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A Priori Error Analysis and Finite Element Approximations for a Coupled Model Under Nonlinear Slip Boundary Conditions

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Abstract. We consider the time-dependent Navier–Stokes system coupled with the heat equation governed by the nonlinear Tresca boundary conditions. We propose a discretization of these equations that combines Euler implicit scheme in time and finite element approximations in space. We present optimal error estimates for velocity, pressure and temperature. Numerical examples are displayed to illustrate the theoretical results.

Keywords. Navier–Stokes equation, Heat equation, Finite element discretization, Nonlinear slip boundary conditions, Variational inequality, *A priori* error analysis.

1. Introduction

Let Ω be a connected bounded open set in $\mathbb{R}^d, d = 2$ or 3, with a Lipschitz-continuous boundary $\partial\Omega$ divided in two parts S and $\Gamma = \partial\Omega\backslash\bar{S}$ with $\bar{\Gamma} \cap \bar{S} = \emptyset$. And let $[0, T]$ denote an interval in $\mathbb{R}$ where T is a positive real number.

We consider the time-dependent Navier–Stokes problem coupled with the heat equation,

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \text{div}\,(2\nu(\theta)\boldsymbol{Du}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in} \quad \Omega \times [0, T], \\ \text{div}\,\boldsymbol{u} = 0 & \text{in} \quad \Omega \times [0, T], \\ \dfrac{\partial \theta}{\partial t} - \text{div}\,(\mu(\theta)\nabla\theta) + (\boldsymbol{u}.\,\nabla)\theta = k & \text{in} \quad \Omega \times [0, T]. \end{cases} \tag{1.1}$$

The unknowns in (1.1) are the velocity $\boldsymbol{u}$, the pressure p and the temperature θ. The function $\boldsymbol{f}$ represents the external volumic forces and the function k represents the external heat source. They depends only on the position vector $(\boldsymbol{x}, t) \in \mathbb{R}^d \times [0, T]$. The functions ν and μ represents respectively the viscosity and the diffusion coefficient, they are both positive, bounded and depends on the temperature.

The first equation in (1.1) represents the balance of forces in the system, while the second equation is the incompressibility of the fluid. The third equation is the heat exchange in the system.

The force within the fluid is the Cauchy stress tensor σ given by the relation $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\theta, \boldsymbol{u}, p)$ given as:

$$\boldsymbol{\sigma} = 2\nu(\theta)\boldsymbol{Du} - p\boldsymbol{I}.$$

Such that $\boldsymbol{I}$ is the identity matrix in $M_d(\mathbb{R})$ and $\boldsymbol{Du}$ is the symmetric tensor defined by

$$\boldsymbol{Du} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\text{Tr}}).$$

The system of equations (1.1) is supplemented by the boundary conditions on the velocity and temperature. For that purpose, we impose the following initial conditions

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0, \qquad \theta(x, 0) = \theta_0 \qquad \text{on} \quad \bar{\Omega}. \tag{1.2}$$

We assume the Dirichlet conditions

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{0} \quad \text{on} \quad \Gamma\times(0,T) \quad \text{and} \quad \theta(\boldsymbol{x},t)=\theta_b \quad \text{on} \quad \partial\Omega\times(0,T), \tag{1.3}$$

θ_b being given and non-negative.

On the part of the boundary S, the velocity is decomposed following its normal and tangential part. We assume the impermeability condition

$$u_N := \boldsymbol{u}\cdot\boldsymbol{n}=\boldsymbol{0} \quad \text{on} \quad S\times(0,T). \tag{1.4}$$

Such that $\boldsymbol{n}$ is the outward unit normal on the boundary $\partial\Omega$ and u_N is the normal component of the velocity. Then the tangential component of the velocity is defined by $\boldsymbol{u_\tau}=\boldsymbol{u}-u_N\boldsymbol{n}$, such that $\boldsymbol{\tau}$ is the tangent vector orthogonal to $\boldsymbol{n}$.

Additionally to (1.4), we also impose on S a threshold slip condition. The threshold slip condition is formulated through a positive function $g : S \to (0,\infty)$ which is called the barrier of threshold function and the tangential part of $\boldsymbol{\sigma n}$ as follows:

$$\left.\begin{array}{llll}\text{If} & |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}|<g & \text{then} & \boldsymbol{u_\tau}=\boldsymbol{0}, \\ \text{If} & |(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}|=g & \text{then} & \boldsymbol{u_\tau}\neq\boldsymbol{0} \quad \text{and} \quad -(\boldsymbol{\sigma n})_{\boldsymbol{\tau}}=g\dfrac{\boldsymbol{u_\tau}}{|\boldsymbol{u_\tau}|}\end{array}\right\} \text{ on } S\times(0,T). \tag{1.5}$$

Many works have been carried out for the system (1.1) with different boundary conditions and different discretization method. In [1] as well in [6], a similar model provided by the homogeneous Dirichlet conditions has been studied. And the authors treated the coupled system by using a spectral discretization. Furthermore, many works have examined the numerical analysis (see for instance [2,7,12]).

In this article, the novelty of our work compared to the existence ones is to propose an unsteady problem under non linear boundary condition by adding an explicit dependency to the temperature for the viscosity and the diffusion coefficient. One of the major challenges in this work is the presence of variational inequality in our problem. Its numerical solutions have been examined by many researchers, see for example, [9,16,19] for a first breakthrough for a systematic mathematical analysis of problems formulated as variational inequalities.

The point of departure of this study related to the numerical part is the work of J.K. and Djoko [8] where some new numerical methods for the Stokes and Navier–Stokes flow driven by nonlinear slip boundary conditions was discussed. To solve the Stokes system, they first reduced the related variational inequality into a saddle-point problem for a well chosen augmented Lagrangian. And to solve this saddle point problem they suggested an alternating direction method of multiplier together with finite element approximations. The solution of the Navier–Stokes was solved by operator splitting. The numerical experiments showed that these methods are scalable, i.e. the number of iterations required for convergence is virtually independent of mesh size.

In this article, the problem is discretized in time by Euler implicit scheme and in order to facilitate its resolution we have used the operator splitting technique. Indeed, it makes it possible to remove the complexity of a problem by reducing it to several simpler sub-problems.

The outline of the paper is as follows:

- In Sect. 2, we introduce some notations and functional spaces useful for the studies of the problem. Then, we study the variational formulation.
- Sect. 3 is devoted to introduce and study the descretized problem then perform the *a priori* corresponding error analysis for $d=2$.
- In Sect. 4, we introduce an operator-splitting method based on the Marchuk-Yanenko's scheme.
- Some numerical experiments are presented in Sect. 5.

2. Analysis of the Continuous Problem

2.1. Variational Formulation

To write the system (1.1) in a variational form, we need some preliminaries. We use the standard definitions for the sobolev spaces [5] and for boldface characters we denote vector quantities

$$\boldsymbol{H}^1(\Omega) = (H^1(\Omega))^d, \quad \boldsymbol{L}^2(\Omega) = (L^2(\Omega))^d \quad \text{and} \quad L^2_+(S) = L^2(S) \cap \mathbb{R}_+.$$

We claim that $H^{1/2}(\partial\Omega)$ is the space of trace for elements of $\boldsymbol{H}^1(\Omega)$. We introduce the space

$$L^2_0(\Omega) = \big\{ q \in L^2(\Omega);\ \int_\Omega q(x)dx = 0 \big\}. \tag{2.1}$$

And we define the following functional spaces

$$\boldsymbol{V}(\Omega) = \big\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \qquad \boldsymbol{v}_{|_\Gamma} = 0, \qquad \boldsymbol{v}\cdot\boldsymbol{n}_{|_S} = 0 \big\}, \tag{2.2}$$

And,

$$\boldsymbol{V}_{\mathrm{div}}(\Omega) = \big\{ \boldsymbol{v} \in \boldsymbol{V}(\Omega) \qquad \mathrm{div}\,\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \Omega \big\}. \tag{2.3}$$

Lemma 2.1. *For $d = 2$, we have the relation*

$$\|\boldsymbol{v}\|^2_{\boldsymbol{L}^4(\Omega)} \le c \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} |\boldsymbol{v}|_{\boldsymbol{H}^1_0(\Omega)}. \tag{2.4}$$

For the mathematical investigations of (1.1) we assume the following hypotheses:

Hypothesis H1. We assume that the viscosity $\nu(\theta)$ and the heat diffusion coefficient $\mu(\theta)$ are two functions which depend on the temperature θ and which satisfy:

$$\begin{cases} \forall s \in \mathbb{R}, & 0 < \nu_0 \le \nu(s) \le \nu_1, \\ \forall s \in \mathbb{R}, & 0 < \mu_0 \le \mu(s) \le \mu_1. \end{cases} \tag{2.5}$$

And

$$\begin{aligned} \forall s, t \in \mathbb{R}, \qquad & |\nu(s) - \nu(t)| \le \nu^* |s - t|, \\ & |\mu(s) - \mu(t)| \le \mu^* |s - t|. \end{aligned} \tag{2.6}$$

where ν_0, ν_1, μ_0, μ_1, ν^* and μ^* are strictly positive constants.

Hypothesis H2. We assume that:

$$\boldsymbol{f} \in L^\infty(0, T; \boldsymbol{L}^2(\Omega)), \quad k \in L^2(0, T; H^{-1}(\Omega)), \quad \theta_b \in L^2(0, T; H^{1/2}(\partial\Omega)) \text{ and } g \in L^2_+(S). \tag{2.7}$$

Hypothesis H3. The data θ_0 belongs to $L^2(\Omega)$ and the data $\boldsymbol{u}_0$ belongs to $\boldsymbol{L}^2(\Omega)$ satisfying the following compatibility condition

$$\mathrm{div}\,\boldsymbol{u}_0 = 0 \qquad \text{in } \Omega. \tag{2.8}$$

We introduce the following functionals that will be used to write the weak form on the problem in abstract setting.

$$
\begin{array}{lcll}
a_1: & \boldsymbol{H}^1(\Omega)\times \boldsymbol{H}^1(\Omega) & \longrightarrow & \mathbb{R}\\
 & (\theta:\boldsymbol{u},\boldsymbol{v}) & \longmapsto & a_1(\theta:\boldsymbol{u},\boldsymbol{v}) = \displaystyle\int_\Omega 2\nu(\theta)\boldsymbol{Du}:\boldsymbol{Dv}\,d\boldsymbol{x}.\\
a_2: & H^1(\Omega)\times H^1(\Omega) & \longrightarrow & \mathbb{R}\\
 & (\theta:\theta,\rho) & \longmapsto & a_2(\theta:\theta,\rho) = \displaystyle\int_\Omega \mu(\theta)\nabla\theta\cdot\nabla\rho\,d\boldsymbol{x}.\\
b: & \boldsymbol{H}^1(\Omega)\times L^2_0(\Omega) & \longrightarrow & \mathbb{R}\\
 & (\boldsymbol{u},q) & \longmapsto & b(\boldsymbol{u},q) = -\displaystyle\int_\Omega \operatorname{div}\boldsymbol{u}\; q\,d\boldsymbol{x}.\\
d_1: & \boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega) & \longrightarrow & \mathbb{R}\\
 & (\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) & \longmapsto & d_1(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = \displaystyle\int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\,\boldsymbol{v}\,d\boldsymbol{x},\\
d_2: & \boldsymbol{H}^1(\Omega)\times H^1(\Omega)\times H^1(\Omega) & \longrightarrow & \mathbb{R}\\
 & (\boldsymbol{v},\theta,\rho) & \longmapsto & d_2(\boldsymbol{v},\theta,\rho) = \displaystyle\int_\Omega (\boldsymbol{v}\cdot\nabla)\theta\rho\,d\boldsymbol{x},\\
J: & \boldsymbol{H}^1(\Omega) & \longrightarrow & \mathbb{R}\\
 & \boldsymbol{u} & \longmapsto & J(\boldsymbol{u}) = \displaystyle\int_S g|\boldsymbol{u}_\tau|\,dS.
\end{array}
\tag{2.9}
$$

where dS being the measure on the surface S.

In order to simplify, we note by

$$\boldsymbol{V} := L^2(0,T;\boldsymbol{V}(\Omega))\cap L^\infty(0,T;\boldsymbol{L}^2(\Omega)),\quad M := L^2(0,T;L^2_0(\Omega))\quad \text{and}\quad Q := L^2(0,T;H^1(\Omega)).$$

We consider the variational problem: For $\theta_b \in L^2(0,T;H^{1/2}(\partial\Omega))$, $\boldsymbol{f} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, $k \in L^2(0,T;H^{-1}(\Omega))$,

$$
\begin{cases}
\text{Find } (\boldsymbol{u},p,\theta)\in \boldsymbol{V}\times M\times Q \text{ such that:}\\
\qquad\qquad \boldsymbol{u}(.,0)=\boldsymbol{u}_0 \ \text{ and } \ \theta(.,0)=\theta_0 \ \text{ in } \ \Omega,\\
\qquad\qquad \theta(.,t)=\theta_b(.,t) \text{ on } \partial\Omega, \quad \forall\, 0\le t\le T,\\
\forall\ (\boldsymbol{v},q,\rho)\in \boldsymbol{V}(\Omega)\times L^2_0(\Omega)\times H^1_0(\Omega),\\
\big(\partial_t\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}\big) + a_1(\theta:\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + b(\boldsymbol{v}-\boldsymbol{u},p) + d_1(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + J(\boldsymbol{v}) - J(\boldsymbol{u}) \ge \langle \boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}\rangle,\\
b(\boldsymbol{u},q)=0,\\
\big(\partial_t\theta,\rho\big) + a_2(\theta:\theta,\rho) + d_2(\boldsymbol{u},\theta,\rho) = \langle k,\rho\rangle,
\end{cases}
\tag{2.10}
$$

Such that, $\langle .,.\rangle$ being the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

Proposition 2.1. *Problems (2.10) and (1.1–1.5) are equivalent. Indeed, any triplet $(\boldsymbol{u},p,\theta)\in\boldsymbol{V}\times M\times Q$ is a solution of (1.1–1.5) in the sense of distribution if and only if it is a solution of (2.10).*

The following standard results will be used for the analysis of problem (2.10) and its corresponding finite element discretization [5,9,21].

The following Poincaré-Friedrich's inequality holds: there is a positive constant c depending on the domain Ω such that

$$\text{for all } \boldsymbol{v}\in\boldsymbol{V}(\Omega),\qquad \|\boldsymbol{v}\|_{H^1(\Omega)} \le c|\boldsymbol{v}|_{H^1(\Omega)}, \tag{2.11}$$

which ensures that the norms $\|.\|_{H^1(\Omega)}$ and $|.|_{H^1(\Omega)}$are equivalent on $\boldsymbol{V(\Omega)}$. We also recall Korn's inequality [9]: there exists $c(\Omega)$ such that

$$\|D\boldsymbol{v}\|_{L^2(\Omega)} \geq c(\Omega)\|\nabla\boldsymbol{v}\|_{L^2(\Omega)} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega). \tag{2.12}$$

We deduce from (2.12) and (2.9) that $a_1(.:.,.)$ is continuous and elliptic on $\boldsymbol{V(\Omega)}$. This means that for $(\boldsymbol{v},\boldsymbol{w})$ element of $\boldsymbol{V(\Omega)} \times \boldsymbol{V(\Omega)}$,

$$a_1(\theta:\boldsymbol{v},\boldsymbol{w}) \leq \nu_1\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}, \qquad a_1(\theta:\boldsymbol{v},\boldsymbol{v}) \geq c\nu_0\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)}. \tag{2.13}$$

From (2.11) and (2.9), we deduce that $a_2(.:.,.)$ is continuous and elliptic on $\boldsymbol{H}^1_0(\Omega)$, this means that for (θ,ρ) element of $H^1_0(\Omega) \times H^1_0(\Omega)$,

$$a_2(\theta:\theta,\rho) \leq c\mu_1\|\theta\|_{H^1(\Omega)}\|\rho\|_{H^1(\Omega)}, \qquad a_2(\theta:\rho,\rho) \geq c\mu_0\|\rho\|^2_{H^1(\Omega)}. \tag{2.14}$$

The space $\boldsymbol{V}_{\mathrm{div}}(\Omega)$ defined in (2.3) characterize the kernel of $b(.,.)$. One easily check that $b(.,.)$ is continuous. That is, for all $(\boldsymbol{v},q) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$

$$b(\boldsymbol{v},q) \leq \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}\|q\|_{L^2_0(\Omega)}. \tag{2.15}$$

The compatibility between velocity and pressure require the inf-sup condition for the study of (2.10), its proof can be seen in [3,13]: there exist a constant $c > 0$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{V}(\Omega), \qquad \sup_{\boldsymbol{v}\in\boldsymbol{V}(\Omega)} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}} \geq c\|q\|_{L^2_0(\Omega)}. \tag{2.16}$$

Lemma 2.2. [17,20] *(i) When $d = 2$, for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ it holds that*

$$|d_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leq C\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{w}\|^{1/2}_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{w}\|^{1/2}_{\boldsymbol{H}^1(\Omega)} \tag{2.17}$$

(ii) When $d = 3$, for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ it holds that

$$|d_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leq C\|\boldsymbol{u}\|^{1/4}_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|^{3/4}_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{w}\|^{1/4}_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{w}\|^{3/4}_{\boldsymbol{H}^1(\Omega)} \tag{2.18}$$

From (2.17) and (2.18), we get

$$|d_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}. \tag{2.19}$$

Lemma 2.3. *For all $\boldsymbol{u} \in \boldsymbol{V}_{div}(\Omega), \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and $\eta,\phi \in H^1(\Omega)$, we have (see [14])*

$$\int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}.\boldsymbol{v}\; d\boldsymbol{x} = 0, \quad \int_\Omega (\boldsymbol{u}\cdot\nabla)\phi\,\phi\; d\boldsymbol{x} = 0, \quad \int_\Omega (\boldsymbol{u}\cdot\nabla)\eta\,\phi\; d\boldsymbol{x} = -\int_\Omega (\boldsymbol{u}\cdot\nabla)\phi\,\eta\; d\boldsymbol{x}. \tag{2.20}$$

