscripta Math. **120**, 91–129 (2006)
[5] Du, L.L., Wang, Y.F.: Blowup criterion for 3-dimensional compressible Navier–Stokes equations involving velocity divergence. Commun. Math. Sci. **12**, 1427–1435 (2014)
[6] Du, L.L., Wang, Y.F.: Mass concentration phenomenon in compressible magnetohydrodynamic flows. Nonlinearity **28**, 2959–2976 (2015)
[7] Du, L.L., Wang, Y.F.: A blowup criterion for viscous, compressible, and heat-conductive magnetohydrodynamic flows. J. Math. Phys. **56**, 091503 (2015)
[8] Duan, Q.: Blowup of smooth solution for non-isentropic magnetohydrodynamic equations without heat conductivity. Math. Meth. Appl. Sci. **40**, 1865–1879 (2017)
[9] Duan, Q., Huang, X.D.: Local weak solution of the isentropic compressible Navier-Stokes equations for half-space. Sci. China Math. **65**(5), 993–1002 (2022)
[10] Duan, Q., Li, H.L.: Global existence of weak solution for the compressible Navier–StokesPoisson system for gaseous stars. J. Differ. Equ. **259**, 5302–5330 (2015)
[11] Duan, Q., Xin, Z.P., Zhu, S.G.: On regular solutions for multi-dimensional full compressible Navier-Stokes equations with degenerate viscosities and far field vacuum. Arch. Rational Mech. Anal. **247**, 1 (2023)
[12] Fan, J.S., Jiang, S., Ou, Y.B.: A blow-up criterion for compressible viscous heatconductive flows. Ann. Inst. H. Poincaré Anal. Non Linéaire **27**, 337–350 (2010)
[13] Feireisl, E., Novotny, A., Petzeltová, H.: On the existence of globally defined weak solutions to the Navier-Stokes equations. J. Math. Fluid Mech. **3**, 358–392 (2001)
[14] Hoff, D.: Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. J. Differ. Equ. **120**, 215–254 (1995)
[15] Hu, X.P., Wang, D.H.: Global solutions to the three-dimensional full compressible magnetohydrodynamic flows. Commun. Math. Phys. **283**, 255–284 (2008)
[16] Hu, X.P., Wang, D.H.: Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. Arch. Rational Mech. Anal. **197**, 203–238 (2010)
[17] Huang, X.D., Li, J.: Serrin-type blowup criterion for viscous, compressible, and heat conducting Navier-Stokes and Magnetohydrodynamic flows. Commun. Math. Phys. **324**(1), 147–171 (2013)
[18] Huang, X.D., Li, J.: Global classical and weak solutions to the three-dimensional full compressible Navier-Stokes system with vacuum and large oscillations. Arch. Ration. Mech. Anal. **227**, 995–1059 (2018)
[19] Huang, X.D., Li, J.: Existence and blowup behavior of global strong solutions to the two-dimensional baratropic compressible Navier-Stokes system with vacuum and large initial data. J. Math. Pures Appl. **106**(9), 123–154 (2016)
[20] Huang, X.D., Li, J., Wang, Y.: Serrin-type blowup criterion for full compressible Navier–Stokes system. Arch. Ration. Mech. Anal. **207**, 303–316 (2013)

