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Existence of Local Solutions to a Free Boundary Problem for Incompressible Viscous Magnetohydrodynamics

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Abstract. We consider the motion of an incompressible magnetohydrodynamics with resistivity in a domain bounded by a free surface which is coupled through the free surface with an electromagnetic field generated by a magnetic field prescribed on an exterior fixed boundary. On the free surface, transmission conditions for the electromagnetic field are imposed. As transmission condition we assume jumps of tangent components of magnetic and electric fields on the free surface. We prove local existence of solutions such that velocity and magnetic fields belong to $H^{2+\alpha,1+\alpha/2}$, $\alpha > 5/8$.

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Keywords. Free boundary, Incompressible magnetohydrodynamics with resistivity, Regularizer technique, Transmission conditions on the free surface, The method of successive approximations.

1. Introduction

We consider a free boundary problem for a viscous incompressible magnetohydrodynamic motion in domain $\overset{1}{\Omega}_t$ bounded by a free surface S_t. The motion interacts with an electromagnetic field located in $\overset{2}{\Omega}_t$ (Fig. 1).

In $\overset{1}{\Omega}_t$ the magnetohydrodynamic motion is described by the following system of equations

$$\begin{aligned}
&v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) - \mu_1\overset{1}{H}\cdot\nabla\overset{1}{H} + \frac{1}{2}\mu_1\nabla\overset{1}{H}{}^2 = f,\\
&\operatorname{div} v = 0,\\
&\mu_1\overset{1}{H}_{,t} = -\operatorname{rot}\overset{1}{E},\\
&\operatorname{rot}\overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v\times\overset{1}{H}),\\
&\operatorname{div}\overset{1}{H} = 0,
\end{aligned}\tag{1.1}$$

where $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $p = p(x,t) \in \mathbb{R}$ is the pressure, $\overset{1}{H} = \overset{1}{H}(x,t) = (\overset{1}{H}_1(x,t), \overset{1}{H}_2(x,t), \overset{1}{H}_3(x,t) \in \mathbb{R}^3$ is the magnetic field, $\overset{1}{E} = \overset{1}{E}(x,t) = (\overset{1}{E}_1(x,t), \overset{1}{E}_2(x,t), \overset{1}{E}_3(x,t)) \in \mathbb{R}^3$ is the electric field, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ is the external force field, and $x = (x_1, x_2, x_3)$ are Cartesian coordinates. Moreover, μ_1 is the constant magnetic permeability, σ_1 is the constant electric conductivity and $\mathbb{T}(v,p)$ is the stress tensor of the form

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - p\mathbb{I},\tag{1.2}$$

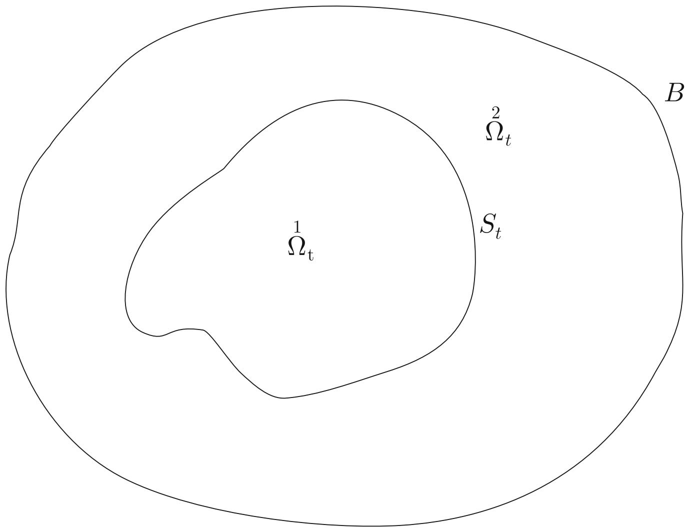

Fig. 1. The two dimensional cross section by a plane of the three-dimensional geometrical description of the considered free boundary problem

where ν is the positive viscosity coefficient, $\mathbb{I}$ is the unit matrix and $\mathbb{D}(v)$ is the dilatation tensor of the form

$$\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}. \tag{1.3}$$

For system (1.1) the following initial and boundary conditions are prescribed

$$\begin{aligned} &\bar{n}\cdot\mathbb{T}(v,p) + \mu_1\bar{n}\cdot\mathbb{T}(\overset{1}{H}) = p_0\bar{n} && \text{on } S_t,\\ &v|_{t=0} = v(0), \quad \overset{1}{H}|_{t=0} = \overset{1}{H}(0), \quad \operatorname{div}\overset{1}{H}(0) = 0 && \text{in } \overset{1}{\Omega}_t,\\ &\overset{1}{\Omega}_t|_{t=0} = \overset{1}{\Omega}_0, \quad S_t|_{t=0} = S_0, \end{aligned} \tag{1.4}$$