From (2.20), the trilinear form $d_2(.,.,.)$ enjoys for all $(\boldsymbol{v},\theta,\rho) \in \boldsymbol{V}_{\mathrm{div}}(\Omega) \times H^1(\Omega) \times H^1(\Omega)$,

$$d_2(\boldsymbol{v},\theta,\rho) = -d_2(\boldsymbol{v},\rho,\theta) \quad \text{and} \quad d_2(\boldsymbol{v},\rho,\rho) = 0. \tag{2.21}$$

The functional $J(.)$ is convexe, lower semi continuous on $\boldsymbol{V}(\Omega)$ but not differentiable at zero.
We define the space

$$\boldsymbol{\Lambda} = \big\{\boldsymbol{\alpha} \quad \text{such as} \;\; \boldsymbol{\alpha} \in \boldsymbol{L}^\infty(S), \quad |\boldsymbol{\alpha}| \leqslant 1 \text{ a.e on } \; S\big\}. \tag{2.22}$$

It can be shown that the solution $(\boldsymbol{u}, p)$ of (2.10) is characterized by:

$$\begin{cases} \text{Find } (\boldsymbol{u}, p, \theta, \lambda) \in \boldsymbol{V} \times M \times Q \times \Lambda \text{ such that:} \\ \qquad \boldsymbol{u}(.,0) = \boldsymbol{u}_0 \text{ and } \theta(.,0) = \theta_0 \text{ in } \Omega, \\ \qquad \theta(.,t) = \theta_b(.,t) \text{ on } \partial\Omega, \quad \forall\, 0 \leq t \leq T, \\ \forall\, (\boldsymbol{v}, q, \rho) \in \boldsymbol{V}(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega), \\ \big(\partial_t \boldsymbol{u}, \boldsymbol{v}\big) + a_1(\theta : \boldsymbol{u}, \boldsymbol{v}) + d_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \int\limits_S g\lambda \cdot \boldsymbol{v}_\tau \, dS = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\ b(\boldsymbol{u}, q) = 0, \\ \lambda \cdot \boldsymbol{u}_\tau = |\boldsymbol{u}_\tau|, \\ \big(\partial_t \theta, \rho\big) + a_2(\theta : \theta, \rho) + d_2(\boldsymbol{u}, \theta, \rho) = \langle k, \rho \rangle. \end{cases} \tag{2.23}$$

Remark 2.1. The existence of $\boldsymbol{\lambda}$ is proved by using the Hahn-Banach theorem in (see [15], page 80, Theorem 5.3) and also by an approach based on a constructive regularization in (see [15], page 96, Theorem 6.3).

Next, we state the following result established in [13]: Let $\mathcal{R}$ denote a lifting operator, i.e., a continuous operator from $H^{1/2}(\partial\Omega)$ into $H^1(\Omega)$. Since θ_b belongs to $L^2(0,T;H^{1/2}(\partial\Omega))$, we denote by $\tilde{\theta}_b$ the function defined for a.e. t, $0 \leq t \leq T$, by

$$\tilde{\theta}_b(t) = \mathcal{R}\theta_b(t).$$

Clearly $\tilde{\theta}_b$ belongs to $L^2(0,T;H^1(\Omega))$ and satisfies the following inequality

$$\forall t \in [0,T], \qquad \|\tilde{\theta}_b\|_{L^2(0,T;H^1(\Omega))} \leq c\|\theta_b\|_{L^2(0,T;H^{1/2}(\partial\Omega))}, \tag{2.24}$$

where the positive constant c depends only on Ω and $\mathcal{R}$. By taking a reading on the temperature at the system (2.10) and setting $\theta^* = \theta - \tilde{\theta}_b$, note that $\theta^*|_{\partial\Omega} = 0$ and $(\boldsymbol{u}, p, \theta^*)$ is the solution of the following variational problem:

$$\begin{cases} \text{Find } (\boldsymbol{u}, p, \theta^*) \in \boldsymbol{V} \times M \times Q_0 \text{ such that :} \\ \boldsymbol{u}(.,0) = \boldsymbol{u}_0 \text{ et } \theta^*(.,0) = \theta_0 - \tilde{\theta}_b(.,0) \text{ in } \Omega, \\ \forall\, (\boldsymbol{v}, q, \rho) \in \boldsymbol{V}(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega), \\ (\partial_t \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + a_1(\theta^* + \tilde{\theta}_b : \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + b(\boldsymbol{v} - \boldsymbol{u}, p) + d_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + J(\boldsymbol{v}) - J(\boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle, \\ b(\boldsymbol{u}, q) = 0, \\ (\partial_t \theta^*, \rho) + a_2(\theta^* + \tilde{\theta}_b : \theta^* + \tilde{\theta}_b, \rho) + d_2(\boldsymbol{u}, \theta^*, \rho) = \langle k, \rho \rangle - (\partial_t \tilde{\theta}_b, \rho) - d_2(\boldsymbol{u}, \tilde{\theta}_b, \rho). \end{cases} \tag{2.25}$$

Such that, $Q_0 = L^2(0,T,H_0^1(\Omega))$.

Moreover, the pair $(\boldsymbol{u}, \theta^*)$ is a solution of the variational problem:

$$\begin{cases} \text{Find } (\boldsymbol{u}, \theta^*) \in \boldsymbol{V}_{\text{div}} \times M \times Q_0 \text{ such that :} \\ \boldsymbol{u}(.,0) = \boldsymbol{u}_0 \text{ et } \theta^*(.,0) = \theta_0 - \tilde{\theta}_b(.,0) \text{ in } \Omega, \\ \forall\, (\boldsymbol{v}, \rho) \in \boldsymbol{V}_{\text{div}}(\Omega) \times H_0^1(\Omega), \\ (\partial_t \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + a_1(\theta^* + \tilde{\theta}_b : \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + d_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + J(\boldsymbol{v}) - J(\boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle, \\ (\partial_t \theta^*, \rho) + a_2(\theta^* + \tilde{\theta}_b : \theta^* + \tilde{\theta}_b, \rho) + d_2(\boldsymbol{u}, \theta^*, \rho) = \langle k, \rho \rangle - (\partial_t \tilde{\theta}_b, \rho) - d_2(\boldsymbol{u}, \tilde{\theta}_b, \rho). \end{cases} \tag{2.26}$$

Such that, $\boldsymbol{V}_{\text{div}} = L^2(0,T;\boldsymbol{V}_{\text{div}}(\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega))$.

2.2. A Priori Estimates

Proposition 2.2. *Assume hypothesis H1–H3, with* $\boldsymbol{u}_0 \in \boldsymbol{H}^1(\Omega)$*, such that (2.8) hold true. Then the following a priori estimate holds for any* $t \in]0, T[$.

$$(1) \quad \|\boldsymbol{u}\|_{L^2(0,T;H^1(\Omega))} \leq c\left(\|\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\right). \tag{2.27}$$

$$(2) \quad \|\partial_t \boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \leq \|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + c\|\boldsymbol{u}_0\|^2_{\boldsymbol{H}^1(\Omega)} + c\|g\|^2_{L^\infty(S)} + c\|g\|_{L^\infty(S)}. \tag{2.28}$$

$$(3) \quad \nu_0\|\nabla \boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \leq \|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + c\|\boldsymbol{u}_0\|^2_{\boldsymbol{H}^1(\Omega)} + c\|g\|^2_{L^\infty(S)} + c\|g\|_{L^\infty(S)}. \tag{2.29}$$

$$(4) \quad \|p\|^2_{L^2(0,T;L^2_0(\Omega))} \leq c\left(\|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{u}_0\|^2_{\boldsymbol{H}^1(\Omega)} + \|g\|^2_{L^\infty(S)} + \|g\|_{L^\infty(S)}\right). \tag{2.30}$$

$$\begin{aligned}(5) \quad \|\theta^*\|_{L^2(0,T;H^1(\Omega))} \leq c\Big[&\|\theta_0\|_{L^2(\Omega)} + \|k\|_{L^2(0,T;H^{-1}(\Omega))} + \|\partial_t\tilde{\theta}_b\|_{L^2(0,T;L^2(\Omega))} \\ &+\|\tilde{\theta}_b\|_{L^2(0,T;H^{1/2}(\partial\Omega))} + \|\tilde{\theta}_b\|^2_{H^1(0,T;H^{1/2}(\partial\Omega))} + \|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{H}^{-1}(\Omega))} + \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}(\Omega)}\Big].\end{aligned} \tag{2.31}$$

Proof. (1) We take successively in (2.26) $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} = 2\boldsymbol{u}$. Comparing the two inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + a_1(\theta^* + \tilde{\theta}_b : \boldsymbol{u}, \boldsymbol{u}) + J(\boldsymbol{u}) = \langle \boldsymbol{f}, \boldsymbol{u}\rangle, \tag{2.32}$$

which after dropping the positive term $J(.)$ and application of the proprieties of $a_1(. :, ., .)$ in (2.13) and Hölder's inequality, we get

$$\frac{d}{dt}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\nu_0\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \leq \frac{1}{\nu_0}\|\boldsymbol{f}\|^2_{\boldsymbol{H}^{-1}(\Omega)} + \nu_0\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}. \tag{2.33}$$

It follows that,

$$\frac{d}{dt}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \nu_0\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \leq \frac{1}{\nu_0}\|\boldsymbol{f}\|^2_{\boldsymbol{H}^{-1}(\Omega)}. \tag{2.34}$$

Integrating between 0 and T, we get

$$\|\boldsymbol{u}(., T)\|^2_{\boldsymbol{L}^2(\Omega)} + \nu_0\int_0^T \|\boldsymbol{u}(., s)\|^2_{\boldsymbol{H}^1(\Omega)}ds \leq \|\boldsymbol{u}(., 0)\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{\nu_0}\int_0^T \|\boldsymbol{f}(., s)\|^2_{\boldsymbol{H}^{-1}(\Omega)}ds. \tag{2.35}$$

Then

$$\|\boldsymbol{u}\|_{L^2(0,T;H^1(\Omega))} \leq \frac{c}{\nu_0}\left(\|\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\right), \tag{2.36}$$

(2) Taking successively in (2.26) $\boldsymbol{v} = \boldsymbol{u} - \partial_t\boldsymbol{u}$ and $\boldsymbol{v} = \boldsymbol{u} + \partial_t\boldsymbol{u}$. Comparing the two inequalities,

$$\|\partial_t\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + a_1(\theta^* + \tilde{\theta}_b : \boldsymbol{u}, \partial_t\boldsymbol{u}) + J(\partial_t\boldsymbol{u}) = \langle \boldsymbol{f}, \partial_t\boldsymbol{u}\rangle, \tag{2.37}$$

Then

$$\|\partial_t\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{dt}\left[\frac{\nu_0}{2}a_1(. : \boldsymbol{u}, \boldsymbol{u}) + J(\boldsymbol{u})\right] \leq \frac{1}{2}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2}\|\partial_t\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{2.38}$$

So

$$\frac{1}{2}\|\partial_t\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{dt}\left[\frac{\nu_0}{2}a_1(. : \boldsymbol{u}, \boldsymbol{u}) + J(\boldsymbol{u})\right] \leq \frac{1}{2}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{2.39}$$

Integration over the time interval $(0, T)$ yields

$$\begin{aligned}&\frac{1}{2}\|\partial_t\boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \frac{\nu_0}{2}a_1(. : \boldsymbol{u}(T), \boldsymbol{u}(T)) + J(\boldsymbol{u}(T)) \\ &\quad\leq \frac{1}{2}\|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \frac{\nu_0}{2}a_1(. : \boldsymbol{u}_0, \boldsymbol{u}_0) + J(\boldsymbol{u}_0).\end{aligned} \tag{2.40}$$

Owing to the positivity of $a_1(\cdot:\cdot,\cdot)$ and $J(\cdot)$ defined in (2.9), we get

$$\|\partial_t \boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \leq \|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \nu_0\|\nabla \boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} + 2\int_S g\left(|\boldsymbol{u}_{\boldsymbol{0},\boldsymbol{\tau}}|+1\right)\, dS. \tag{2.41}$$

So

$$\|\partial_t \boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \leq \|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + c\|\boldsymbol{u}_0\|^2_{\boldsymbol{H}^1(\Omega)} + c\|g\|^2_{L^\infty(S)} + c\|g\|_{L^\infty(S)}. \tag{2.42}$$

(3) Back to (2.40), we have

$$\nu_0\|\boldsymbol{D}\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + 2J(\boldsymbol{u}(T)) \leq \|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \nu_0\|\nabla \boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} + 2\int_S g\left(|\boldsymbol{u}_{\boldsymbol{0},\boldsymbol{\tau}}|+1\right)\, dS. \tag{2.43}$$

So

$$\nu_0\|\nabla \boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \leq \|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + c\|\boldsymbol{u}_0\|^2_{\boldsymbol{H}^1(\Omega)} + c\|g\|^2_{L^\infty(S)} + c\|g\|_{L^\infty(S)}. \tag{2.44}$$

(4) As for the pressure, we take successively in (2.25) $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} = 2\boldsymbol{u}$, such that $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$. Comparing the two inequalities,

$$b(\boldsymbol{u}, p) = \langle \boldsymbol{f}, \boldsymbol{u}\rangle - (\partial_t \boldsymbol{u}, \boldsymbol{u}) - a_1(\theta^* + \tilde{\theta}_b : \boldsymbol{u}, \boldsymbol{u}), \tag{2.45}$$

which with the inf-sup condition and the proprieties of $a_1(.:,.,.)$ in (2.13) gives

$$c\|p\|_{L^2_0(\Omega)} \leq \sup_{\boldsymbol{v}\in \boldsymbol{H}^1_0(\Omega)} \frac{b(\boldsymbol{u}, p)}{\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}} \leq \|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\partial_t \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + 2\nu_1\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}. \tag{2.46}$$

(5) Concerning the temperature, by taking $\rho = \theta^*$ in (2.26) we find

$$\left(\partial_t \theta^*, \theta^*\right) + a_2(\theta^* + \tilde{\theta}_{b,} : \theta^* + \tilde{\theta}_{b,}, \theta^*) + d_2(\boldsymbol{u}, \theta^*, \theta^*) = \langle k, \theta^*\rangle - \left(\partial_t \tilde{\theta}_b, \theta^*\right) - d_2(\boldsymbol{u}, \tilde{\theta}_b, \theta^*). \tag{2.47}$$

From (2.20) and application of the proprieties of $a_2(.:,.,.)$ in (2.14), one finds

$$(\partial_t \theta^*, \theta^*) + (\mu(\theta^* + \tilde{\theta}_b)\nabla\theta^*, \nabla\theta^*) = \langle k, \theta^*\rangle - (\partial_t \tilde{\theta}_b\,, \theta^*) - ((\boldsymbol{u}\cdot\nabla)\tilde{\theta}_b, \theta^*) - (\mu(\theta^* + \tilde{\theta}_b)\nabla\tilde{\theta}_b, \nabla\theta^*).$$

Using Hölder's inequality on the right-hand side yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta^*\|^2_{L^2(\Omega)} + \mu_0 c\|\theta^*\|^2_{H^1(\Omega)} \leq c\Big(&\|k\|_{H^{-1}(\Omega)}\|\theta^*\|_{H^1(\Omega)} + \|\partial_t\tilde{\theta}_b\|_{L^2(\Omega)}\|\theta^*\|_{L^2(\Omega)} \\ &+\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\tilde{\theta}_b\|_{H^1(\Omega)}\|\theta^*\|_{H^1(\Omega)} + \mu_1\|\tilde{\theta}_b\|_{H^1(\Omega)}\|\theta^*\|_{H^1(\Omega)}\Big).\end{aligned} \tag{2.48}$$

Young's inequality yields the following bounds

$$\|k\|_{H^{-1}(\Omega)}\|\theta^*\|_{H^1(\Omega)} \leq \frac{1}{2}\Big[\|k\|^2_{H^{-1}(\Omega)} + \|\theta^*\|^2_{H^1(\Omega)}\Big], \tag{2.49}$$

$$\|\partial_t\tilde{\theta}_b\|_{L^2(\Omega)}\|\theta^*\|_{L^2(\Omega)} \leq \|\partial_t\tilde{\theta}_b\|_{L^2(\Omega)}\|\theta^*\|_{H^1(\Omega)} \leq \frac{1}{2}\Big(\|\partial_t\tilde{\theta}_b\|^2_{L^2(\Omega)} + \|\theta^*\|^2_{H^1(\Omega)}\Big), \tag{2.50}$$

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\tilde{\theta}_b\|_{H^1(\Omega)}\|\theta^*\|_{H^1(\Omega)} \leq \frac{c'}{2}\left[\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\tilde{\theta}_b\|^2_{H^1(\Omega)}\right]^2 + \frac{\|\theta^*\|^2_{H^1(\Omega)}}{2}, \tag{2.51}$$

and

$$\mu_1\|\tilde{\theta}_b\|_{H^1(\Omega)}\|\theta^*\|_{H^1(\Omega)} \leq \frac{\mu_1}{2}\left[\|\tilde{\theta}_b\|^2_{H^1(\Omega)} + \|\theta^*\|^2_{H^1(\Omega)}\right]. \tag{2.52}$$

So returning to (2.48), one gets

$$\frac{d}{dt}\|\theta^*\|^2_{L^2(\Omega)} + c\|\theta^*\|^2_{H^1(\Omega)} \leq c\left(\|k\|^2_{H^{-1}(\Omega)} + \|\partial_t\tilde{\theta}_b\|^2_{L^2(\Omega)} + \|\tilde{\theta}_b\|^2_{H^1(\Omega)} + \frac{c'}{2}\left[\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\tilde{\theta}_b\|^2_{H^1(\Omega)}\right]^2\right).$$