[21] Huang, X.D., Li, J., Xin, Z.P.: Serrin type criterion for the three-dimensional viscous compressible flows. SIAM J. Math. Anal. **43**, 1872–1886 (2011)
[22] Huang, X.D., Li, J., Xin, Z.P.: Blowup criterion for viscous barotropic flows with vacuum states. Commun. Math. Phys. **301**, 23–35 (2011)
[23] Huang, X.D., Li, J., Xin, Z.P.: Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier- Stokes equations. Comm. Pure Appl. Math. **65**, 549–585 (2012)
[24] Huang, X.D., Li, J.: A blow-up criterion for classical solutions to the compressible Navier-Stokes equations. Sci. China Math. **53**, 671–686 (2010)
[25] Huang, X.D., Xin, Z.P.: On formation of singularity for non-isentropic Navier-Stokes equations without heat-conductivity. Discrete Contin. Dyn. Syst. **36**, 4477–4493 (2016)
[26] Ji, R.H., Wang, Y.F.: Mass concentration phenomenon to the 2D Cauchy problem of the compressible Navier-Stokes equations. Discrete Contin. Dyn. Syst. **39**, 1117–1133 (2019)
[27] Jiang, S., Ou, Y.B.: A blow-up criterion for compressible viscous heat-conductive flows, Acta Math. Sci. Ser. B Engl. Ed. **30**(6), 1851–1864 (2010)
[28] Jiang, S., Zhang, P.: On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations. Commun. Math. Phys. **215**, 559–581 (2001)
[29] Jiang, S., Zhang, P.: Axisymmetric solutions of the 3-D Navier-Stokes equations for compressible isentropic fluids. J. Math. Pure Appl. **82**, 949–973 (2003)
[30] Jiu, Q.S., Wang, Y., Xin, Z.P.: Global well-posedness of Cauchy problem of two-dimensional compressible Navier-stokes equations in weighted spaces. J. Differ. Equ. **255**, 351–404 (2013)
[31] Li, H.L., Xu, X.Y., Zhang, J.W.: Global classical solutions to 3D compressible Magnetohydrodynamic equations with large oscillations and vacuum. SIAM J. Math. Anal. **45**, 1356–1387 (2013)
[32] Li, J., Liang, Z.L.: On local classical solutions to the Cauchy problem of the two-dimensional barotropic compressible Navier-Stokes equations with vacuum. J. Math. Pures Appl. **4**, 640–671 (2014)
[33] Li, J., Xin, Z.P.: Global well-posedness and large time asymptotic behavior of classical solutions to the compressible Navier-Stokes equations with vacuum. Ann. PDE **5**(1), 7 (2019)
[34] Liang, Z.L., Shi, X.D.: Classical solutions to the Cauchy problem for 2D viscous polytropic fluids with vacuum and zero heat-conduction. Commun. Math. Sci. **13**, 327–345 (2015)
[35] Lions, P.L.: Mathematical Topics in Fluid Mechanics. Compressible Models, vol. 2. Oxford University Press, New York (1998)
[36] Lu, L., Huang, B.: On local strong solutions to the Cauchy problem of the two-dimensional full compressible magnetohydrodynamic equations with vacuum and zero heat conduction. Nonlinear Anal. Real World Appl. **31**, 409–430 (2016)
[37] Lü, B.Q., Shi, X.D., Xu, X.Y.: Global well-posedness and large time asymptotic behavior of strong solutions to the compressible magnetohydrodynamic equations with vacuum. Indiana Univ. Math. J. **65**, 925–975 (2016)
[38] Lü, B.Q., Xu, Z.H., Zhong, X.: Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent magnetohydrodynamic equations with vacuum. J. Math. Kyoto Univ. **108**, 41–62 (2017)
[39] Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. **20**, 67–104 (1980)
[40] Nash, J.: Le probleme de Cauchy pour les'equations différentielles d'un fluide général. Bull. Soc. Math. France **90**, 487–497 (1962)
[41] Nirenberg, L.: On elliptic partial differential equations. Ann. Scuola Norm. Sup. Pisa **13**(3), 115–162 (1959)
[42] Ponce, G.: Remarks on a paper: Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Commun. Math. Phys. **98**, 349–353 (1985)
[43] Seerin, J.: On the uniqueness of compressible fluid motion. Arch. Ration. Math. Anal. **3**, 271–288 (1959)
[44] Serrin, J.: On the interior regularity of weak solutions of Navier-Stokes equation. Arch. Ration. Mech. Anal. **9**, 187–195 (1962)
[45] Sun, Y.Z., Wang, C., Zhang, Z.F.: A Beale-Kato-Majda blow-up criterion for the 3-D compressible Navier-Stokes equations. J. Math. Pures Appl. **95**, 36–47 (2011)
[46] Sun, Y.Z., Wang, C., Zhang, Z.F.: A Beale-Kato-Majda criterion for three dimensional compressible viscous heat-conductive flows. Arch Ration. Mech. Anal. **201**, 727–742 (2011)
[47] Wang, Y.F.: A Beale-Kato-Majda criterion for three dimensional compressible viscous non-isentropic magnetohydrodynamic flows without heat-conductivity. J. Differ. Equ. **280**, 66–98 (2021)
[48] Wang, Y.: One new blowup criterion for the 2D full compressible Navier-Stokes system. Nonlinear Anal. Real World Appl. **16**, 214–226 (2014)
[49] Wen, Y.H., Zhu, C.J.: Blow-up criterions of strong solutions to 3D compressible Navier-Stokes equations with vacuum. Adv. Math. **248**, 534–572 (2013)
[50] Xin, Z.P.: Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density. Commun. Pure Appl. Math. **51**(2), 229–240 (1998)
[51] Xin, Z.P., Yan, W.: On blowup of classical solutions to the compressible Navier-Stokes Equations. Commun. Math. Phys. **231**, 29–541 (2013)
[52] Xu, X.Y., Zhang, J.W.: A blow-up criterion for 3D compressible magnetohydrodynamic equations with vacuum. Math. Models and Meth. Appl. Sci. **22**(2), 1150010 (2012)

[53] Zhong, X.: On formation of singularity of the full compressible magnetohydrodynamic equations with zero heat conduction. Indiana Univ. Math. J. **68**, 1379–1407 (2019)
[54] Zhong, X.: Singularity formation to the 2D Cauchy problem of the full compressible Navier-Stokes equations with zero heat conduction. Adv. Math. (China) **52**, 290–304 (2023)
[55] Zhong, X.: Singularity formation to the two-dimensional full compressible Navier-Stokes equations with zero heat conduction in a bounded domain. arXiv:1810.01265

Yongfu Wang
School of Mathematics
Southwestern University of Finance and Economics
Chengdu 611130
People's Republic of China
e-mail: wyf1247@163.com

(accepted: July 1, 2024; published online: July 15, 2024)