where $\bar{n}$ is the unit vector outward to $\overset{1}{\Omega}_t$ and normal to S_t, the constant exterior pressure p_0 can be absorbed by pressure p and

$$\mathbb{T}(\overset{1}{H}) = \left\{\overset{1}{H}_i\overset{1}{H}_j - \frac{1}{2}\overset{1}{H}{}^2\delta_{ij}\right\}_{i,j=1,2,3}. \tag{1.5}$$

The boundary conditions $(1.4)_1$ implies the compatibility condition

$$\bar{n}(0)\cdot\mathbb{D}(v(0))\cdot\bar{\tau}(0) + \mu_1\bar{n}(0)\cdot\overset{1}{H}(0)\bar{\tau}(0)\cdot\overset{1}{H}(0) = 0 \quad \text{on } S_0,$$

where $\bar{n}(0) = \bar{n}|_{t=0}$, $\bar{\tau}(0) = \bar{\tau}|_{t=0}$ and $\bar{\tau}$ is a tangent vector to S_t.

In $\overset{2}{\Omega}_t$ we have a motionless dielectric gas under a constant pressure p_0. Therefore, we only have an electromagnetic field described by the system of equations

$$\begin{aligned} &\mu_2\overset{2}{H}_{,t} = -\operatorname{rot}\overset{2}{E},\\ &\sigma_2\overset{2}{E} = \operatorname{rot}\overset{2}{H},\\ &\operatorname{div}\overset{2}{H} = 0. \end{aligned} \tag{1.6}$$

For system (1.6) the following initial and boundary conditions are prescribed:

$$\begin{aligned} &\overset{2}{H}|_{t=0} = \overset{2}{H}(0), \quad \operatorname{div} \overset{2}{H}(0) = 0, \quad \overset{2}{\Omega}_t|_{t=0} = \overset{2}{\Omega}_0, \\ &\overset{2}{H} \cdot \bar{\tau}_\alpha|_B = H_{*\alpha}, \quad \alpha = 1,2, \quad \operatorname{div} \overset{2}{H}|_B = 0, \end{aligned} \tag{1.7}$$

where $\bar{\tau}_\alpha$, $\alpha = 1,2$, are tangent vectors to B.

The magnetohydrodynamic system (1.1) is composed of two problems. For a given magnetic field $\overset{1}{H}$ system $(1.1)_{1,2}$ determines velocity v and pressure p under appropriate initial and boundary conditons (1.4). To formulate a problem for $\overset{1}{H}$ we have to recall that a motion in $\overset{1}{\Omega}_t$ interacts with an electromagnetic field in $\overset{2}{\Omega}_t$ through the free surface S_t by transmission conditions. Therefore, system $(1.1_{3,4,5})$ and (1.6) is a problem for $\overset{1}{H}$ and $\overset{2}{H}$ which are coupled by transmission conditions on the interface S_t. Therefore, for a given v, we have the following problem for $\overset{1}{H}$ and $\overset{2}{H}$:

$$\begin{aligned} &\mu_1 \overset{1}{H}_{,t} = -\operatorname{rot} \overset{1}{E} && \text{in } \overset{1}{\Omega}_t, \\ &\operatorname{rot} \overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v \times \overset{1}{H}) && \text{in } \overset{1}{\Omega}_t, \\ &\operatorname{div} \overset{1}{H} = 0 && \text{in } \overset{1}{\Omega}_t, \\ &\mu_2 \overset{2}{H}_{,t} = -\operatorname{rot} \overset{2}{E} && \text{in } \overset{2}{\Omega}_t, \\ &\operatorname{rot} \overset{2}{H} = \sigma_2 \overset{2}{E} && \text{in } \overset{2}{\Omega}_t, \\ &\operatorname{div} \overset{2}{H} = 0 && \text{in } \overset{2}{\Omega}, \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} &\overset{1}{H}|_{t=0} = \overset{1}{H}(0), \quad \overset{2}{H}|_{t=0} = \overset{2}{H}(0), \quad \overset{2}{H} \cdot \bar{\tau}_\alpha|_B = H_{*\alpha}, \quad \alpha = 1,2, \\ &\operatorname{div} \overset{2}{H}|_B = 0, \end{aligned}$$

and $\bar{\tau}_\alpha$, $\alpha = 1,2$, are tangent vectors to B. Electric vectors $\overset{1}{E}$ and $\overset{2}{E}$ are auxiliary.