Integrating between 0 and T and recalling (2.24), we obtain

$$\|\theta^*(.,T)\|^2_{L^2(\Omega)} + c\int_0^T \|\theta^*(.,s)\|^2_{H^1(\Omega)} ds \leq c\Bigg(\|\theta^*(.,0)\|^2_{L^2(\Omega)} + \int_0^T \|k(.,s)\|^2_{H^{-1}(\Omega)} ds$$
$$+\int_0^T \|\partial_t\tilde{\theta}_b(.,s)\|^2_{L^2(\Omega)} ds + \int_0^T \|\tilde{\theta}_b(.,s)\|^2_{H^{1/2}(\partial\Omega)} ds$$
$$+\frac{c'}{2}\left[\int_0^T \|\boldsymbol{u}(.,s)\|^2_{\boldsymbol{H}^1(\Omega)} ds + \int_0^T \|\tilde{\theta}_b(.,s)\|^2_{H^{1/2}(\partial\Omega)} ds\right]^2\Bigg).$$

Thus from (2.36), it results

$$\|\theta^*\|_{L^2(0,T;H^1(\Omega))} \leq c\Big[\|\theta_0\|_{L^2(\Omega)} + \|k\|_{L^2(0,T;H^{-1}(\Omega))} + \|\partial_t\tilde{\theta}_b\|_{L^2(0,T;L^2(\Omega))}$$
$$+\|\tilde{\theta}_b\|_{L^2(0,T;H^{1/2}(\partial\Omega))} + \|\tilde{\theta}_b\|^2_{H^1(0,T;H^{1/2}(\partial\Omega))} + \|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{H}^{-1}(\Omega))} + \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)}\Big]. \tag{2.53}$$

Which completes the proof. □

Theorem 2.1. *Assume that the data* $(\boldsymbol{f}, k, \theta_b, g)$ *satisfying (2.7) and the initial velocity* $\boldsymbol{u}_0$ *belongs to* $\boldsymbol{H}^1(\Omega)$ *and checks (2.8). That the initial temperature in* $L^2(\Omega)$ *and the temperature on the boundary* θ_b *in* $H^{1/2}(\partial\Omega)$*. Then the problem (2.10) admits at least one solution* $(\boldsymbol{u}, p, \theta)$ *such that*

$$\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{V}(\Omega)), \quad p \in L^2(0,T;L^2_0(\Omega)), \quad \theta \in L^2(0,T;H^1_0(\Omega)),$$

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T;\boldsymbol{V}'(\Omega)) \quad \text{if } d=2, \tag{2.54}$$

and

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^{4/3}(0,T;\boldsymbol{V}'(\Omega)) \quad \text{if } d=3. \tag{2.55}$$

Proof. **Step** 1*: Regularization:*
Problem (2.10) is equivalent to

$$\begin{cases} \text{Find } (\boldsymbol{u},\theta^*) \in \boldsymbol{V}_{\text{div}} \times Q_0 \text{ such that:} \\ \boldsymbol{u}(.,0) = \boldsymbol{u}_0 \text{ and } \theta^*(.,0) = \theta_0 - \bar{\theta}_b(.,0) \text{ in } \Omega, \\ \forall\ (\boldsymbol{v},\rho) \in \boldsymbol{V}_{\text{div}}(\Omega) \times H^1_0(\Omega), \\ \big(\partial_t \boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}\big) + a_1(\theta^*+\bar{\theta}_b : \boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + d_1(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + J(\boldsymbol{v}) - J(\boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle, \\ \big(\partial_t\theta^*,\rho\big) + a_2(\theta^*+\bar{\theta}_b : \theta^*+\bar{\theta}_b,\rho) + d_2(\boldsymbol{u},\theta^*,\rho) = \langle k,\rho\rangle - \big(\partial_t\bar{\theta}_b,\rho\big) - d_2(\boldsymbol{u},\bar{\theta}_b,\rho). \end{cases} \tag{2.56}$$

We introduce the functional

$$J_\varepsilon : \boldsymbol{V}(\Omega) \longrightarrow \mathbb{R}$$
$$\boldsymbol{v} \longrightarrow J_\varepsilon(\boldsymbol{v}) = \int_S g\sqrt{|\boldsymbol{v_\tau}|^2+\varepsilon^2}\ dS, \quad 0<\varepsilon<<1. \tag{2.57}$$

We observe that

(a) J_ε is convex and differentiable, with Gateaux-derivative $J'_\varepsilon : \boldsymbol{V}(\Omega) \longrightarrow \boldsymbol{V}'(\Omega)$ given by

$$\langle J'_\varepsilon(\boldsymbol{u}),\boldsymbol{v}\rangle = \int_S g\frac{\boldsymbol{u}_\tau\cdot\boldsymbol{v}_\tau}{\sqrt{|\boldsymbol{u_\tau}|^2+\varepsilon^2}}\ dS. \tag{2.58}$$

.

(b) $J_\varepsilon^{'}$ is monotone, that is

$$\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(\Omega), \quad \langle J_\varepsilon^{'}(\boldsymbol{u}) - J_\varepsilon^{'}(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v}\rangle \geq 0.$$

(c) For all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, $J_\varepsilon(\boldsymbol{v}) \xrightarrow[\varepsilon \longrightarrow 0]{} J(\boldsymbol{v})$.
Indeed,

$$J_\varepsilon(\boldsymbol{v}) - J(\boldsymbol{v}) = \int_S g \frac{\varepsilon^2}{\sqrt{|\boldsymbol{v_\tau}|^2 + \varepsilon^2} + |\boldsymbol{v_\tau}|} \, dS.$$

The regularized problem then takes the following form:

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon, \theta_\varepsilon^*) \in \boldsymbol{V}_{\text{div}} \times Q_0 \text{ such that:} \\ \boldsymbol{u}_\varepsilon(.,0) = \boldsymbol{u}_0 \text{ and } \theta_\varepsilon^*(.,0) = \theta_0 - \bar{\theta}_b(.,0) \text{ in } \Omega, \\ \forall\ (\boldsymbol{v}, \rho) \in \boldsymbol{V}_{\text{div}}(\Omega) \times H_0^1(\Omega), \\ \big(\partial_t \boldsymbol{u}_\varepsilon, \boldsymbol{v}\big) + a_1(\theta_\varepsilon^* + \bar{\theta}_b : \boldsymbol{u}_\varepsilon, \boldsymbol{v}) + d_1(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{v}) + \langle J_\varepsilon^{'}(\boldsymbol{u}_\varepsilon), \boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \\ \big(\partial_t \theta_\varepsilon^*, \rho\big) + a_2(\theta_\varepsilon^* + \bar{\theta}_b : \theta_\varepsilon^* + \bar{\theta}_b, \rho) + d_2(\boldsymbol{u}_\varepsilon, \theta_\varepsilon^*, \rho) = \langle k, \rho\rangle - \big(\partial_t \bar{\theta}_b, \rho\big) - d_2(\boldsymbol{u}_\varepsilon, \bar{\theta}_b, \rho). \end{cases} \tag{2.59}$$

Step 2*: Galerkin approximation:*
The spaces $\boldsymbol{V}_{\text{div}}(\Omega)$ and $H_0^1(\Omega)$ are separable, there exists an increasing sequence $(\boldsymbol{V}_{\text{div}}^m(\Omega))_m$ of finite dimensional subspaces of $\boldsymbol{V}_{\text{div}}(\Omega)$ and an increasing sequence $(W^m(\Omega))_m$ of finite-dimensional subspaces of $H_0^1(\Omega)$ such that $\bigcup_{m \in \mathbb{N}} \boldsymbol{V}_{\text{div}}^m(\Omega) \times W^m(\Omega)$ is dense in $\boldsymbol{V}_{\text{div}}(\Omega) \times H_0^1(\Omega)$. The Galerkin Problem is

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon^m, \theta_\varepsilon^{*m}) \in L^2(0,T; \boldsymbol{V}_{\text{div}}^m(\Omega)) \cap L^\infty(0,T; \boldsymbol{L}^2(\Omega)) \times L^2(0,T; W^m(\Omega)) \text{ such that:} \\ \boldsymbol{u}_\varepsilon^m(.,0) = \boldsymbol{u}_0 \text{ and } \theta_\varepsilon^{*m}(.,0) = \theta_0 - \bar{\theta}_b(.,0) \text{ in } \Omega, \\ \forall\ (\boldsymbol{v}, \rho) \in \boldsymbol{V}_{\text{div}}^m(\Omega) \times W^m(\Omega), \\ \big(\partial_t \boldsymbol{u}_\varepsilon^m, \boldsymbol{v}\big) + a_1(\theta_\varepsilon^{*m} + \bar{\theta}_b : \boldsymbol{u}_\varepsilon^m, \boldsymbol{v}) + d_1(\boldsymbol{u}_\varepsilon^m, \boldsymbol{u}_\varepsilon^m, \boldsymbol{v}) + \langle J_\varepsilon^{'}(\boldsymbol{u}_\varepsilon^m), \boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \\ \big(\partial_t \theta_\varepsilon^{*m}, \rho\big) + a_2(\theta_\varepsilon^{*m} + \bar{\theta}_b : \theta_\varepsilon^{*m} + \bar{\theta}_b, \rho) + d_2(\boldsymbol{u}_\varepsilon^m, \theta_\varepsilon^{*m}, \rho) = \langle k, \rho\rangle - \big(\partial_t \bar{\theta}_b, \rho\big) - d_2(\boldsymbol{u}_\varepsilon^m, \bar{\theta}_b, \rho). \end{cases} \tag{2.60}$$

The mapping

$$(\boldsymbol{w}, z) \mapsto \begin{pmatrix} \boldsymbol{f} + \text{div } (2\nu(z + \bar{\theta}_b)\boldsymbol{D}\boldsymbol{w}) - (\boldsymbol{w} \cdot \nabla)\boldsymbol{w} - J_\varepsilon^{'}(\boldsymbol{w}) \\ k - \partial_t \bar{\theta}_b - (\boldsymbol{w} \cdot \nabla)\bar{\theta}_b - \text{div } (\mu(z + \bar{\theta}_b)\nabla(z + \bar{\theta}_b)) - (\boldsymbol{w} \cdot \nabla) z \end{pmatrix} \tag{2.61}$$

is locally Lipschitz in $\boldsymbol{H}^1(\Omega) \times H^1(\Omega)$. It follows then from Cauchy-Lipschitz's theorem that (2.60) has a unique solution $(\boldsymbol{u}_\varepsilon^m, \theta_\varepsilon^{*m})$.
Step 3: ***A priori* estimates and passage to the limit:**

The *a priori* estimates obtained in proposition 2.2 will also hold in the discrete setting $\boldsymbol{V}_{\text{div}}^m(\Omega) \times W^m(\Omega)$. Hence one can pass to the limit exactly as in reference [1] with respect to m.
The result will give us now a sequence depending only on ε, that is

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon, \theta_\varepsilon^*) \in \boldsymbol{V}_{\text{div}} \times Q_0 \text{ such that:} \\ \boldsymbol{u}_\varepsilon(.,0) = \boldsymbol{u}_0 \text{ and } \theta_\varepsilon^*(.,0) = \theta_0 - \bar{\theta}_b(.,0) \text{ in } \Omega, \\ \forall\ (\boldsymbol{v}, \rho) \in \boldsymbol{V}_{\text{div}}(\Omega) \times H_0^1(\Omega), \\ \big(\partial_t \boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon\big) + a_1(\theta_\varepsilon^* + \bar{\theta}_b : \boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon) + d_1(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon) + J_\varepsilon(\boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon\rangle \geq J_\varepsilon(\boldsymbol{u}_\varepsilon), \\ \big(\partial_t(\theta_\varepsilon^* + \bar{\theta}_b), \rho\big) + a_2(\theta_\varepsilon^* + \bar{\theta}_b : \theta_\varepsilon^* + \bar{\theta}_b, \rho) + d_2(\boldsymbol{u}_\varepsilon, \theta_\varepsilon^* + \bar{\theta}_b, \rho) = \langle k, \rho\rangle. \end{cases} \tag{2.62}$$

which is equivalent to (2.59).

Again the *a priori* estimates obtained in Proposition 2.2 are valid. Hence one can pass to the limit as $\varepsilon \to 0$ with the same process in [10]. And by arguing as in [11] (see p 56–57), we ensure the existence of solutions claimed in Theorem 2.1. □

3. Discretization

In this section, we propose a space-time discretization of the problem (2.10), derive and prove an *a priori* error estimation. To discretize the time derivative of velocity and temperature, we did call to the semi-implicit Euler method. We introduce a partition of the interval $[0,T]$ into intervals $[t^{n-1},t^n],\ n\geq 1$ and we denote by

$$\Delta t = t^n - t^{n-1}$$

the time discretization step such that $t^n = n\Delta t$. We approximate the derivative $\partial_t \boldsymbol{u}(t^n)$ and $\partial_t\theta(t^n)$ by the differential quotient of order Δt,

$$\partial_t \boldsymbol{u}^n \approx \frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\Delta t} \quad \text{and} \quad \partial_t \theta^n \approx \frac{\theta^n - \theta^{n-1}}{\Delta t}. \tag{3.1}$$

For the space discretization, we propose the finite element method. We assume that Ω is a polygon when $d=2$ or polyhedron when $d=3$, so it can be completely meshed. We denote by $\mathcal{T}_h$ a regular family of triangulations of Ω by closed triangles K. The elements $\mathcal{T}_h$ is a set of closed non degenerate triangles or tetrahedra, satisfying,

- For each h, Ω is the union of all elements of $\mathcal{T}_h$,
- The intersection of two distinct elements of $\mathcal{T}_h$ is either empty, a common vertex, or an entire common edge or face,
- The ratio of the diameter of an element K in $\mathcal{T}_h$ to the diameter of its inscribed circle or ball is bounded by a constant independent of h such that $h = \max\limits_{K\in\mathcal{T}_h} \mathrm{diam} K$ denotes the mesh size.

We define the space $\mathcal{P}_l(K)$ of polynomials of degree less than or equal to l on K. And we use "$P2/P1$" pair of finite element, that is

$$\begin{aligned}
\boldsymbol{V}_h(\Omega) &= \left\{\boldsymbol{v}_h \in \mathcal{C}(\bar{\Omega})^2 \cap \boldsymbol{V}(\Omega),\ \text{for all } K\in\mathcal{T}_h,\ \boldsymbol{v}_h|_K \in \mathcal{P}_2(K)^2\right\},\\
L^2_h(\Omega) &= \left\{q_h \in L^2(\Omega)\cap\mathcal{C}(\bar{\Omega}),\ \text{for all } K\in\mathcal{T}_h,\ q_h|_K\in\mathcal{P}_1(K)\right\},\\
H^1_h(\Omega) &= \left\{\rho_h \in H^1(\Omega)\cap\mathcal{C}(\bar{\Omega}),\ \text{for all } K\in\mathcal{T}_h,\ \rho_h|_K\in\mathcal{P}_1(K)\right\}.
\end{aligned}$$

We let

$$L^2_{0h}(\Omega) = L^2_h(\Omega)\cap L^2_0(\Omega) \quad \text{and} \quad H^1_{0h}(\Omega) = H^1_h(\Omega)\cap H^1_0(\Omega).$$

We introduce the interpolation operator I_h such that, for any function φ continue on $\bar{\Omega}$, I_h checks

$$I_h\varphi(\boldsymbol{x},.) = \varphi(\boldsymbol{x},.)$$

We note $i_h^{\partial\Omega}$ the lagrange interpolation operator such that, for any function φ continue on $\partial\Omega$, $i_h^{\partial\Omega}$ checks

$$i_h^{\partial\Omega}\varphi(\boldsymbol{x},.) = \varphi(\boldsymbol{x},.)$$

There exists an approximation operator $P_h \in \mathcal{L}(\boldsymbol{H}^1_0(\Omega), \boldsymbol{V}_h(\Omega))$ and $Q_h \in \mathcal{L}(\boldsymbol{H}^1(\Omega), L^2_{0h}(\Omega))$ such that (see for instance Takahito in [22] and V.Girault and P.Raviart in [14])

$$\forall \boldsymbol{v}\in H^1_0(\Omega), \quad \forall q_h \in L^2_h(\Omega), \int_\Omega q_h \mathrm{div}(P_h(\boldsymbol{v}) - \boldsymbol{v})\, d\boldsymbol{x} = 0. \tag{3.2}$$

For $k=0$ or $k=1$,

$$\forall \boldsymbol{v}\in[\boldsymbol{H}^{1+k}(\Omega)\cap\boldsymbol{H}^1_0(\Omega)], \quad \|P_h(\boldsymbol{v}) - \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq C_1 h^{1+k}|\boldsymbol{v}|_{\boldsymbol{H}^{1+k}(\Omega)}. \tag{3.3}$$

For all $r\geq 2$, $k=0$ or $k=1$,

$$\forall \boldsymbol{v}\in[\boldsymbol{W}^{1+k,r}(\Omega)\cap\boldsymbol{H}^1_0(\Omega)], \quad |P_h(\boldsymbol{v}) - \boldsymbol{v}|_{\boldsymbol{W}^{1,r}(\Omega)} \leq C_2 h^{k}|\boldsymbol{v}|_{\boldsymbol{W}^{1+k,r}(\Omega)}. \tag{3.4}$$

$$\forall q\in H^1(\Omega), \quad \|Q_h q - q\|_{L^2(\Omega)} \leq C_3 h |q|_{H^1(\Omega)}. \tag{3.5}$$

In addition, there exists an approximation operator R_h in $\mathcal{L}(W^{1,p}(\Omega), H^1_h(\Omega))$,(see for instance, Scott and Zhang [18]) such that for $m = 0, 1, l = 0, 1$ and for all $p \geq 2$,

$$\forall \rho \in W^{l+1,p}(\Omega), \quad |R_h(\rho) - \rho|_{W^{m,p}(\Omega)} \leq C_3 h^{l+1-m} |\rho|_{W^{l+1,p}(\Omega)}. \tag{3.6}$$

In order to introduce the discrete scheme, we define the following forms:

For all $\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{V}_h(\Omega)$ and $\theta_h, \rho_h \in \boldsymbol{H}^1_h(\Omega)$

$$d_{1h}(\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h) = d_1(\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h) + \frac{1}{2}\left((\operatorname{div} \boldsymbol{u}_h)\, \boldsymbol{v}_h, \boldsymbol{w}_h\right),$$

$$d_{2h}(\boldsymbol{u}_h, \theta_h, \rho_h) = d_2(\boldsymbol{u}_h, \theta_h, \rho_h) + \frac{1}{2}\left((\operatorname{div} \boldsymbol{u}_h)\, \theta_h, \rho_h\right).$$

The finite element problem reads;

$$\begin{cases} \text{Find}\, \boldsymbol{u}^n_h \in \boldsymbol{V}_h(\Omega),\, p^n_h \in L^2_{0h}(\Omega)\, \text{and}\, \theta^n \in \boldsymbol{H}^1_h(\Omega) \text{ such that:} \\ \boldsymbol{u}^0_h = \boldsymbol{I}_h \boldsymbol{u}_0 \ \text{ and } \ \theta^0_h = I_h \theta_0 \ \text{ in } \ \Omega, \\ \theta^n_h = i^{\partial\Omega}_h \theta^n_b \ \text{ in } \partial\Omega, \quad \forall\, 1 \leq n \leq N, \\ \forall\, (\boldsymbol{v}_h, q_h, \rho_h) \in \boldsymbol{V}_h(\Omega) \times L^2_{0h}(\Omega) \times \boldsymbol{H}^1_{0h}(\Omega), \\ \dfrac{1}{\Delta t}(\boldsymbol{u}^n_h, \boldsymbol{v}_h - \boldsymbol{u}^n_h) + a_1(\theta^{n-1}_h : \boldsymbol{u}^n_h, \boldsymbol{v}_h - \boldsymbol{u}^n_h) + b(\boldsymbol{v}_h - \boldsymbol{u}^n_h, p^n_h) \\ \qquad + d_{1h}(\boldsymbol{u}^{n-1}_h, \boldsymbol{u}^n_h, \boldsymbol{v}_h - \boldsymbol{u}^n_h) + J(\boldsymbol{v}_h) - J(\boldsymbol{u}^n_h) \geq \langle \boldsymbol{f}^n, \boldsymbol{v}_h - \boldsymbol{u}^n_h \rangle - \dfrac{1}{\Delta t}(\boldsymbol{u}^{n-1}_h, \boldsymbol{v}_h - \boldsymbol{u}^n_h), \\ b(\boldsymbol{u}^n_h, q_h) = 0, \\ \dfrac{1}{\Delta t}(\theta^n_h, \rho_h) + a_2(\theta^{n-1}_h : \theta^n_h, \rho_h) + d_{2h}(\boldsymbol{u}^n_h, \theta^n_h, \rho_h) = \langle k^n, \rho_h \rangle - \dfrac{1}{\Delta t}(\theta^{n-1}_h, \rho_h). \end{cases} \tag{3.7}$$

where $\boldsymbol{f}^n$ and k^n are given as

$$\boldsymbol{f}^n = \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \boldsymbol{f}(s) ds \quad \text{and} \quad k^n = \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} k(s) ds. \tag{3.8}$$

with

Remark 3.1. The trilinear forms $d_{1\,h}(\cdot, \cdot, \cdot)$ and $d_{2\,h}(\cdot, \cdot, \cdot)$ enjoys due to the stability relations

$$d_{1h}(\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{v}_h) = d_{2h}(\boldsymbol{u}_h, \theta_h, \theta_h) = 0 \quad \forall\, (\boldsymbol{u}_h, \boldsymbol{v}_h) \in \boldsymbol{V}_h(\Omega)^2, \theta_h \in \boldsymbol{H}^1_h(\Omega).$$

▹ We recall that the discrete version of inf-sup condition (2.46) holds: there exists $C > 0$ independent of h such that

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega)} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h(\Omega)}} \geqslant C \|q_h\|_{L^2(\Omega)} \quad \forall q_h \in L^2_{0h}(\Omega). \tag{3.9}$$

Theorem 3.1. *Under hypothesis H1–H3, at each time step n, for a given $\boldsymbol{u}^{n-1}_h \in \boldsymbol{V}_h(\Omega), p^{n-1}_h \in L^2_{0\,h}(\Omega)$ and $\theta^{n-1}_h \in \boldsymbol{H}^1_h(\Omega)$, the variational problem (3.7) admits at least one solution $(\boldsymbol{u}^n_h, p^n_h, \theta^n_h)$ which verifies, for $m = 1, ..., N$ the following bounds*

$$\|\boldsymbol{u}^m\|^2_{\boldsymbol{H}^1(\Omega)} + 2\gamma\nu_0 C_\Omega \Delta t \sum_{n=1}^{m} \|\boldsymbol{D}\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{u}_0\|^2_{\boldsymbol{H}^1(\Omega)} + \frac{2\gamma\Delta t}{\nu_0} \sum_{n=1}^{m} \|\theta^{n-1}\|^2_{\boldsymbol{H}^1(\Omega)}. \tag{3.10}$$

and

$$\|\theta^m\|^2_{H^1(\Omega)} + C_\Omega \mu_0 \Delta t \sum_{n=1}^{m} \|\theta^n\|^2_{H^1(\Omega)} \leq \|\theta_0\|^2_{H^1(\Omega)} + \frac{\Delta t}{C_\Omega \mu_0} \sum_{n=1}^{m} \|k\|^2_{H^{-1}(\Omega)}. \tag{3.11}$$

Similar to the continuous problem, the following characterization of the solution $(\boldsymbol{u}, p, \theta)$ of the problem (2.23) holds. Then, problem (3.7) is equivalent to

$$\begin{cases} \text{Find } \boldsymbol{u}_h^n \in \boldsymbol{V}_h(\Omega), p_h^n \in L^2_{0h}(\Omega),\, \theta_h^n \in \boldsymbol{H}^1_h(\Omega) \text{ and } \boldsymbol{\lambda}_h^n \in \boldsymbol{\Lambda}_h \text{such that:} \\ \boldsymbol{u}_h^0 = \boldsymbol{I}_h \boldsymbol{u}_0 \;\text{ and }\; \theta_h^0 = I_h\theta_0 \;\text{ in }\; \Omega, \\ \theta_h^n = i_h^{\partial\Omega}\theta_b^n \;\text{ in } \partial\Omega, \quad \forall\, 1 \le n \le N, \\ \forall\, (\boldsymbol{v}_h, q_h, \rho_h) \in \boldsymbol{V}_h(\Omega) \times L^2_{0h}(\Omega) \times \boldsymbol{H}^1_{0h}(\Omega), \\ \dfrac{1}{\Delta t}(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + a_1(\theta_h^{n-1} : \boldsymbol{u}_h^n, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h^n) + d_{1h}(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h) \\ \qquad\qquad = \langle \boldsymbol{f}^{n-1}, \boldsymbol{v}_h\rangle - \displaystyle\int_S g\boldsymbol{\lambda}_h^{n-1}\cdot \boldsymbol{v}_\tau \, dS - \dfrac{1}{\Delta t}(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h), \\ b(\boldsymbol{u}_h^n, q_h) = 0, \\ \boldsymbol{u}_h^n \cdot \boldsymbol{\lambda}_h^{n-1} = |\boldsymbol{\lambda}_h^n|, \\ \dfrac{1}{\Delta t}(\theta_h^n, \rho_h) + a_2(\theta_h^{n-1} : \theta_h^n, \rho_h) + d_{2h}(\boldsymbol{u}_h^n, \theta_h^n, \rho_h) = \langle k, \rho_h\rangle - \dfrac{1}{\Delta t}(\theta_h^{n-1}, \rho_h). \end{cases}$$

And we let $\boldsymbol{\Lambda}_h = \boldsymbol{\Lambda} \cap \boldsymbol{L}_h$.

Such that,

$$\boldsymbol{L}_h = \big\{\boldsymbol{\alpha}_h | \boldsymbol{\alpha}_h, \boldsymbol{\alpha}_h = \sum_e \boldsymbol{\alpha}_e \mathcal{I}_e, \boldsymbol{\alpha}_e \in \mathbb{R}^2,\ \forall \text{edge } e \text{ in } S\big\},$$

where $\mathcal{I}_e$ is the characteristic function of edge e.

3.1. A Priori Error Estimate

In this section, we establish the *a priori* estimates corresponding to the proposed numerical schemes. We begin by establishing the error estimates corresponding to the temperature, and then we will establish the corresponding to the velocity and the pressure.

In all the rest of the paper, we denote by $\boldsymbol{u}^n = \boldsymbol{u}(t^n), p^n = p(t^n)$ and $\theta^n = \theta(t^n)$. And for the simplicity of the establishment of the *a priori* error estimates, we consider from now on $\theta_h^0 = R_h\theta(0)$ and $\boldsymbol{u}_h^0 = P_h\boldsymbol{u}(0)$.

Theorem 3.2. *Let $(\boldsymbol{u}, p, \theta)$ be the solution of problem (2.10) and $(\boldsymbol{u}_h^n, p_h^n, \theta_h^n)$ be the solution of problem (3.7). Under hypothesis $H1 - H3$, there exist two positive constants C_1^θ and C_2^θ independent on h and Δt such that, for all $m \le N$,*

$$\begin{aligned} \frac{1}{2}\,\|\theta_h^m - R_h\theta^m\|^2_{L^2(\Omega)} &+ \mu_0 \sum_{n=1}^m \Delta t\, |\theta_h^n - R_h\theta^n|^2_{H^1(\Omega)} \\ &+ \frac{1}{2}\sum_{n=1}^m \big\|\theta_h^n - R_h\theta^n - \big(\theta_h^{n-1} - R_h\theta^{n-1}\big)\big\|^2_{L^2(\Omega)} \\ &\le C_1^\theta\left(h^2 + \Delta t^2\right) + C_2^\theta \sum_{n=1}^m \Delta t\, |\mathbf{u}_h^n - P_h\mathbf{u}^n|^2_{\boldsymbol{H}^1(\Omega)} \end{aligned} \tag{3.12}$$

Proof. We consider the heat equation of the continuous variational problem (2.10), for all $\rho_h \in H^1_{0h}(\Omega)$:

$$\left(\frac{\partial\theta}{\partial t}, \rho_h\right) + a_2(\theta : \theta, \rho_h) + d_2(\boldsymbol{u}, \theta, \rho_h) = \langle k, \rho_h\rangle. \tag{3.13}$$

Setting $\rho_h = \rho_h^n = \theta_h^n - R_h\theta^{\rm n}$ in (3.13) integrated over $[t^{n-1}, t^n]$ by the definition of the operator R_h gives

$$\left(\theta^n - \theta^{n-1}, \rho_h^n\right) + \left(\int_{t^{n-1}}^{t^n} \mu(\theta)\nabla\theta(t)dt, \nabla\rho_h^n\right) + \left(\int_{t^{n-1}}^{t^n} (\mathbf{u}(t)\nabla)\cdot\theta(t)dt, \rho_h^n\right) = \left(\int_{t^{n-1}}^{t^n} k(t)dt, \rho_h^n\right) \tag{3.14}$$

Then, we subtract (3.14) with the third equation of (3.7) after multiplying it by the time step Δt and by using the definition of k^n by (3.8), we obtain

$$\begin{aligned} &\left(\left(\theta_h^{\rm n} - \theta_h^{n-1}\right) - \left(\theta^n - \theta^{\rm n-1}\right), \rho_h^n\right) + \left\{\Delta t\, a_2(\theta_h^{n-1} : \theta_h^n, \rho_h^n) - \left(\int_{t^{n-1}}^{t^n} \mu(\theta)\nabla\theta(t)dt, \nabla\rho_h^n\right)\right\} \\ &+ \left\{\Delta t\, d_{2h}(\boldsymbol{u}_h^n, \theta_h^n, \rho_h^n) - \left(\int_{t^{n-1}}^{t^n} (\mathbf{u}(t)\nabla)\cdot\theta(t)dt, \rho_h^n\right)\right\} = 0. \end{aligned} \tag{3.15}$$

We denote respectively by b_1, b_2 and b_3 the terms in the left-hand side of (3.15) as following:

$$b_1 = \left(\left(\theta_h^{\rm n} - \theta_h^{n-1}\right) - \left(\theta^n - \theta^{\rm n-1}\right), \rho_h^n\right),$$

$$b_2 = \Delta t\, a_2(\theta_h^{n-1} : \theta_h^n, \rho_h^n) - \left(\int_{t^{n-1}}^{t^n} \mu(\theta)\nabla\theta(t)dt, \nabla\rho_h^n\right),$$

and

$$b_3 = \Delta t\, d_{2h}(\boldsymbol{u}_h^n, \theta_h^n, \rho_h^n) - \left(\int_{t^{n-1}}^{t^n} (\mathbf{u}(t)\nabla)\cdot\theta(t)dt, \rho_h^n\right).$$

The first term b_1 can be bounded by inserting $\pm R_h\theta^n$ and $\pm R_h\theta^{n-1}$, we obtain

$$\begin{aligned} \left(\left(\theta_h^n - \theta_h^{n-1}\right) - \left(\theta^n - \theta^{n-1}\right), \rho_h^n\right) &= \frac{1}{2}\left\|\rho_h^n\right\|_{L^2(\Omega)}^2 - \frac{1}{2}\left\|\rho_h^{n-1}\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|\rho_h^n - \rho_h^{n-1}\right\|_{L^2(\Omega)}^2 \\ &+ \left(\left(\theta^{n-1} - R_h\theta^{n-1}\right) - \left(\theta^n - R_h\theta^n\right), \rho_h^n\right). \end{aligned} \tag{3.16}$$

Noting that $(R_h\theta)' = R_h\theta'$, the last term of the previous equality can be bounded for any $\tilde{\varepsilon}_1 > 0$ as follows:

$$\begin{aligned} \left|\left(\left(\theta^{n-1} - R_h\theta^{n-1}\right) - \left(\theta^n - R_h\theta^n\right), \rho_h^n\right)\right| &= \left|-\left(\int_{t^{n-1}}^{t^n} \left(R_h\theta' - \theta'\right)(t)dt, \rho_h^n\right)\right| \\ &\le \frac{c_1^2h^2}{2\tilde{\varepsilon}_1}\int_{t^{n-1}}^{t^n} \|\theta'(t)\|_{H^1(\Omega)}^2\,dt + \frac{\Delta t\,\tilde{\varepsilon}_1}{2}\|\rho_h^n\|_{L^2(\Omega)}^2 \\ &\le \frac{c_1^2h^2}{2\tilde{\varepsilon}_1}\left\|\frac{\partial\theta}{\partial t}\right\|_{L^2(t^{n-1},t^n;H^1(\Omega))}^2 + C_\Omega\frac{\Delta t\,\tilde{\varepsilon}_1}{2}\left|\rho_h^n\right|_{H_0^1(\Omega)}^2. \end{aligned}$$

where C_Ω^1 is the square of the norm of the imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$.
The second term b_2 can be bounded, by inserting

$$\pm\left(\Delta t\,\mu\left(\theta_h^{n-1}\right)\nabla R_h\theta^n, \nabla\rho_h^n\right), \pm\left(\int_{t^{n-1}}^{t^n} \mu\left(\theta_h^{n-1}\right)\nabla R_h\theta(t), \nabla\rho_h^n\right) \text{ and } \pm\left(\int_{t^{n-1}}^{t^n} \mu(\theta(t))\nabla R_h\theta(t)dt, \nabla\rho_h^n\right).$$

Then

$$\begin{aligned} b_2 &= \left(\Delta t\, \mu\left(\theta_h^{n-1}\right) \nabla\left(\theta_h^n - R_h\theta^n\right), \nabla\rho_h^n\right) \\ &\quad + \left(\Delta t\, \mu\left(\theta_h^{n-1}\right) \nabla R_h\theta^n - \int_{t^{n-1}}^{t^n} \mu\left(\theta_h^{n-1}\right) \nabla R_h\theta(t)dt, \nabla\rho_h^n\right) \\ &\quad + \left(\int_{t^{n-1}}^{t^n} \mu\left(\theta_h^{n-1}\right) \nabla R_h\theta(t)dt - \int_{t^{n-1}}^{t^n} \mu(\theta(t))\nabla R_h\theta(t)dt, \nabla\rho_h^n\right) \\ &\quad + \left(\int_{t^{n-1}}^{t^n} \mu(\theta(t))\nabla R_h\theta(t)dt - \int_{t^{n-1}}^{t^n} \mu(\theta(t))\nabla\theta(t)dt, \nabla\rho_h^n\right) \\ &= b_{2,1} + b_{2,2} + b_{2,3} + b_{2,4}. \end{aligned}$$

Thus,

$$b_{2,1} = \left(\Delta t\, \mu\left(\theta_h^{n-1}\right) \nabla\rho_h^n, \nabla\rho_h^n\right) \geq \Delta t\, \mu_0 \left|\rho_h^n\right|_{H^1(\Omega)}^2 .$$