To make system (1.8) complete we have to add transmission conditions. The conditions are recessary for a proof of existence of solutions to (1.1)–(1.8). Satisfying them we can derive such energy equality which can imply all necessary estimates for the proof of existence in the L_2-approach.

Lemma 1.1. *Assume that near any point of S_t there exists an orthonormal system $(\bar{\tau}_1, \bar{\tau}_2, \bar{n})$, where $\bar{\tau}_\alpha$, $\alpha = 1,2$, is tangent to S_t and $\bar{n}$ is normal.*
Let a_i, $i = 1,2$, be given positive numbers. Let $H_{*\alpha} = 0$, $\alpha = 1,2$. Assume that

$$\sum_{\alpha=1}^{2} (a_1 \overset{1}{E} \cdot \bar{\tau}_\alpha \bar{n} \times \bar{\tau}_\alpha \cdot \overset{1}{H} - a_2 \overset{2}{E} \cdot \bar{\tau}_\alpha \bar{n} \times \bar{\tau}_\alpha \cdot \overset{2}{H}) = 0. \tag{1.9}$$

Then the following energy equality

$$\sum_{i=1}^{2} \left[a_i \cdot \mu_i \int_{\overset{i}{\Omega}_t} \overset{i}{H}_{,t} \cdot \overset{i}{H} dx + a_i \int_{\overset{i}{\Omega}_t} \overset{i}{E} \cdot \operatorname{rot} \overset{i}{H} dx \right] = 0 \tag{1.10}$$

holds.

Proof. From (1.8)$_1$ and (1.8)$_4$ we have

$$\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_t} a_i\mu_i\overset{i}{H}_{,t}\cdot\overset{i}{H}dx+\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_t} a_i\mathrm{rot}\,\overset{i}{E}\cdot\overset{i}{H}dx=0. \tag{1.11}$$

Recall the identity

$$\int_{\Omega_t}\mathrm{rot}\,H\cdot\psi dx=\int_{\Omega_t}H\cdot\mathrm{rot}\,\psi dx-\int_{S_t}\bar{n}\times H\cdot\psi dS_t, \tag{1.12}$$

where $S_t=\partial\Omega_t$, ψ is a sufficiently regular function and $\bar{n}$ is the unit exterior vector to Ω_t and normal to S_t.

From (1.12) we have

$$\int_{\overset{1}{\Omega}_t}\mathrm{rot}\,\overset{1}{E}\cdot\overset{1}{H}dx=\int_{\overset{1}{\Omega}_t}\overset{1}{E}\cdot\mathrm{rot}\,\overset{1}{H}dx-\int_{S_t}\overset{1}{\bar{n}}\times\overset{1}{E}\cdot\overset{1}{H}dS_t, \tag{1.13}$$

$$\int_{\overset{2}{\Omega}_t}\mathrm{rot}\,\overset{2}{E}\cdot\overset{2}{H}dx=\int_{\overset{2}{\Omega}_t}\overset{2}{E}\cdot\mathrm{rot}\,\overset{2}{H}dx-\int_{S_2}\overset{2}{\bar{n}}\times\overset{2}{E}\cdot\overset{2}{H}dS_t, \tag{1.14}$$

where $\overset{i}{\bar{n}}$ is exterior to $\overset{i}{\Omega}_t$ and $\overset{2}{\bar{n}}=-\overset{1}{\bar{n}}$.

Using (1.13) and (1.14) in (1.11) yields

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_t} a_i\mu_i\overset{i}{H}_{,t}\cdot\overset{i}{H}dx+\sum_{i=1}^{2}\int_{\overset{i}{\Omega}_t} a_i\overset{i}{E}\cdot\mathrm{rot}\,\overset{i}{H}dx\\&-\int_{S_t}(a_1\bar{n}\times\overset{1}{E}\cdot\overset{1}{H}-a_2\bar{n}\times\overset{2}{E}\cdot\overset{2}{H})dx=0,\end{aligned} \tag{1.15}$$

where $\bar{n}=\overset{1}{\bar{n}}$. Using the orthonormal system $(\bar{\tau}_1,\bar{\tau}_2,\bar{n})$ we have

$$\overset{i}{E}=\sum_{\alpha=1}^{2}\overset{i}{E}\cdot\bar{\tau}_\alpha\bar{\tau}_\alpha+\overset{i}{E}\cdot\bar{n}\bar{n},\quad i=1,1. \tag{1.16}$$