For the term $b_{2,2}$,

$$\begin{aligned} |b_{2,2}| &= \left|\left(\int_{t^{n-1}}^{t^n} \mu\left(\theta_h^{n-1}\right) \nabla R_h\theta^n\, dt - \int_{t^{n-1}}^{t^n} \mu\left(\theta_h^{n-1}\right) \nabla R_h\theta(t)dt, \nabla\rho_h^n\right)\right| \\ &= \left|\left(\int_{t^{n-1}}^{t^n} \mu\left(\theta_h^{n-1}\right) \int_t^{t^n} \nabla R_h\theta'(s)\, ds\, dt, \nabla\rho_h^n\right)\right|, \end{aligned}$$

by applying Fubini's theorem and using Hölder's inequalities

$$|b_{2,2}| \leq \mu_1 \int_{t^{n-1}}^{t^n} \left|R_h\theta'(s)\right|_{H^1(\Omega)} \left|\rho_h^n\right|_{H^1(\Omega)} \left(s - t^{n-1}\right) ds.$$

The Cauchy Schwarz's inequality yields for any $\tilde{\varepsilon}_1 > 0$

$$|b_{2,2}| \leq \frac{1}{6\tilde{\varepsilon}_2} \mu_1^2 c_2^2 \Delta t^2 \left\|\theta'\right\|_{L^2(t^{n-1},t^n;H^1(\Omega))}^2 + \frac{\Delta t\, \tilde{\varepsilon}_2}{2} \left|\rho_h^n\right|_{H^1(\Omega)}^2 .$$

We now deal with the term $b_{2,3}$:

By inserting $\pm\left(\int_{t^{n-1}}^{t^n} \mu\left(\theta^{n-1}\right) \nabla R_h\theta(t)dt, \nabla\rho_h^n\right)$, then $\pm\left(\int_{t^{n-1}}^{t^n} \left(\mu\left(\theta_h^{n-1}\right) - \mu\left(\theta^{n-1}\right)\right) \nabla\theta(t)dt, \nabla\rho_h^n\right)$ and $\pm\left(\int_{t^{n-1}}^{t^n} \left(\mu\left(\theta_h^{n-1}\right) - \mu\left(\theta(t)\right)\right) \nabla\theta(t)dt, \nabla\rho_h^n\right)$, we get

$$\begin{aligned} |b_{2,3}| &\leq \left(\int_{t^{n-1}}^{t^n} \left(\mu\left(\theta_h^{n-1}\right) - \mu\left(\theta^{n-1}\right)\right) \nabla\left(R_h\theta(t) - \theta(t)\right) dt, \nabla\rho_h^n\right) \\ &\quad + \left|\left(\int_{t^{n-1}}^{t^n} \left(\mu\left(\theta_h^{n-1}\right) - \mu\left(\theta^{n-1}\right)\right) \nabla\theta(t)dt, \nabla\rho_h^n\right)\right| \end{aligned}$$

$$+\left|\left(\int_{t^{n-1}}^{t^n}\left(\mu\left(\theta^{n-1}\right)-\mu(\theta(t))\right)\nabla\left(R_h\theta(t)-\theta(t)\right)dt,\nabla\rho_h^n\right)\right|$$
$$+\left|\left(\int_{t^{n-1}}^{t^n}\left(\mu\left(\theta^{n-1}\right)-\mu(\theta(t))\right)\nabla\theta(t)dt,\nabla\rho_h^n\right)\right|$$

Since μ satisfies the propreties (2.5) and (2.6) we get for any $\tilde{\varepsilon}_3>0$ and $\tilde{\varepsilon}_4>0$ the following:

$$\begin{aligned}|b_{2,3}| &\le \frac{1}{\tilde{\varepsilon}_3}8\mu_1^2c_3^2h^2\|\theta\|^2_{L^2(0,T;H^2(\Omega))}+\frac{\Delta t\,\tilde{\varepsilon}_3}{2}\left|\rho_h^n\right|^2_{H^1(\Omega)}\\ &+\frac{1}{2\tilde{\varepsilon}_4}\left(\mu^*\right)^2\Delta t\|\theta\|^2_{L^\infty(0,T;W^{1,d}(\Omega))}\left\|\theta_h^{n-1}-\theta^{n-1}\right\|^2_{L^4(\Omega)}+\frac{\Delta t\tilde{\varepsilon}_4}{2}\left|\rho_h^n\right|^2_{H^1(\Omega)}\\ &+\mu^*\int_\Omega\int_{t^{n-1}}^{t^n}\left|\theta\left(x,t^{n-1}\right)-\theta(x,t)\right|\cdot\left|\nabla\theta(x,t)\right|\cdot\left|\nabla\rho_h^n(x)\right|dt\,dx\end{aligned}$$

For the last term,

$$\begin{aligned}&\mu^*\int_\Omega\int_{t^{n-1}}^{t^n}\left|\theta\left(x,t^{n-1}\right)-\theta(x,t)\right|\cdot\left|\nabla\theta(x,t)\right|\cdot\left|\nabla\rho_h^n(x)\right|dt\,dx\\ &\quad\le\mu^*(C_2^\Omega)^{1/2}\|\theta\|_{L^\infty(0,T;W^{1,4}(\Omega))}\int_{t^{n-1}}^{t^n}\left|\theta'(s)\right|_{H^1(\Omega)}\left|\rho_h^n\right|_{H^1(\Omega)}\left(t^n-s\right)ds.\end{aligned}$$

From Cauchy Schwarz's inequality for any $\tilde{\varepsilon}_5>0$

$$\begin{aligned}&\mu^*\int_\Omega\int_{t^{n-1}}^{t^n}\left|\theta^{n-1}(x)-\theta(x,t)\right|\cdot\left|\nabla\theta(x,t)\right|\cdot\left|\nabla\rho_h^n(x)\right|dt\,dx\\ &\le\mu^*\Delta t\left(C_2^\Omega\Delta t\right)^{\frac{1}{2}}\|\theta\|_{L^\infty(0,T;W^{1,4}(\Omega))}\left\|\theta'\right\|_{L^2(t^{n-1},t^n;H^1(\Omega))}\left|\rho_h^n\right|_{H^1(\Omega)}\\ &\le\frac{1}{2\tilde{\varepsilon}_5}\left(\mu^*\right)^2C_2^\Omega\Delta t^2\|\theta\|^2_{L^\infty(0,T;W^{1,4}(\Omega))}\left\|\theta'\right\|^2_{L^2(t^{n-1},t^n;H^1(\Omega))}+\frac{\Delta t\tilde{\varepsilon}_5}{2}\left|\rho_h^n\right|^2_{H^1(\Omega)}\end{aligned}$$

Such that, C_2^Ω is the norm of the imbedding of $H_0^1(\Omega)$ into $L^4(\Omega)$. It résult

$$\begin{aligned}|b_{2,3}| &\le \frac{1}{\tilde{\varepsilon}_3}8(\mu^*)^2c_3^2h^2\|\theta\|^2_{L^2(t^{n-1},t^n;H^2(\Omega))}+\frac{\Delta t\tilde{\varepsilon}_3}{2}\left|\rho_h^n\right|^2_{H^1(\Omega)}+\frac{\Delta t\tilde{\varepsilon}_4}{2}\left|\rho_h^n\right|^2_{H^1(\Omega)}\\ &+\frac{\left(\mu^*\right)^4}{16\tilde{\varepsilon}_4^2}\|\theta\|^4_{L^\infty(0,T;W^{1,4}(\Omega))}\frac{\Delta t}{2}\left\|\theta_h^{n-1}-\theta^{n-1}\right\|^2_{L^2(\Omega)}+\frac{\Delta t}{2}\left|\theta_h^{n-1}-\theta^{n-1}\right|^2_{H^1(\Omega)}\\ &\frac{\Delta t\tilde{\varepsilon}_5}{2}\left|\rho_h^n\right|^2_{H^1(\Omega)}+\frac{1}{2\tilde{\varepsilon}_5}\left(\mu^*\right)^2C_2^\Omega\Delta t^2\|\theta\|^2_{L^\infty\left(0,\bar{T};W^{1,4}(\Omega)\right)}\left\|\theta'\right\|^2_{L^2(t^{n-1},t^n;H^1(\Omega))}.\end{aligned}$$

Furthermore, for the term $b_{2,4}$ can be bounded for every positive real number $\tilde{\varepsilon}_6>0$ as:

$$\begin{aligned}|b_{2,4}| &\le \mu_1\int_{t^{n-1}}^{t^n}\left\|\nabla\left(R_h\theta-\theta\right)(t)\right\|_{L^2(\Omega)}\left|\rho_h^n\right|_{H^1(\Omega)}dt\\ &\le\frac{\left(\mu_1\right)^2c_4^2h^2}{2\tilde{\varepsilon}_6}\|\theta\|^2_{L^2(t^{n-1},t^n;H^2(\Omega))}+\frac{\Delta t\,\tilde{\varepsilon}_6}{2}\left|\rho_h^n\right|^2_{H^1(\Omega)}.\end{aligned}$$

Finally, the third term b_3

$$b_3 = \Delta t\, d_2\left(\boldsymbol{u}_h^n, \theta_h^n, \rho_h^n\right) + \frac{\Delta t}{2}\left(\left(\operatorname{div} \boldsymbol{u}_h^n\right) \theta_h^n, \rho_h^n\right) - \left(\int_{t^{n-1}}^{t^n} (\mathbf{u}(t)\nabla)\cdot\theta(t)dt, \rho_h^n\right).$$

can be bounded by inserting

$$\pm\Delta t\, d_2(\boldsymbol{u}_h^n, R_h\theta^n, \rho_h^n) \quad \text{and} \quad \pm\frac{\Delta t}{2}\left(\left(\operatorname{div} \boldsymbol{u}_h^n\right) R_h\theta^n, \rho_h^n\right).$$

b_3 becomes

$$b_3 = \Delta t\, d_2\left(\boldsymbol{u}_h^n, R_h\theta^n, \rho_h^n\right) + \frac{\Delta t}{2}\left(\left(\operatorname{div} \boldsymbol{u}_h^n\right) R_h\theta^n, \rho_h^n\right) - \left(\int_{t^{n-1}}^{t^n} (\mathbf{u}(t)\nabla)\cdot\theta(t)dt, \rho_h^n\right).$$

Furthermore, by inserting $\pm\Delta t\left(P_h\boldsymbol{u}^n\nabla R_h\theta^n, \rho_h^n\right)$, $\pm\frac{\Delta t}{2}\left(\operatorname{div}\left(P_h\boldsymbol{u}^n\right) R_h\theta^n, \rho_h^n\right)$, $\pm\int_{t^{n-1}}^{t^n}\left((P_h\boldsymbol{u}^n\nabla)\cdot\theta(t), \rho_h^n\right)dt$ and $\pm\frac{1}{2}\int_{t^{n-1}}^{t^n}\left(\operatorname{div} P_h\boldsymbol{u}^n\theta(t), \rho_h^n\right)dt$, we obtain

$$\begin{aligned}
*b_3 = {} & \Delta t\left(\left(\left(\mathbf{u}_h^n - P_h\mathbf{u}^n\right)\nabla\right)\cdot R_h\theta^n, \rho_h^n\right) + \int_{t^{n-1}}^{t^n}\left((P_h\mathbf{u}^n\nabla)\cdot\left(R_h\theta^n - \theta(t)\right), \rho_h^n\right)dt \\
& + \int_{t^{n-1}}^{t^n}\left(\left(\left(P_h\boldsymbol{u}^n - \boldsymbol{u}(t)\right)\nabla\right)\cdot\theta(t), \rho_h^n\right)dt + \frac{\Delta t}{2}\left(\operatorname{div}\left(\boldsymbol{u}_h^n - P_h\boldsymbol{u}^n\right) R_h\theta^n, \rho_h^n\right) \\
& + \frac{1}{2}\int_{t^{n-1}}^{t^n}\left(\operatorname{div}\left(P_h\boldsymbol{u}^n\right)\left(R_h\theta^n - \theta(t)\right), \rho_h^n\right)dt + \frac{1}{2}\int_{t^{n-1}}^{t^n}\left(\operatorname{div}\left(P_h\boldsymbol{u}^n - \mathbf{u}(t)\right)\theta(t), \rho_h^n\right)dt.
\end{aligned} \tag{3.17}$$

We denote by $b_{3,1}, \ldots, b_{3,6}$ the terms of the right hand side of (3.17) which can be bounded as following: The term $b_{3,1}$ can be bounded thanks to Holder's inequality for every positive real number $\tilde{\varepsilon}_7 > 0$ as:

$$|b_{3,1}| \leq \frac{\Delta t}{2\tilde{\varepsilon}_7} c_5^2\left(C_2^\Omega\right)^2\|\theta\|^2_{L^\infty(0,T,H^1(\Omega))}\left|\boldsymbol{u}_h^n - P_h\boldsymbol{u}^n\right|_{\boldsymbol{H}^1(\Omega)} + \frac{\tilde{\varepsilon}_7\Delta t}{2}\left|\rho_h^n\right|^2_{H^1(\Omega)}.$$

The term $b_{3,2}$ can be bounded for every positive real number $\tilde{\varepsilon}_8 > 0$ and $\tilde{\varepsilon}_9 > 0$ as:

$$\begin{aligned}
|b_{3,2}| \leq {} & \frac{1}{2}c_6^2\left(C_2^\Omega\right)^2\|\boldsymbol{u}\|^2_{L^\infty(0,T;H^1(\Omega))}\left(\frac{c_5^2\Delta t^2}{3\tilde{\varepsilon}_8}\|\theta'\|^2_{L^\infty(0,T;H^1(\Omega))} + \frac{c_1^2h^2}{2\tilde{\varepsilon}_9}\|\theta\|^2_{L^2(t^{n-1},t^n;H^2(\Omega))}\right) \\
& + \frac{\Delta t\left(\tilde{\varepsilon}_8 + \tilde{\varepsilon}_9\right)}{2}\left|\rho_h^n\right|^2_{H^1(\Omega)}.
\end{aligned}$$

For the term $b_{3,3}$, we insert $\pm\int_{t^{n-1}}^{t^n}\left((P_h\boldsymbol{u}^n\nabla R_h\theta(t), \rho_h^n\right)dt$. It results for every positive real number $\tilde{\varepsilon}_{10} > 0$ and $\tilde{\varepsilon}_{11} > 0$:

$$\begin{aligned}
|b_{3,3}| \leq {} & \frac{1}{6\tilde{\varepsilon}_{10}}\left(C_2^\Omega\right)^2 c_6^2\Delta t^2\|\theta\|^2_{L^\infty(0,T;H^1(\Omega))}\|\boldsymbol{u}'\|^2_{L^2(t^{n-1},t^n;\boldsymbol{H}^1(\Omega))} \\
& + \frac{1}{2\tilde{\varepsilon}_{11}}\left(C_2^\Omega\right)^2 c_5^2h^2\|\theta\|^2_{L^\infty(0,T;H^1(\Omega))}\|\boldsymbol{u}\|^2_{L^2(t^{n-1},t^n;\boldsymbol{H}^2(\Omega))} + \frac{\Delta t\left(\tilde{\varepsilon}_{10} + \tilde{\varepsilon}_{11}\right)}{2}\left|\rho_h^n\right|^2_{H^1(\Omega)}.
\end{aligned}$$

The term $b_{3,4}$ can be bounded for every positive real number $\tilde{\varepsilon}_{12} > 0$ as:

$$|b_{3,4}| \leq \frac{c_7}{2\tilde{\varepsilon}_{12}} \Delta t \left(C_2^{\Omega}\right)^2 \|\theta\|^2_{L^\infty(0,T;H^1(\Omega))} \left|\boldsymbol{u}_h^n - P_h \boldsymbol{u}^n\right|^2_{\boldsymbol{H}^1(\Omega)} + \frac{\Delta t}{2} \tilde{\varepsilon}_{12} \left|\rho_h^n\right|^2_{H^1(\Omega)} .$$

The terms $b_{3,5}$ and $b_{3,6}$ can be treated exactly like $b_{3,2}$ and $b_{3,3}$ and we get for every positive real numbers $\tilde{\varepsilon}_{13} > 0$, $\tilde{\varepsilon}_{14} > 0$, $\tilde{\varepsilon}_{15} > 0$ and $\tilde{\varepsilon}_{16} > 0$,

$$\begin{aligned} |b_{3,5}| \leq \frac{c_8 (C_2^{\Omega})^2}{2} \|\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{H}^1(\Omega))} \left(\frac{\Delta t^2}{\tilde{\varepsilon}_{13}} \|\theta'\|^2_{L^2(0,T;H^1(\Omega))} + \frac{h^2}{\tilde{\varepsilon}_{14}} \|\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{H}^1(\Omega))} \right) \\ + \frac{\Delta t \left(\tilde{\varepsilon}_{13} + \tilde{\varepsilon}_{14}\right)}{2} \left|\rho_h^n\right|^2_{H^1(\Omega)} , \end{aligned}$$

and

$$\begin{aligned} |b_{3,6}| \leq \frac{c_9 (C_2^{\Omega})^2}{2} \|\theta\|^2_{L^\infty(0,T;H^1(\Omega))} \left(\frac{\Delta t^2}{\tilde{\varepsilon}_{15}} \|\mathbf{u}'\|^2_{L^2(0,T;\boldsymbol{H}^1(\Omega))} + \frac{h^2}{\tilde{\varepsilon}_{16}} \|\mathbf{u}\|_{L^2(0,T;\boldsymbol{H}^2(\Omega))} \right) \\ + \frac{\Delta t \left(\tilde{\varepsilon}_{15} + \tilde{\varepsilon}_{16}\right)}{2} \left|\rho_h^n\right|^2_{H^1(\Omega)} . \end{aligned}$$

Finally, by using the obtained bounds and summing over n from 1 to $m \leq N$, we get

$$\begin{aligned} &\frac{1}{2} \|\rho_h^m\|^2_{L^2(\Omega)} + \frac{1}{2} \sum_{n=1}^{m} \left\|\rho_h^n - \rho_h^{n-1}\right\|^2_{L^2(\Omega)} + \mu_0 \sum_{n=1}^{m} \Delta t \left|\rho_h^n\right|^2_{H^1(\Omega)} \\ &\leq \bar{\xi}_1 \left(h^2 + \Delta t^2\right) + \bar{\xi}_2 \sum_{n=1}^{m} \Delta t \left|\mathbf{u}_h^n - P_h \mathbf{u}^n\right|^2_{\boldsymbol{H}^1(\Omega)} \\ &\quad + \bar{\xi}_3 \sum_{n=1}^{m} \Delta t \left|\rho_h^n\right|^2_{H^1(\Omega)} \end{aligned}$$

where $\bar{\xi}_1, \bar{\xi}_2$, and $\bar{\xi}_3$ depending of $\tilde{\varepsilon}_i, i = 1 \ldots 16$. After a suitable choice of $\tilde{\varepsilon}_i$, we conclude the following bound

$$\begin{aligned} &\frac{1}{2} \|\rho_h^m\|^2_{L^2(\Omega)} + \mu_0 \sum_{n=1}^{m} \Delta t \left|\rho_h^n\right|^2_{H^1(\Omega)} + \frac{1}{2} \sum_{n=1}^{m} \left\|\rho_h^n - \rho_h^{n-1}\right\|^2_{L^2(\Omega)} \\ &\leq C_1^{\theta} \left(h^2 + \Delta t^2\right) + C_2^{\theta} \sum_{n=1}^{m} \Delta t \left|\mathbf{u}_h^n - P_h \mathbf{u}^n\right|^2_{\boldsymbol{H}^1(\Omega)} . \end{aligned} \tag{3.18}$$

□

To establish the *a priori* error estimates corresponding to the numerical scheme (3.7), we use Lemma 2.1 which is valid only for $d = 2$.

Theorem 3.3. *Let $(\boldsymbol{u}, p, \theta)$ be the solution of problem (2.10) and $(\boldsymbol{u}_h^n, p_h^n, \theta_h^n)$ be the solution of problem (3.7). Under hypothesis $H1 - H3$, there exist three positive constants $C_1^{\boldsymbol{u}}$, $C_2^{\boldsymbol{u}}$ and $C_3^{\boldsymbol{u}}$ independ on h and Δt such that, for all $m \leq N$,*

$$\begin{aligned} &\frac{1}{2} \|\boldsymbol{u}_h^m - P_h \boldsymbol{u}^m\|^2_{L^2(\Omega)} + \mu_0 \sum_{n=1}^{m} \Delta t \left|\boldsymbol{u}_h^n - P_h \boldsymbol{u}^n\right|^2_{H^1(\Omega)} \\ &\quad + \frac{1}{2} \sum_{m=1}^{m} \left\|(\boldsymbol{u}_h^n - P_h \boldsymbol{u}^n) - (\boldsymbol{u}_h^{n-1} - P_h \boldsymbol{u}^{n-1})\right\|^2_{L^2(\Omega)} \\ &\leq C_1^{\boldsymbol{u}} \left(h^2 + \Delta t^2\right) + C_2^{\boldsymbol{u}} \sum_{n=1}^{m} \Delta t \left|\theta_h^n - \theta(t^n)\right|^2_{H^1(\Omega)} + C_3^{\boldsymbol{u}} \sum_{n=1}^{m} \Delta t \left\|\theta_h^{n-1} - \theta(t^{n-1})\right\|^2_{L^2(\Omega)} \end{aligned} \tag{3.19}$$

Proof. We consider the fluid inequality of the continuous variational problem (2.10). Let $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$, we take $\boldsymbol{v} - \boldsymbol{u} = \pm \boldsymbol{w}$ in (2.10) and obtain

$$\left(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{w}\right) + a_1(\theta : \boldsymbol{u}, \boldsymbol{w}) + b(\boldsymbol{w}, p) + d_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) = \langle \boldsymbol{f}, \boldsymbol{w} \rangle, \tag{3.20}$$

We denote by $\boldsymbol{H}_{0h}^1(\Omega)$ a conforming finite element discretization of $\boldsymbol{H}_0^1(\Omega)$. In a similar way we have from (3.7), for all $\boldsymbol{w}_h \in \boldsymbol{H}^1_{0\,h}(\Omega)$

$$\frac{1}{\Delta t}(\boldsymbol{u}_h^n, \boldsymbol{w}_h) + a_1(\theta_h^{n-1} : \boldsymbol{u}_h^n, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p_h^n) + d_{1h}(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{w}_h) = \langle \boldsymbol{f}^n, \boldsymbol{w}_h \rangle - \frac{1}{\Delta t}(\boldsymbol{u}_h^{n-1}, \boldsymbol{w}_h), \tag{3.21}$$

Setting $\boldsymbol{w}_h = \boldsymbol{w}_h^n = \boldsymbol{u}_h^n - P_h \boldsymbol{u}^n$, in (3.20) integrated over $[t^{n-1}, t^n]$ by the definition of the operator P_h gives

$$(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \boldsymbol{w}_h^n) + \left(\int_{t^{n-1}}^{t^n} \nu(\theta) \nabla \boldsymbol{u}\, dt, \nabla \boldsymbol{w}_h^n\right) - \left(\int_{t^{n-1}}^{t^n} p(t)\, dt, \operatorname{div} \boldsymbol{w}_h^n\right) = \left(\int_{t^{n-1}}^{t^n} \boldsymbol{f}(t)\, dt, \boldsymbol{w}_h^n\right) \tag{3.22}$$

Then, we subtract (3.22) with (3.21) after multiplying it by the time step Δt, we obtain

$$\begin{aligned}
&\left((\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}) - (\boldsymbol{u}^n - \boldsymbol{u}^{n-1}), \boldsymbol{w}_h^n\right) + \Delta t\, a_1(\theta_h^{n-1} : \boldsymbol{u}_h^n, \boldsymbol{w}_h) \\
&- \left(\int_{t^{n-1}}^{t^n} \nu(\theta) \nabla \boldsymbol{u}\, dt, \nabla \boldsymbol{w}_h^n\right) + \Delta t\, b(\boldsymbol{w}_h, p_h^n) - \left(\int_{t^{n-1}}^{t^n} p(t)\, dt, \operatorname{div} \boldsymbol{w}_h^n\right) \\
&+ \Delta t\, d_1(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{w}_h^n) + \frac{1}{2}\left(\Delta t \operatorname{div}(\boldsymbol{u}_h^{n-1}) \boldsymbol{u}_h^n, \boldsymbol{w}_h^n\right) - \left(\int_{t^{n-1}}^{t^n} (\boldsymbol{u}(t) \nabla) \cdot \boldsymbol{u}(t)\, dt, \boldsymbol{w}_h^n\right) = 0
\end{aligned}$$

We denote respectively by A_1, A_2, A_3 and A_4 the following terms

$$\begin{aligned}
A_1 &= \left((\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}) - (\boldsymbol{u}^n - \boldsymbol{u}^{n-1}), \boldsymbol{w}_h^n\right), \\
A_2 &= \Delta t\, a_1(\theta_h^{n-1} : \boldsymbol{u}_h^n, \boldsymbol{w}_h) - \left(\int_{t^{n-1}}^{t^n} \nu(\theta) \nabla \boldsymbol{u}\, dt, \nabla \boldsymbol{w}_h^n\right), \\
A_3 &= \Delta t\, b(\boldsymbol{w}_h, p_h^n) - \left(\int_{t^{n-1}}^{t^n} p(t)\, dt, \operatorname{div} \boldsymbol{w}_h^n\right),
\end{aligned}$$

and

$$A_4 = \Delta t\, d_1(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{w}_h^n) + \frac{1}{2}\left(\Delta\, t \operatorname{div}(\boldsymbol{u}_h^{n-1}) \boldsymbol{u}_h^n, \boldsymbol{w}_h^n\right) - \left(\int_{t^{n-1}}^{t^n} (\boldsymbol{u}(t) \nabla) \cdot \boldsymbol{u}(t)\, dt, \boldsymbol{w}_h^n\right).$$

The term A_1 can be expressed as

$$\begin{aligned}
\left(\left(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\right) - \left(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\right), \boldsymbol{w}_h^n\right) &= \frac{1}{2}\|\boldsymbol{w}_h^n\|^2_{L^2(\Omega)} - \frac{1}{2}\left\|\boldsymbol{w}_h^{n-1}\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\boldsymbol{w}_h^n - \boldsymbol{w}_h^{n-1}\right\|^2_{L^2(\Omega)} \\
&+ \left(\left(\boldsymbol{u}^{n-1} - P_h \boldsymbol{u}^{n-1}\right) - \left(\boldsymbol{u}^n - P_h \boldsymbol{u}^n\right), \boldsymbol{w}_h^n\right).
\end{aligned}$$

And similar to b_1, for any $\varepsilon_1 > 0$:

$$\left|\left(\left(\boldsymbol{u}^{n-1} - P_h \boldsymbol{u}^{n-1}\right) - \left(\boldsymbol{u}^n - P_h \boldsymbol{u}^n\right), \boldsymbol{w}_h^n\right)\right| \le \frac{c_1^2 h^2}{2\varepsilon_1}\left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2_{L^2(t^{n-1}, t^n; \boldsymbol{H}^1(\Omega))} + C_\Omega \frac{\Delta t\, \varepsilon_1}{2}\, |\boldsymbol{w}_h^n|^2_{H^1(\Omega)}.$$

To bound the term A_2, we follow the same approach in b_2 to get:

$$
\begin{aligned}
|A_2| \le & \frac{(\nu_1)^2}{6\varepsilon_2}\Delta t^2 \ \|\boldsymbol{u}'\|^2_{L^2(0,T;\boldsymbol{H}^1(\Omega))} + \frac{\Delta t\, \varepsilon_2}{2}\|\boldsymbol{w}_h^n\|^2_{\boldsymbol{H}^1(\Omega)} + \frac{c_1}{\varepsilon_3}\nu_1^2 h^2 \|\boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{H}^2(\Omega))} + \frac{\Delta t\, \varepsilon_3}{2}\left|\boldsymbol{w}_h^n\right|^2_{\boldsymbol{H}^1(\Omega)} \\
& + \frac{c_2}{\varepsilon_4}(\nu^*)^2 C_2^{\Omega}\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{1,4}(\Omega))}\Delta t \left(\frac{1}{2}\left\|\theta_h^{n-1} - \theta\left(t^{n-1}\right)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left|\theta_h^{n-1} - \theta\left(t^{n-1}\right)\right|^2_{H^1(\Omega)}\right) \\
& + \frac{\Delta t\, \varepsilon_4}{2}\left|\boldsymbol{w}_h^n\right|^2_{\boldsymbol{H}^1(\Omega)} + \frac{c_3\left(\nu^* C_2^{\Omega}\right)^2}{\varepsilon_5}\|\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{W}^{1,4}(\Omega))}\left\|\theta'\right\|^2_{L^2(t^{n-1},t^n;H^1(\Omega))} + \frac{\Delta t\, \varepsilon_5}{2}\left|\boldsymbol{w}_h^{\mathrm{n}}\right|^2_{\boldsymbol{H}^1(\Omega)}.
\end{aligned}
$$

For the term A_3, since $Q_h p \in L^2_{0\,h}(\Omega)$ and $p_h^n \in L^2_{0\,h}(\Omega)$ then

$$(Q_h p(t), \operatorname{div}(\boldsymbol{w}_h^n)) = (p_h^n, \operatorname{div}(\boldsymbol{w}_h^n)) = 0.$$

Hence,

$$|A_3| = \left|\int_{t^{n-1}}^{t^n} (Q_h p(t) - p(t), \operatorname{div}(\boldsymbol{w}_h^n))\, dt\right| \le \frac{c_4 h^2}{\varepsilon_6}\|p\|^2_{L^2(0,T;H^1(\Omega))} + \frac{\varepsilon_6 \Delta t}{2}\left|\boldsymbol{w}_h^n\right|_{\boldsymbol{H}^1(\Omega)}$$

The term A_4 can be bounded by inserting $\pm\Delta t\,(\boldsymbol{u}_h^n \nabla P_h \boldsymbol{u}^n, \boldsymbol{w}_h^n)$ and $\pm\Delta t\left(\frac{1}{2}\operatorname{div}\boldsymbol{u}_h^{n-1} P_h \boldsymbol{u}^n, \boldsymbol{w}_h^n\right)$ then insering $\pm\left(\int_{t^{n-1}}^{t^n} P_h \boldsymbol{u}^{n-1}\nabla\cdot P_h\boldsymbol{u}^n, \boldsymbol{w}_h^n\right)$ and $\pm\frac{1}{2}\left(\int_{t^{n-1}}^{t^n} P_h \boldsymbol{u}^{n-1}\nabla\cdot P_h\boldsymbol{u}^n, \boldsymbol{w}_h^n\right)$. We have

$$
\begin{aligned}
A_4 = & \Delta t\left(\left(\left(\boldsymbol{u}_h^{n-1} - P_h\boldsymbol{u}\left(t^{n-1}\right)\right)\nabla\right)\cdot P_h\boldsymbol{u}\left(t^n\right), \boldsymbol{w}_h^n\right) + \int_{t^{n-1}}^{t^n}\left(\left(P_h\boldsymbol{u}\left(t^{n-1}\right)\nabla\left(P_h\boldsymbol{u}\left(t^n\right) - \boldsymbol{u}(t)\right), \boldsymbol{w}_h^n\right)\right) dt \\
& + \int_{t_{n-1}}^{t^n}\left(\left(\left(P_h\boldsymbol{u}\left(t^{n-1}\right) - \boldsymbol{u}(t)\right)\nabla\boldsymbol{u}(t), \boldsymbol{w}_h^n\right) dt + \frac{\Delta t}{2}\left(\operatorname{div}\left(\boldsymbol{u}_h^{n-1} - P_h\boldsymbol{u}\left(t^{n-1}\right)\right)P_h\boldsymbol{u}\left(t^n\right), \boldsymbol{w}_h^n\right)\right. \\
& + \frac{1}{2}\int_{t^{n-1}}^{t^n}\left(\operatorname{div}\left(P_h\boldsymbol{u}\left(t^{n-1}\right)\right)\left(P_h\boldsymbol{u}\left(t^n\right) - \boldsymbol{u}(t)\right), \boldsymbol{w}_h^n\right) dt + \frac{1}{2}\int_{t^{n-1}}^{t^n}\left(\operatorname{div}\left(P_h\boldsymbol{u}\left(t^{n-1}\right) - \boldsymbol{u}(t)\right)\boldsymbol{u}(t), \boldsymbol{w}_h^n\right) dt
\end{aligned}
$$

The terms $A_{4,2}, A_{4,3}, A_{4,5}$ and $A_{4,6}$ can be treated exactly by following the steps of the corresponding terms in the proof of Theorem 3.2. Then, we have to treat only the sum of the first term $A_{4,1}$ and the fourth term $A_{4,4}$ as following:

$$
\begin{aligned}
A_{4,1} + A_{4,4} &= \Delta t\left(\left(\left(\boldsymbol{u}_h^{n-1} - P_h\boldsymbol{u}\left(t^{n-1}\right)\right)\nabla\right)\cdot P_h\boldsymbol{u}\left(t^n\right), \boldsymbol{w}_h^n\right) + \frac{\Delta t}{2}\left(\operatorname{div}\left(\boldsymbol{u}_h^{n-1} - P_h\boldsymbol{u}\left(t^{n-1}\right)\right)P_h\boldsymbol{u}\left(t^n\right), \boldsymbol{w}_h^n\right) \\
&= \frac{\Delta t}{2}\left(\left(\left(\boldsymbol{u}_h^{n-1} - P_h\boldsymbol{u}\left(t^{n-1}\right)\right)\nabla\right)\cdot P_h\boldsymbol{u}\left(t^n\right), \boldsymbol{w}_h^n\right) - \frac{\Delta t}{2}\left(\left(\left(\boldsymbol{u}_h^{n-1} - P_h\boldsymbol{u}\left(t^{n-1}\right)\right)\nabla\right)\cdot \boldsymbol{w}_h^n, P_h\boldsymbol{u}\left(t^n\right)\right).
\end{aligned}
$$

Then, for every positive real number ε_7,

$$
\begin{aligned}
& |A_{4,1} + A_{4,4}| \\
& \quad \le c_5\Delta t\left\|\boldsymbol{u}_h^{n-1} - P_h\boldsymbol{u}\left(t^{n-1}\right)\right\|_{\boldsymbol{L}^4(\Omega)}\left|\boldsymbol{w}_h^n\right|_{\boldsymbol{H}^1(\Omega)}\left|\boldsymbol{u}\left(t^n\right)\right|_{\boldsymbol{H}^1(\Omega)} \\
& \quad \le c_5\|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{H}^1(\Omega))} \\
& \quad \left[\frac{1}{2\varepsilon_7}\left(\frac{\Delta t}{2}\left|\boldsymbol{u}_h^{n-1} - P_h\boldsymbol{u}\left(t^{n-1}\right)\right|^2_{\boldsymbol{H}^1(\Omega)} + \frac{\Delta t}{2}\left\|\boldsymbol{u}_h^{n-1} - P_h\boldsymbol{u}\left(t^{n-1}\right)\right\|^2_{\boldsymbol{L}^2(\Omega)}\right) + \frac{\varepsilon_7}{2}\left|\boldsymbol{w}_h^n\right|^2_{\boldsymbol{H}^1(\Omega)}\right]
\end{aligned}
$$