In view of (1.16) the boundary term in (1.15) equals

$$\sum_{\alpha=1}^{2}\int_{S_t}(a_1\overset{1}{E}\cdot\bar{\tau}_\alpha\bar{n}\times\bar{\tau}_\alpha\cdot\overset{1}{H}-a_2\overset{2}{E}\cdot\bar{\tau}_\alpha\bar{n}\times\bar{\tau}_\alpha\cdot\overset{2}{H})dS_t \tag{1.17}$$

Hence (1.9) implies (1.10) and concludes the proof. □

Remark 1.2. The boundary term in (1.15) needs more regularity than the second term. If the transmission condition does not hold equality (1.15) does not imply any estimate.

Remark 1.3. (Some discussion of transmission conditions can be found in [1]). There is many different transmission conditions.

1. Let $a_1^{\nu_1}\overset{1}{E}\cdot\bar{\tau}_\alpha=a_2^{\nu_2}\overset{2}{E}\cdot\bar{\tau}_\alpha$, $\;a_1^{\nu_2}\bar{n}\times\bar{\tau}_\alpha\cdot\overset{1}{H}=a_2^{\nu_2}\bar{n}\times\bar{\tau}_\alpha\cdot\overset{2}{H}$, $\;\alpha=1,2$, where $\nu_1+\nu_2=1$ and a_1, a_2 are positive numbers. Then (1.9) holds.
2. Let $a_1=a_2=1$, $\overset{1}{E}\cdot\bar{\tau}_\alpha=\overset{2}{E}\cdot\bar{\tau}_\alpha$, $\bar{n}\times\bar{\tau}_\alpha\cdot\overset{1}{H}=\bar{n}\times\bar{\tau}_\alpha\cdot\overset{2}{H}$, $\alpha=1,2$. Then (1.9) also holds.
3. Let $a_1=a_2=1$, $\overset{1}{E}\cdot\bar{\tau}_\alpha=a\overset{2}{E}\cdot\bar{\tau}_\alpha$, $\bar{n}\times\bar{\tau}_\alpha\cdot\overset{1}{H}=\frac{1}{a}\bar{n}\times\bar{\tau}_\alpha\cdot\overset{2}{H}$, $\alpha=1,2$, where a is an arbitrary positive number. Then (1.9) is satisfied too.

4. We can also define anisotropic transmission conditions.

In cases 1 and 3 we have jumps of tangent components of electric and magnetic fields.

We have to recall that in magnetohydrodynamics the displacement current $E_{,t}$ is omitted.

To prove the existence of solutions to problem (1.1)–(1.9) we transform it into two problems: the problem for the fluid motion and the problem for the electromagnetic field. Therefore, for given $\overset{1}{H}$ we have the problem for (v, p):

$$\begin{aligned} & v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f + \mu_1 \operatorname{div}\mathbb{T}(\overset{1}{H}) && \text{in } \overset{1}{\Omega}_t, \\ & \operatorname{div} v = 0 && \text{in } \overset{1}{\Omega}_t, \\ & \bar{n}\cdot\mathbb{T}(v,p) = -\mu_1\bar{n}\cdot\mathbb{T}(\overset{1}{H}) && \text{on } S_t, \\ & v|_{t=0} = v(0) && \text{in } \Omega_0, \end{aligned} \tag{1.18}$$

where we assumed that p_0 is absorbed by p.

Next for a given v, the electromagnetic field is determined by the problem:

$$\begin{aligned} & \left.\begin{aligned} & \mu_1\overset{1}{H}_{,t} = -\operatorname{rot}\overset{1}{E}, \quad \operatorname{rot}\overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v\times\overset{1}{H}), \\ & \operatorname{div}\overset{1}{H} = 0 \end{aligned}\right. && \text{in } \overset{1}{\Omega}_t, \\ & \mu_2\overset{2}{H}_{,t} = -\operatorname{rot}\overset{2}{E}, \quad \sigma_2\overset{2}{E} = \operatorname{rot}\overset{2}{H}, \quad \operatorname{div}\overset{2}{H} = 0 && \text{in } \overset{2}{\Omega}_t, \\ & \overset{1}{H}|_{t=0} = \overset{1}{H}(0), \quad \operatorname{div}\overset{1}{H}(0) = 0 && \text{in } \overset{1}{\Omega}_0, \\ & \overset{2}{H}|_{t=0} = \overset{2}{H}(0), \quad \operatorname{div}\overset{2}{H}(0) = 0 && \text{in } \overset{2}{\Omega}_0, \\ & \overset{2}{H}\cdot\bar{\tau}'_\alpha = H_{*\alpha}, \quad \operatorname{div}\overset{2}{H}|_B = 0, \quad \bar{\tau}'_\alpha, \quad \alpha = 1,2, && \text{is a tangent vector to } B, \\ & \text{transmission condition (1.9).} \end{aligned} \tag{1.19}$$

Let v be a solution to (1.18). Then Lagrangian coordinates are the Cauchy data to the problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi.$$

Then

$$x(\xi,t) = \xi + \int_0^t \bar{v}(\xi,t')dt',$$

where $\bar{v}(\xi,t) = v(x(\xi,t),t)$ describes a relation between Cartesian and Lagrangian coordinates.

Therefore, the transmission condition (1.9) holds on the surface

$$S_t = \left\{x\in\mathbb{R}^3 \colon x = \xi + \int_0^t \bar{v}(\xi,t'), dt', \ \xi\in S_0\right\}.$$

The structure of problems (1.18), (1.19) suggests an applying a method of successive approximations.

Eliminating the electric field in (1.19) yields

$$\begin{aligned}
&\mu_1 \overset{1}{H}_{,t} + \frac{1}{\sigma_1}\mathrm{rot}^2 \overset{1}{H} = \mu_1 \mathrm{rot}\,(v \times \overset{1}{H}) && \text{in } \overset{1}{\Omega}_t,\\
&\mathrm{div}\, \overset{1}{H} = 0 && \text{in } \overset{1}{\Omega}_t,\\
&\mu_2 \overset{2}{H}_t + \frac{1}{\sigma_2}\mathrm{rot}^2 \overset{2}{H} = 0 && \text{in } \overset{2}{\Omega}_t,\\
&\mathrm{div}\, \overset{2}{H} = 0 && \text{in } \overset{2}{\Omega}_t,\\
&\overset{1}{H}|_{t=0} = \overset{1}{H}(0), \quad \mathrm{div}\, \overset{1}{H}(0) = 0 && \text{in } \overset{1}{\Omega}_0,\\
&\overset{2}{H}|_{t=0} = \overset{2}{H}(0), \quad \mathrm{div}\, \overset{2}{H}(0) = 0 && \text{in } \overset{2}{\Omega}_0,\\
&\overset{2}{H} \cdot \bar{\tau}'_\alpha = H_{*\alpha}, \quad \mathrm{div}\, \overset{2}{H}|_B = 0, \quad \bar{\tau}'_\alpha, \quad \alpha = 1,2, &&\\
&\quad \text{is a tangent vector to } B && \text{on } B,\\
&\text{transmission conditions (1.9) holds.}
\end{aligned} \tag{1.20}$$

Now, we formulate the main result of this paper

Theorem 1.4. *Assume that* $\Omega = \overset{1}{\Omega}_t \cup S_t \cup \overset{2}{\Omega}_t$. *Assume that* $f \in H^{\alpha,\alpha/2}(\Omega^t)$, $H_{*\beta} \in H^{3/2+\alpha,3/4+\alpha/2}(B^t)$, $S_0 \in H^{3/2+\alpha}$, *where* $\frac{5}{8} < \alpha < 1$, $\beta = 1,2$. *Assume that* $v(0) \in H^{1+\alpha}(\overset{1}{\Omega}_0)$, $\overset{i}{H}(0) \in H^{1+\alpha}(\overset{i}{\Omega}_0)$, $i = 1,2$.

Then for T *sufficiently small there exists a local solution to problem* (1.18)–(1.20) *such that* $\bar{v} \in H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)$, $\overset{i}{\bar{H}} \in H^{2+\alpha,1+\alpha/2}(\overset{i}{\Omega}{}_0^t)$, $i = 1,2$, $\bar{p}_\xi, \bar{p} \in L_2(\overset{1}{\Omega}{}_0^t)$, $\bar{p}|_{S_0} \in H^{1/2+\alpha,1/4+\alpha/2}(S_0^t)$ *and the estimate*

$$\begin{aligned}
X(t) \equiv{}& \|\bar{v}\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)} + \|\bar{p}_\xi\|_{L_2(\overset{1}{\Omega}{}_0^t)} + \|\bar{p}\|_{L_2(\overset{1}{\Omega}{}_0^t)}\\
&+ \|\bar{p}|_{S_0}\|_{H^{1/2+\alpha,1/4+\alpha/2}(S_0^t)} \le c\bigg(\|\bar{f}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)}\\
&+ \sum_{\beta=1}^{2} \|H_{*\beta}\|_{H^{3/2+\alpha,3/4+\alpha/2}(B^t)} + \|v(0)\|_{H^{1+\alpha}(\overset{1}{\Omega}_0)}\\
&+ \sum_{i=1}^{2} \|\overset{i}{H}(0)\|_{H^{1+\alpha}(\overset{i}{\Omega}_0)}\bigg)
\end{aligned} \tag{1.21}$$