By collecting the above bounds and summing over n from 1 to $m \leq N$, we get

$$\frac{1}{2}\left\|\boldsymbol{w}_h^m\right\|_{\boldsymbol{L}^2(\Omega)}^2+\frac{1}{2}\sum_{n=1}^m\left\|\boldsymbol{w}_h^n-\boldsymbol{w}_h^{n-1}\right\|_{\boldsymbol{L}^2(\Omega)}^2+\nu_0\sum_{n=1}^m\Delta t\left|\boldsymbol{w}_h^n\right|_{\boldsymbol{H}^1(\Omega)}^2$$
$$\leq \xi_1\left(h^2+\Delta t^2\right)+\xi_2\sum_{n=1}^m\frac{\Delta t}{2}\left\|\theta_h^{n-1}-\theta\left(t^{n-1}\right)\right\|_{L^2(\Omega)}^2$$
$$+\xi_3\sum_{n=1}^m\frac{\Delta t}{2}\left|\theta_h^n-\theta\left(t^n\right)\right|_{H^1(\Omega)}^2+\xi_4\sum_{n=1}^m\Delta t\left\|\boldsymbol{w}_h^n\right\|_{\boldsymbol{L}^2(\Omega)}^2+\xi_5\sum_{n=1}^m\Delta t\left\|\boldsymbol{w}_h^n\right\|_{\boldsymbol{H}^1(\Omega)}^2$$

where $\xi_1\xi_2,\xi_3,\xi_4$ and ξ_5 depending of $\varepsilon_i, i=1..7$. The last term can be bounded following the relation

$$\sum_{n=1}^m\left\|\boldsymbol{w}_h^n\right\|_{\boldsymbol{H}^1(\Omega)}\leq\sum_{n=1}^m\left\|\boldsymbol{w}_h^n-\boldsymbol{w}_h^{m-1}\right\|_{\boldsymbol{H}^1(\Omega)}+\sum_{n=1}^m\left\|\boldsymbol{w}_h^{n-1}\right\|_{\boldsymbol{H}^1(\Omega)},$$

After a suitable choice of $\varepsilon_i, i=1..7$ and $\xi_i, i=1..5$ and applyication of Gronwall's lemma, we get

$$\frac{1}{2}\left\|\boldsymbol{w}_h^m\right\|_{\boldsymbol{L}^2(\Omega)}^2+\frac{1}{2}\sum_{n=1}^m\left\|\boldsymbol{w}_h^n-\boldsymbol{w}_h^{n-1}\right\|_{\boldsymbol{L}^2(\Omega)}^2+\frac{\nu_0}{2}\sum_{n=1}^m\Delta t\left|\boldsymbol{w}_h^n\right|_{\boldsymbol{H}^1(\Omega)}^2$$
$$\leq C_1^{\boldsymbol{u}}\left(h^2+\Delta t^2\right)+C_2^{\boldsymbol{u}}\sum_{n=1}^m\frac{\Delta t}{2}\left\|\theta_h^{n-1}-\theta\left(t^{n-1}\right)\right\|_{L^2(\Omega)}^2+C_3^{\boldsymbol{u}}\sum_{n=1}^m\frac{\Delta t}{2}\left|\theta_h^n-\theta\left(t^n\right)\right|_{H^1(\Omega)}^2.$$

□

Theorem 3.4. *Under the assumptions of Theorems 3.2 and 3.3, and by choosing $\theta\in L^\infty(0,T,H^2(\Omega))$, there exists constants k_0, $\tilde{C}^{\boldsymbol{u}}$ and $\tilde{C}^\theta$, independent of h and Δt, such that for $\Delta t<k_0$*

$$\sup_{1\leq n\leq N}\left\|\theta_h^n-\theta(t^n)\right\|_{L^2(\Omega)}^2+\mu_0\sum_{n=1}^N\Delta t\left|\theta_h^n-\theta(t^n)\right|_{H^1(\Omega)}^2\leq\tilde{C}^\theta\left(h^2+\Delta t^2\right). \tag{3.23}$$

$$\sup_{1\leq n\leq N}\left\|\boldsymbol{u}_h^n-\boldsymbol{u}(t^n)\right\|_{\boldsymbol{L}^2(\Omega)}^2+\nu_1\sum_{n=1}^N\Delta t\left|\boldsymbol{u}_h^n-\boldsymbol{u}(t^n)\right|_{\boldsymbol{H}^1(\Omega)}^2\leq\tilde{C}^{\boldsymbol{u}}\left(h^2+\Delta t^2\right). \tag{3.24}$$

Proof. We insert $\pm R_h\theta^{n-1}$ in (3.19), use Theorem 3.2, choose $\theta\in L^\infty(0,T,H^2(\Omega))$, and apply the properties of the operator R_h to get the following bound

$$\frac{1}{2}\left\|\rho_h^m\right\|_{L^2(\Omega)}^2+\mu_0\sum_{\mathrm{n}=1}^m\Delta t\left|\rho_h^n\right|_{H^1(\Omega)}^2+\frac{1}{2}\sum_{n=1}^m\left\|\rho_h^n-\rho_h^{n-1}\right\|_{L^2(\Omega)}^2$$
$$\leq C_1\left(h^2+\Delta t^2\right)+C_2\sum_{n=1}^m\Delta t\left|\rho_h^{n-1}\right|_{H^1(\Omega)}^2+C_3\sum_{n=1}^m\Delta t\left\|\rho_h^{n-1}\right\|_{L^2(\Omega)}^2.$$

with $\rho_h^n=\theta_h^n-R_h\theta^{\mathrm{n}}$ and C_1, C_2 and C_3 are constants independent of h and Δt. From the continuous injection from $H^1(\Omega)$ into $L^2(\Omega)$, we obtain

$$\frac{1}{2}\left\|\rho_h^m\right\|_{L^2(\Omega)}^2+\mu_0\sum_{\mathrm{n}=1}^m\Delta t\left|\rho_h^n\right|_{H^1(\Omega)}^2\leq C_1\left(h^2+\Delta t^2\right)+C_2^{'}\sum_{n=1}^m\Delta t\left\|\rho_h^{n-1}\right\|_{L^2(\Omega)}^2.$$

It follows from the discrete Gronwall Lemma the following inequality

$$\frac{1}{2}\left\|\rho_h^m\right\|_{L^2(\Omega)}^2+\mu_0\sum_{\mathrm{n}=1}^m\Delta t\left|\rho_h^n\right|_{H^1(\Omega)}^2\leq C_1\left(h^2+\Delta t^2\right).$$

To get the relation (3.23), it suffices to apply

$$\left|\theta_h^n-\theta(t^n)\right|_{H^1(\Omega)}\leq\left|\theta_h^n-R_h\theta^n\right|_{H^1(\Omega)}+\left|R_h\theta^n-\theta(t^n)\right|_{H^1(\Omega)}$$

Finally, the bound (3.24) can be directly deduced from Theorem 3.3. □

4. Operator Splitting

The operator splitting is a decomposition technique that consists, as its name suggests, in treating each of the operators of a partial differential equation separately, in order to facilitate its resolution. Indeed, it makes it possible to remove the complexity of a problem by reducing it to several simpler sub-problems. We recall that (2.10) can be decomposed as variational equation, an inequality and a variational equation for the heat. The two variational equations are written as the following initial value problem:
Find ϕ and ψ such that

$$\begin{cases} \phi(0) = \phi_0, \psi(0) = \psi_0. \\ \dfrac{d\phi}{dt} + \displaystyle\sum_{i=1}^{q} \mathcal{A}_i \phi = 0 \quad \text{on}(0,T), \\ \dfrac{d\psi}{dt} + \displaystyle\sum_{i=1}^{q} \mathcal{B}_i \psi = 0 \quad \text{on}(0,T), \end{cases} \tag{4.1}$$

Let $\phi_n \approx \phi(t_n)$ and $\psi_n \approx \psi(t_n)$. Then Marchuk-Yanenko's method to solve (2.10) borrows mainly from [15]:

$$\phi^0 = \phi_0, \quad \psi^0 = \psi_0 \tag{4.2}$$

for $n \geq 1$, knowing ϕ_n, ψ_n and for $i = 1, 2, 3, \ldots, q$ we compute ϕ^{n+1} via

$$\frac{\phi^{n+i/q} - \phi^{n+(i-1)/q}}{\Delta t} + \mathcal{A}_i \left(\phi^{n+i/q}, t^{n+1}\right) = 0. \tag{4.3}$$

Then compute ψ^{n+1} via

$$\frac{\psi^{n+i} - \psi^{n+i-1}}{\Delta t} + \mathcal{B}_i \left(\psi^{n+i}, t^{n+1}\right) = 0. \tag{4.4}$$

For $n \leq 0$, with $t^{n+\alpha} = (n+\alpha)\Delta t$ and for non negative γ_1 and γ_2, such that $\gamma_1 + \gamma_2 = 1$, we compute $\boldsymbol{u}^{n+1/2}, (\boldsymbol{u}^{n+1}, p^{n+1})$ then θ^{n+1} as in the following Algorithm:

Algorithm: Marchuk–Yanenko operator splitting algorithm

$n = 0$: Initialization: $\boldsymbol{u}^0 = \boldsymbol{u}_0,\ \theta^0 = \theta_0$

$n \geq 0$: Knowing $\boldsymbol{u}^n$ and θ^n compute $\boldsymbol{u}^{n+1/2}$ and $(\boldsymbol{u}^{n+1}, p^{n+1})$ as follows:

Step 1. Linear step without a constraint: compute $\boldsymbol{u}^{n+1/2}$ such that, for all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$

$$\frac{1}{\Delta t}(\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^n, \boldsymbol{v}) + \gamma_1 a_1(\theta^n : \boldsymbol{u}^{n+1/2}, \boldsymbol{v}) + d_1(\boldsymbol{u}^n, \boldsymbol{u}^{n+1/2}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle. \tag{4.5}$$

Step 2. Nonlinear elliptic variational inequality step with a constraint: compute $(\boldsymbol{u}^{n+1}, p^{n+1})$ such that, for all $(\boldsymbol{v}, q) \in \boldsymbol{V}(\Omega) \times L_0^2(\Omega)$

$$\begin{aligned} \frac{1}{\Delta t}(\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1/2}, \boldsymbol{v} - \boldsymbol{u}^{n+1}) + \gamma_2 a_1(\theta^n : \boldsymbol{u}^{n+1}, \boldsymbol{v} - \boldsymbol{u}^{n+1}) \\ + b(\boldsymbol{v} - \boldsymbol{u}^{n+1}, p^{n+1}) + J(\boldsymbol{v}) - J(\boldsymbol{u}^{n+1}) \geq 0, \end{aligned} \tag{4.6}$$

$$b(\boldsymbol{u}^{n+1}, q) = 0. \tag{4.7}$$

Step 3. Knowing θ^n, compute θ^{n+1} such that, for all $\rho \in H_0^1(\Omega)$

$$\frac{1}{\Delta t}(\theta^{n+1} - \theta^n, \rho) + a_2(\theta^n, \theta^{n+1}, \rho) + d_2(\boldsymbol{u}^{n+1}, \theta^{n+1}, \rho) = \langle k, \rho \rangle. \tag{4.8}$$

Theorem 4.1. *Suppose that $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\Omega)$; then the scheme (4.5)-(4.8) is unconditionally stable in the following sense: for $n = 0, 1, \ldots, m$ there exists a positive constant C_Ω such that*

$$\|\boldsymbol{u}^m\|^2_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{C_\Omega t_m}{\gamma_1\nu_0}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}, \tag{4.9}$$

$$\Delta t \sum_{n=0}^{m-1} \|(\boldsymbol{D}\boldsymbol{u}^{n+1/2})\|^2_{\boldsymbol{L}^2(\Omega)} \leq \frac{1}{3\nu_0\gamma_1}\left(\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{C_\Omega t_m}{\gamma_1\nu_0}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}\right) \tag{4.10}$$

$$\Delta t \sum_{n=0}^{m-1} \|(\boldsymbol{D}\boldsymbol{u}^{n+1})\|^2_{\boldsymbol{L}^2(\Omega)} \leq \frac{1}{4\nu_0\gamma_2}\left(\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{C_\Omega t_m}{\gamma_1\nu_0}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}\right) \tag{4.11}$$

$$\sum_{n=0}^{m-1} \|\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{C_\Omega t_m}{\gamma_1\nu_0}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} \tag{4.12}$$

$$\sum_{n=0}^{m-1} \|\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1/2}\|^2_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{C_\Omega t_m}{\gamma_1\nu_0}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} \tag{4.13}$$

$$\Delta t \sum_{n=0}^{m-1} J(\boldsymbol{u}^{n+1}) \leq \frac{1}{2}\left(\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{C_\Omega t_m}{\gamma_1\nu_0}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}\right) \tag{4.14}$$

$$\|\boldsymbol{u}^{m+1/2}\|^2_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{C_\Omega t_m}{\gamma_1\nu_0}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} \tag{4.15}$$

Proof. Let $\boldsymbol{v} = \boldsymbol{u}^{n+1/2}$ in (4.5), one has

$$2(\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^n, \boldsymbol{u}^{n+1/2}) + 2\gamma_1\Delta t\ a_1(\theta^n : \boldsymbol{u}^{n+1/2}, \boldsymbol{u}^{n+1/2}) = 2\Delta t\langle \boldsymbol{f}, \boldsymbol{v}\rangle. \tag{4.16}$$

The second term on the right hand side of (4.16) can be treated with Cauchy–Schwarz, Korn and Young inequalities as follows

$$2\Delta t\langle \boldsymbol{f}, \boldsymbol{v}\rangle \leq \frac{C_\Omega \Delta t}{\nu_0\gamma_1}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \nu_0\gamma_1\Delta t\|\boldsymbol{D}\boldsymbol{u}^{n+1/2}\|^2_{\boldsymbol{L}^2(\Omega)} \tag{4.17}$$

Throughout

$$(\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^n, \boldsymbol{u}^{n+1/2}) = \frac{1}{2}\big(\|\boldsymbol{u}^{n+1/2}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}\big) \tag{4.18}$$

Dropping (4.17) and (4.18) in (4.16), we find that

$$\|\boldsymbol{u}^{n+1/2}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + 3\gamma_1\nu_0\Delta t\|\boldsymbol{D}\boldsymbol{u}^{n+1/2}\|^2_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_\Omega\Delta t}{\nu_0\gamma_1}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} \tag{4.19}$$

Next, we take successively $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} = 2\boldsymbol{u}^{n+1}$ in (4.6), and after comparison of the resulting relations, we find

$$\frac{1}{\Delta t}(\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1/2}, \boldsymbol{u}^{n+1}) + \gamma_2 a_1(\theta^n : \boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}) + J(\boldsymbol{u}^{n+1}) = 0 \tag{4.20}$$

Applying 4.18 we find

$$\|\boldsymbol{u}^{n+1}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\boldsymbol{u}^{n+1/2}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1/2}\|^2_{\boldsymbol{L}^2(\Omega)} + 4\gamma_2\nu_0\Delta t\|\boldsymbol{D}\boldsymbol{u}^{n+1}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\Delta t J(\boldsymbol{u}^{n+1}) \leq 0 \tag{4.21}$$

We do (4.19)+(4.21) for $n = 0, 1, 2, ..., m-1$, we obtain

$$\|\boldsymbol{u}^m\|^2_{\boldsymbol{L}^2(\Omega)} + 3\gamma_1\nu_0\Delta t\sum_{n=0}^{m-1}\|\boldsymbol{D}\boldsymbol{u}^{n+1/2}\|^2_{\boldsymbol{L}^2(\Omega)} + 4\gamma_2\nu_0\Delta t\sum_{n=0}^{m-1}\|\boldsymbol{D}\boldsymbol{u}^{n+1}\|^2_{\boldsymbol{L}^2(\Omega)}$$

$$+\sum_{n=0}^{m-1}\|\boldsymbol{u}^{n+1/2}-\boldsymbol{u}^{n}\|^2_{\boldsymbol{L}^2(\Omega)}+\sum_{n=0}^{m-1}\|\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n+1/2}\|^2_{\boldsymbol{L}^2(\Omega)}+2\Delta t\sum_{n=0}^{m-1}J(\boldsymbol{u}^{n+1})$$
$$\leq \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{C_\Omega t_m}{\gamma_1\nu_0}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} \tag{4.22}$$

□

From which we obtain (4.9)-(4.14).
By adding the relation (4.19) for $n=0,1,2,...,m$ and dropping some positive terms, we obtain

$$\sum_{n=0}^{m}\|\boldsymbol{u}^{n+1/2}\|^2_{\boldsymbol{L}^2(\Omega)}-\sum_{n=0}^{m}\|\boldsymbol{u}^{n}\|^2_{\boldsymbol{L}^2(\Omega)}\leq\frac{C_\Omega t_m}{\nu_0\gamma_1}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} \tag{4.23}$$

By adding the relation (4.21) for $n=0,1,2,...,m-1$ and dropping some positive terms, we obtain

$$\sum_{n=0}^{m-1}\|\boldsymbol{u}^{n+1}\|^2_{\boldsymbol{L}^2(\Omega)}-\sum_{n=0}^{m-1}\|\boldsymbol{u}^{n+1/2}\|^2_{\boldsymbol{L}^2(\Omega)}\leq 0 \tag{4.24}$$

Adding (4.23) and (4.24) gives (4.15).