holds, where $t \le T$ *and* $\bar{v}$, $\overset{i}{\bar{H}}$, $\bar{p}$, $\bar{f}$ *are equal to* v, $\overset{i}{H}$, p, f *expressed in Lagrangian coordinates. Moreover, the Sobolev–Slobodetskii space* $H^{k+\alpha,k/2+\alpha/2}$, $k \in \mathbb{N}$, $\alpha \in (0,1)$ *is defined in* (2.2.1).

In this paper we prove existence of local solutions to problem (1.18), (1.20). The formulation of problem (1.18), (1.20) suggests that the method of successive approximations should be used. The method is described in Sect. 3 in problems (3.8) and (3.9). The Stokes system (3.8) with the Neumann boundary conditions determines a relation between v_{n+1}, p_{n+1} at the $n+1$-th step of iteration and v_n, p_n, $\overset{1}{H}_n$ at the n-th step. Similarly, $\overset{1}{H}_{n+1}$, $\overset{2}{H}_{n+1}$ are solutions to problem (3.9) for given v_n $\overset{1}{H}_n$, $\overset{2}{H}_n$. The existence of solutions to problem (3.8) follows from Lemma 2.3.2. Problem (3.9) describes $\overset{1}{H}_{n+1}$ and $\overset{2}{H}_{n+1}$ in domains $\overset{1}{\Omega}_0$ and $\overset{2}{\Omega}_0$, respectively. Moreover, $\overset{1}{H}_{n+1}$ and $\overset{2}{H}_{n+1}$ are coupled through the free surface S_0 by transmission conditions (see Lemma 1.1) and $\overset{2}{H}_{n+1}$ satisfies some boundary conditions on B. Existence of solutions to problem (3.9) follows from Lemma 2.5.1. The proof of this lemma exploits the technique of regularizer introduced in [2]. Since Lagrangian coordinates are used domains $\overset{1}{\Omega}_0$, $\overset{2}{\Omega}_0$ are the initial domains and S_0 is the initial free boundary.

Exploiting Lagrangian coordinates the r.h.s. of problems (3.8) and (3.9) are strongly nonlinear and complicated.

In Sect. 4 and 5 we derive the inequality (see Corollaries 4.5 and 5.4)

$$X_{n+1}(t) \le \phi(t^a X_n(t), \text{data}), \tag{1.22}$$

where

$$X_n(t) = \|\bar{v}_n\|_{V_2^{2+\alpha}(\Omega_0^t)} + \sum_{i=1}^{2} \|\overset{i}{\bar{H}}_n\|_{V_2^{2+\alpha}(\overset{i}{\Omega}{}_0^t)}, \tag{1.23}$$

where $V_2^{2+\alpha}(\Omega^t)$ is defined by

$$\|u\|_{V_2^{2+\alpha}(\Omega^t)} = \sup_t \|u(t)\|_{H^{1+\alpha}(\Omega)} + \|u\|_{H^{2+\alpha,1+\alpha/2}(\Omega^t)},$$

$\alpha > 5/8$, ϕ is a strongly nonlinear increasing function and $a > 0$.

Thanks to the coefficient t^a and t sufficiently small we derive the estimate (see Lemma 6.1)

$$X_n(t) \le \phi(\text{data}) \quad \text{for any} \quad n \in \mathbb{N}. \tag{1.24}$$

Using differences introduced in Lemma 6.2 we prove in Sect. 6 that the sequence $(\bar{v}_n, \overset{1}{\bar{H}}_n, \overset{2}{\bar{H}}_n)$ converges. Hence, we prove the existence of solutions to problem (1.18), (1.20).

We should explain some points of the proof of Lemma 2.5.1 which needs the technique of regularizer. This technique needs to examine the following local problems derived from (3.9):

1. near an interior point of $\overset{1}{\Omega}_0$,
2. near an interior point of $\overset{2}{\Omega}_0$,
3. near a point of S_0,
4. near a point of B.

The local problems in cases 3 and 4 are considered in Sects. 7 and 8, respectively. To solve the local problems we need Besov spaces $H^{2+\alpha,1+\alpha/2}$ expressed in the Fourier-Laplace transforms. This is possible because $H^{2+\alpha,1+\alpha/2}$ are L_2-Besov spaces.