Theorem 4.2. *Suppose that $\boldsymbol{u}_0\in\boldsymbol{L}^2(\Omega)$; then we have the following heat stability results: for $n=0,1,\dots,m$ there exists a positive constant C_Ω such that*

$$\|\theta^m\|^2_{H^1(\Omega)}\leq\|\theta_0\|^2_{H^1(\Omega)}+\frac{t_m}{C_\Omega\mu_0}\|k\|^2_{H^{-1}(\Omega)}. \tag{4.25}$$

$$\sum_{n=0}^{m-1}\|\theta^{n+1}-\theta^n\|^2_{H^1(\Omega)}\leq\|\theta_0\|^2_{H^1(\Omega)}+\frac{t_m}{C_\Omega\mu_0}\|k\|^2_{H^{-1}(\Omega)}. \tag{4.26}$$

$$\Delta t\sum_{n=0}^{m-1}\|\theta^{n+1}\|^2_{H^1(\Omega)}\leq\frac{1}{C_\Omega\mu_0}\|\theta_0\|^2_{H^1(\Omega)}+t_m\|k\|^2_{H^{-1}(\Omega)}. \tag{4.27}$$

Let $\rho=\theta^{n+1}$ in (4.8), one has

$$\frac{1}{\Delta t}(\theta^{n+1}-\theta^n,\theta^{n+1})+a_2(\theta^n:\theta^{n+1},\theta^{n+1})=\langle k,\theta^{n+1}\rangle. \tag{4.28}$$

We have

$$\langle k,\theta^{n+1}\rangle\leq\frac{1}{2}\left[\frac{1}{C_\Omega\mu_0}\|h\|^2_{H^{-1}(\Omega)}+C_\Omega\mu_0\|\theta^{n+1}\|^2_{H^1(\Omega)}\right]. \tag{4.29}$$

Thanks to Cauchy-Schwarz and Young's inequalities and the proprieties of $a_2(.:,.,.)$ in (2.14),

$$\|\theta^{n+1}\|^2_{H^1(\Omega)}-\|\theta^n\|^2_{H^1(\Omega)}+\|\theta^{n+1}-\theta^n\|^2_{H^1(\Omega)}+C_\Omega\mu_0\Delta t\|\theta^{n+1}\|^2_{H^1(\Omega)}\leq\frac{\Delta t}{C_\Omega\mu_0}\|k\|^2_{H^{-1}(\Omega)}. \tag{4.30}$$

For $n=0,1,2,...,m-1$, we obtain

$$\|\theta^m\|^2_{H^1(\Omega)}+\sum_{n=0}^{m-1}\|\theta^{n+1}-\theta^n\|^2_{H^1(\Omega)}+C_\Omega\mu_0\Delta t\sum_{n=0}^{m-1}\|\theta^{n+1}\|^2_{H^1(\Omega)}\leq\|\theta_0\|^2_{H^1(\Omega)}+\frac{t_m}{C_\Omega\mu_0}\|k\|^2_{H^{-1}(\Omega)}. \tag{4.31}$$

This completes the proof of the theorem.

5. Numerical Results and Discussion

All computations were performed using Matlab on DELL $i7$ with 16 GB RAM. The test problems used are designed to illustrate the behavior of the algorithm more than to model an actual phenomenon. For the discretization, w use the the MINI-element (P_1-Bubble/P_1) for the velocity-pressure pair and the P_1 element for the temperature. The algorithm described will be tested computationally. We stop the computations when the following condition is satisfied

$$\frac{\left\|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\right\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})} + \left\|\theta^n - \theta^{n-1}\right\|^2_{L^2(\Omega)}}{\left\|\boldsymbol{u}^{n-1}\right\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})} + \left\|\theta^{n-1}\right\|^2_{L^2(\Omega)}} < Tol \equiv 10^{-6}.$$

5.1. Test Case with Stationary Solution

We consider the unit square $\Omega = (0,1)^2$ and we assume that its boundary consists of two portions Γ and S defined as follows

$$\Gamma = \{0\} \times (0,1) \cup (0,1) \times \{0\} \cup \{1\} \times (0,1)$$
$$S = (0,1) \times \{1\}.$$

We consider

$$\nu(\theta) = \frac{1}{2}e^{-\theta} + \frac{1}{4} \quad \text{for which one has} \quad \frac{1}{4} \le \nu(\theta) \le \frac{3}{4}.$$

and

$$\mu(\theta) = \frac{1}{2} + 1.7\theta - 6.46 \times 10^{-3}\theta^2$$

In order to obtain a stationary solution of the problem

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \text{div } (2\nu(\theta)\boldsymbol{D}\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } & \Omega \times [0,T], \\ \dfrac{\partial \theta}{\partial t} - \text{div } (\mu(\theta)\nabla\theta) + (\boldsymbol{u}.\ \nabla)\theta = k \qquad\quad \text{in} & \Omega \times [0,T], \end{cases} \tag{5.1}$$

the right-hand side is adjusted accordingly, i.e.

$$f_1(x,y) = 80x^2(1-x)^2 - 20(2+12x^2-12x)y(1-2y) + 2(2y-1), \tag{5.2}$$
$$f_2(x,y) = 20(12x-6)y^2(1-y)^2 + 20x(1-2x)(1-x)(2+12y^2-12y), \tag{5.3}$$
$$k(x,y) = 2x(1-x) + 2y(1-y). \tag{5.4}$$

We take the time step $\Delta t = 0.01$. Figures 1, 2 show the velocity fields and the heat distribution for $g = 0.5$ and $g = 1.75$. One can notice in Fig. 1 (left) that the friction occurs on S for $g = 0.5$.

Since we do not have the exact solution, we assume that the solution obtained for $h = 1/512$ is the reference solution. For the velocity-pressure pair, we compute the following error

$$e_h(\boldsymbol{u}, p) = \|\boldsymbol{u}_{\text{ref}} - \boldsymbol{u}_h\|_{\boldsymbol{H}^1(\Omega)} + \|p_{\text{ref}} - p_h\|_{L^2(\Omega)}.$$

For the MINI-element or the stabilized P_1/P_1 element, $e_h(\boldsymbol{u}, p)$ converges linearly, see e.g. [4]. We report, in the Tables 1, 2, the errors and convergence rates for both stick/slip cases. From the obtained results we can conclude that the convergence rates are almost independent from the stick bound g. We also notice that the convergence rates are linear for all components.

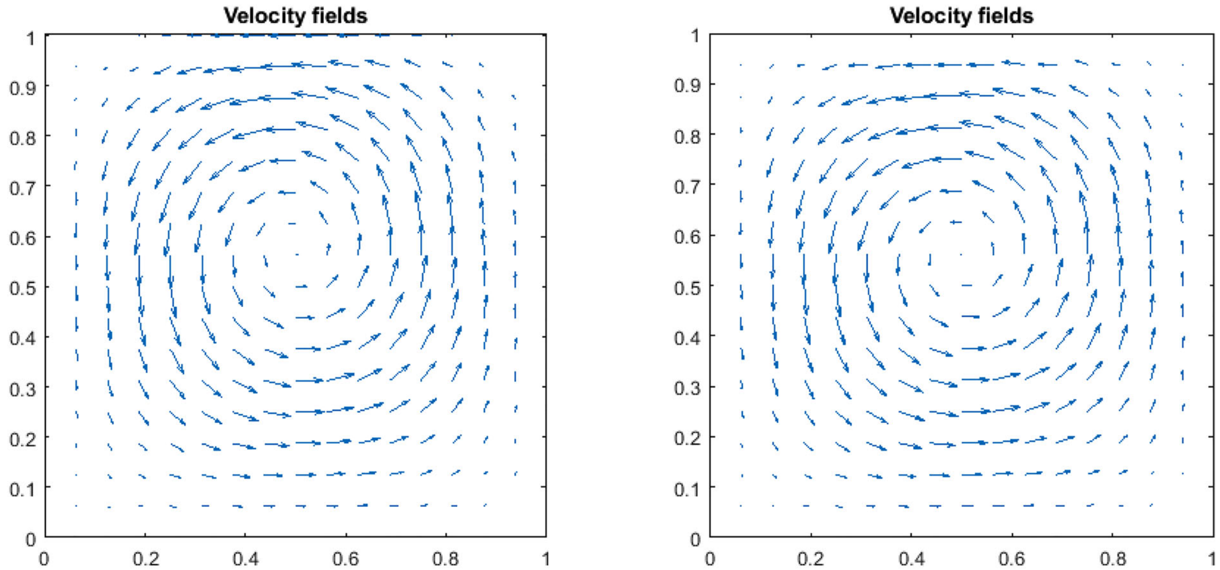

Fig. 1. Velocity fields for $g = 0.5$ and $g = 1.75$ with $h = 1/16$

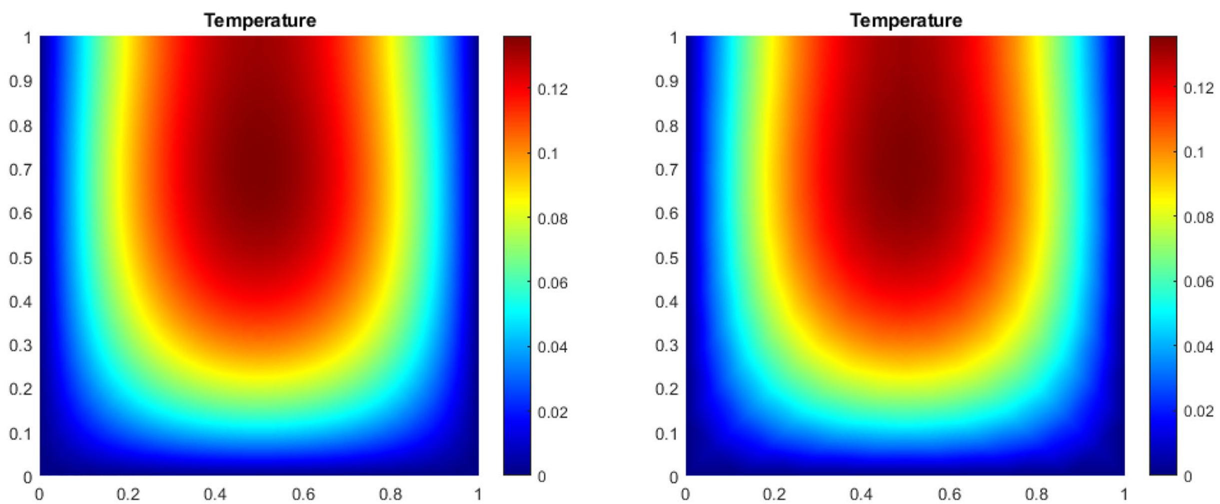

Fig. 2. Heat diffusion for $g = 0.5$ and $g = 1.75$ with $h = 1/16$

5.2. Test Case with Time-Dependent Solution

In order to verify time convergence rates, we consider a test problem with the same data as in the previous section except the right-hand side (now time-dependent)

$$f_1(x, y, t) = t^2 f_1(x, y),$$
$$f_2(x, y, t) = t^2 f_2(x, y),$$
$$k(x, y, t) = tk(x, y),$$

where $(f_1(x, y),\ f_2(x, y),\ k(x, y))$ is the right-hand side (5.2)-(5.4). We compute a reference solution using $\Delta t = 2^{-10}$ as time step and $h = 1/256$ as mesh size. To check the results of Theorem 3.4 we compute the

Table 1. *Convergence rates for* $(\boldsymbol{u}, p, \theta)$ *with* $g = 0.5$

h	$e_h(\boldsymbol{u}, p)$	Rate	$\|\theta_{\mathrm{ref}} - \theta_h\|_{H^1(\Omega)}$	Rate
1/8	4.68523e−01		1.91945e−02	
		1.0486		0.9821
1/16	2.26496e−01		9.71706e−03	
		1.2191		0.9780
1/32	9.72887e−02		4.93324e−03	
		1.2417		0.9861
1/64	4.114120e−02		2.49056e−03	
		1.1974		0.9924
1/128	1.79401e−02		1.25186e−03	
		1.1926		0.9960
1/256	7.84930e−03		6.27667e−04	

Table 2. *Convergence rates for* $(\boldsymbol{u}, p, \theta)$ *with* $g = 1.75$

h	$e_h(\boldsymbol{u}, p)$	Rate	$\|\theta_{\mathrm{ref}} - \theta_h\|_{H^1(\Omega)}$	Rate
1/8	5.14279e−01		1.91938e−02	
		0.9161		0.9821
1/16	2.72539e−01		9.71702e−03	
		1.1830		0.9780
1/32	1.20033e−01		4.93320e−03	
		1.2689		0.9861
1/64	4.98103e−02		2.49054e−03	
		1.2719		0.9924
1/128	2.06278e−02		1.25185e−03	
		1.2457		0.9960
1/256	8.69858e−03		6.27661e−04	

following errors at $T = 0.5$.

$$e_h(\boldsymbol{u}) = \sup_{1 \le n \le N} \| \boldsymbol{u}_h^n - \boldsymbol{u}_{ref} \|_{L^2(\Omega)} + \nu_1 \Delta t \sum_{n=1}^{N} |\boldsymbol{u}_h^n - \boldsymbol{u}_{ref}|_{H^1(\Omega)}, \tag{5.5}$$

$$e_h(\theta) = \sup_{1 \le n \le N} \| \theta_h^n - \theta_{ref} \|_{L^2(\Omega)} + \mu_0 \Delta t \sum_{n=1}^{N} |\theta_h^n - \theta_{ref}|_{H^1(\Omega)}. \tag{5.6}$$

where $(\boldsymbol{u}_{ref}, \theta_{ref})$ is a reference (computed) solution. In (5.5–5.6), $\nu_1 = 1/4$ and $\mu_0 = 1/2$, assuming $\theta \ge 0$. Figure 3 shows the velocity fields for $g = 0.1$ and $g = 0.5$ at $T = 0.5$. We notice that the slip occurs for $g = 0.1$.

To investigate experimentally the convergence rates for $e_h(\boldsymbol{u})$ and $e_h(\theta)$, we compute a reference solution with $h = 1/256$ and $\Delta t = 2^{-10}$. We report in Tables 3, 4 $e_h(\boldsymbol{u})$, $e_h(\theta)$ and convergence rates for various time-steps. We notice again that the convergence rates are almost independent of the stick/slip bound g. Moreover, $e_h(\boldsymbol{u})$ and $e_h(\theta)$ converge linearly as predicted by Thorem 3.4.

6. Concusion

The purpose of this work was to deal with the elements finite approximation of the time-dependent Navier–Stokes system coupled with the heat equation and governed by the nonlinear Tresca boundary conditions where both the viscosity and conductivity depend on the temperature. We present optimal error estimates for velocity, pressure and temperature. We have formulated and established the convergence of the Marchuk-Yanenko's algorithm associated to the finite element equations. And finally established some

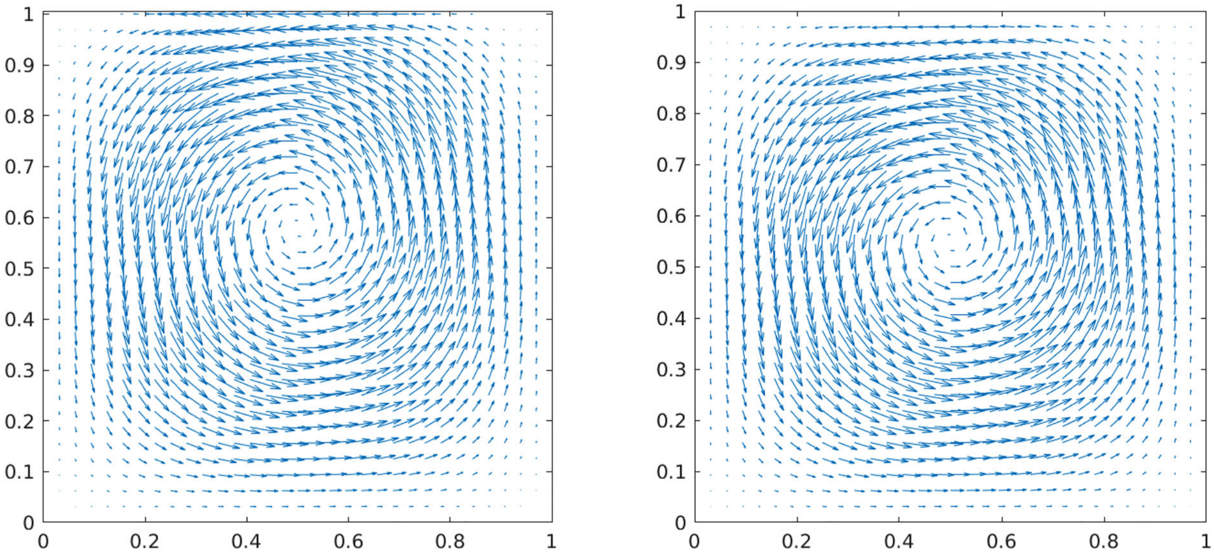

Fig. 3. Velocity fields for $g = 0.1$ (left) and $g = 0.5$ (right) at $T = 0.5$

Table 3. *Convergence rates at* $T = 0.5$ *with* $g = 0.1$, $\Delta t = 2^{-1-\ell}$ *and* $h = 1/256$

ℓ	$e_h(\boldsymbol{u})$	Rate	$e_h(\theta)$	Rate
1	2.48187e−03		2.21158e−03	
		0.7434		0.3903
2	1.48253e−03		1.68737e−03	
		0.9607		0.7995
3	7.61719e−04		9.69494e−04	
		1.0209		0.9290
4	3.75377e−04		5.09215e−04	
		1.0362		0.9944
5	1.83038e−04		2.55599e−04	
		1.0983		1.0718
6	8.54908e−05		1.21591e−04	

Table 4. *Convergence rates at* $T = 0.5$ *with* $g = 0.5$, $\Delta t = 2^{-1-\ell}$ *and* $h = 1/256$

ℓ	$e_h(\boldsymbol{u})$	Rate	$e_h(\theta)$	Rate
1	1.18213e−03		2.21160e−03	
		0.4826		0.3903
2	8.46044e−04		1.68737e−03	
		0.8565		0.7995
3	4.67252e−04		9.69494e−04	
		0.9759		0.9290
4	2.37555e−04		5.09215e−04	
		1.0340		0.9944
5	1.16014e−04		2.55599e−04	
		1.0947		1.0718
6	5.43232e−05		1.21591e−04	

numerical simulations which confirm the theoretical estimate. The extension of this work is in process, we aim to proceed for experiments on realistic examples, such as a lake heated by the sun.

Declarations

Conflict of interest The author(s) declares that they have no competing interests.

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