The equations of magnetohydrodynamics (mhd) can be found in [3,4].

The first result on solvability of mhd equations appeared in [5]. Later free boundary problems to incompressible viscous mhd with resistivity were considered in [6].

Free boundary problems for mhd equations were also considered in [7,8]. In [7,8] the external magnetic field satisfies the elliptic system

$$\operatorname{rot} \overset{2}{H} = 0, \quad \operatorname{div} \overset{2}{H} = 0$$

However in those papers the boundary condition on the free surface contains the surface tension.

The existence of local solutions to problem (1.18), (1.20) has been already considered in [2,9,10]. Comparing to [2] in this paper we proved existence of solutions with the lowest possible regularity. Moreover, we used the L_2-approach because we are going to prove global existence of solutions using appropriate differential inequalities (see [11]). In [9,10] the existence of solutions to problem (1.18), (1.20) is proved by the Faedo–Galerkin method. The applied energy method to solutions to problem (1.20) in [9,10] implies very strong restrictions on the transmission coefficients. In this approach the proof of existence with optimal regularity is not possible.

In this paper we express the results in [9] in a more explicit and appropriate way.

In [12,13], using the Weis theory of Fourier multipliers, two different mhd fluids inteacting through a free surface are considered.

The result similar to Lemma 2.5.1 is also proved in [14].

In [15–17] the global existence of solutions to problem (1.18), (1.20) is proved by the energy method so only Sobolev spaces are used.

Moreover, the methods used in [15–17] imposes strong restrictions on the transmission coefficients. In the forthcoming paper we will relax the restrictions.

2. Notation and Auxiliary Results

2.1. Partition of Unity

To prove the existence of solutions to problem (1.18), (1.20) we need a partition of unity. We consider two collections of open subsets $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$, $k \in \mathfrak{M} \cup \mathfrak{N}$, such that $\bar{\omega}^{(k)} \subset \Omega^{(k)} \subset \Omega_0 = \overset{1}{\Omega}_0 \cup \overset{2}{\Omega}_0$, $\bigcup_k \omega^{(k)} = \bigcup_k \Omega^{(k)} = \Omega_0$, $\bar{\Omega}^{(k)} \cap S_0 = \phi_0$, where ϕ_0 describes the empty set for $k \in \mathfrak{M}_1 \cup \mathfrak{M}_2$, $\bar{\Omega}^{(k)} \cap S_0 \neq \phi_0$, where ϕ_0 describes the empty set for $k \in \mathfrak{N}_1$ and $\bar{\Omega}^{(k)} \cap B \neq \phi_0$, where ϕ_0 describes the emnpty set for $k \in \mathfrak{N}_2$, $\mathfrak{N} = \mathfrak{N}_1 \cup \mathfrak{N}_2$. Moreover, subdomains with $k \in \mathfrak{M}_i$ are contained in $\overset{i}{\Omega}_0$, $i = 1, 2$.

We assume that at most N_0 of the $\Omega^{(k)}$ have nonempty intersections, $\sup_k \operatorname{diam} \Omega^{(k)} \le 2\lambda$, $\sup_k \operatorname{diam} \omega^{(k)} \le \lambda$ for some $\lambda > 0$. Let $\zeta^{(k)}(x)$ be a smooth function such that $0 \le \zeta^{(k)}(x) \le 1$, $\zeta^{(k)}(x) = 1$ for $x \in \omega^{(k)}$, $\zeta^{(k)}(x) = 0$ for $x \in \Omega_0 \setminus \Omega^{(k)}$ and $|D_x^\nu \zeta^{(k)}(x)| \le c/\lambda^{|\nu|}$. Then $1 \le \sum_k (\zeta^{(k)}(x))^2 \le N_0$. Introducing the function

$$\eta^{(k)}(x) = \frac{\zeta^{(k)}(x)}{\sum_l (\zeta^{(l)}(x))^2}$$

we have that $\eta^{(k)}(x) = 0$ for $x \in \Omega_0 \setminus \Omega^{(k)}$, $\sum_k \eta^{(k)}(x)\zeta^{(k)}(x) = 1$ and $|D_x^\nu \eta^{(k)}(x)| \le c/\lambda^{|\nu|}$.

We denote by $\xi^{(k)}$ an interior point of $\omega^{(k)}$ and $\Omega^{(k)}$ for $k \in \mathfrak{M}$ and an interior point of $\bar{\omega}^{(k)} \cap S_0$ and of $\bar{\Omega}^{(k)} \cap S_0$ for $k \in \mathfrak{N}_1$ and an interior point of $\bar{\omega}^{(k)} \cap B$ and of $\bar{\Omega}^{(k)} \cap B$ for $k \in \mathfrak{N}_2$. For $k \in \mathfrak{M}_i$, $\xi^{(k)} \in \Omega_0^i$, $i = 1, 2$. Let $x = (x_1, x_2, x_3)$ be the Cartesian system of coordinates with the origin located in the interior of Ω_0.

Then by translations and rotations we introduce a local coordinate system $y = (y_1, y_2, y_3)$ with the origin at $\xi^{(k)} \in \Omega^{(k)} \cap S_0$, $k \in \mathfrak{N}_1$, such that the part $\tilde{S}_0^{(k)} = S_0 \cap \bar{\Omega}^{(k)}$ of the boundary S_0 is described by $y_3 = F_k(y_1, y_2)$. We denote the transformation as $y = Y_k(x)$. Then we introduce new coordinates defined by

$$z_i = y_i, \quad i = 1, 2, \quad z_3 = y_3 - F_k(y_1, y_2), \quad k \in \mathfrak{N}_1.$$

We will denote this transformation by $\hat{\Omega}^{(k)} \ni z = \Phi_k(y)$, where $y \in \Omega^{(k)}$.

We assume that the sets $\hat{\omega}^{(k)}$, $\hat{\Omega}^{(k)}$ are described in local coordinates at $\xi^{(k)}$ by the inequalities

$$\begin{aligned} &|y_i| < \lambda, \quad i = 1, 2, \quad |y_3 - F_k(y_1, y_2)| < \lambda, \\ &|y_i| < 2\lambda, \quad i = 1, 2, \quad |y_3 - F_k(y_1, y_2)| < 2\lambda, \end{aligned}$$

respectively. Moreover, $(y_1, y_2, y_3) \in \overset{1}{\Omega}_0$ if $y_3 > F_k(y_1, y_2)$ and $(y_1, y_2, y_3) \in \overset{2}{\Omega}_0$ for $y_3 < F_k(y_1, y_2)$. Let $\Psi_k = \Phi_k \circ Y_k$. Then $z = \Psi_k(x)$ and

$$\hat{u}^{(k)}(z, t) = u(\Psi_k^{-1}(z), t), \quad \tilde{u}^{(k)}(z, t) = \hat{u}^{(k)}(z, t)\hat{\zeta}^{(k)}(z, t).$$

For $k \in \mathfrak{M}$ we have

$$\tilde{u}^{(k)}(x, t) = u^{(k)}(x, t)\zeta^{(k)}(x).$$

For $k \in \mathfrak{N}_2$ we introduce new local coordinates with origin at $\xi^{(k)} \in B \cap \bar{\Omega}^{(k)}$ such that $y_3 = F_k(y_1, y_2)$ describes locally $B \cap \bar{\Omega}^{(k)}$. We also introduce the transfomation $z_i = y_i$, $i = 1, 2$, $z_3 = y_3 - F_k(y_1, y_2)$ and assume that $z = \Phi_k(y)$ belongs to $\hat{\Omega}^{(k)}$ for $y \in \Omega^{(k)}$.

Finally, $\hat{\omega}^{(k)}$, $\hat{\Omega}^{(k)}$ are described by the inequalities

$$|y_i| < \lambda, \quad i = 1,2, \quad 0 < y_3 - F_k(y_1,y_2) < \lambda,$$
$$|y_i| < 2\lambda, \quad i = 1,2, \quad 0 < y_3 - F_k(y_1,y_2) < 2\lambda,$$

respectively.

Moreover, we introduce the notation: r.h.s. (l.h.s.) right-hand side (left-hand side).

By ϕ we denote an increasing positive function such that $\phi(0) \neq 0$ and it can change its form from formula to formula.

2.2. Spaces

We prove the existence of local solutions to problem (1.18), (1.20) in L_2-Sobolev-Slobodetskii spaces with the norm

$$\begin{aligned}\|u\|^2_{H^{k+\alpha,k/2+\alpha/2}(\Omega^T)} &= \sum_{|\beta|+2i\le k} \|D_x^\beta \partial_t^i u\|^2_{L_2(\Omega^T)} \\ &+ \sum_{|\beta|=k} \int_0^T \int_\Omega \int_\Omega \frac{|D^\beta_{x'} u(x',t) - D^\beta_{x''} u(x'',t)|^2}{|x'-x''|^{3+2\alpha}} dx'dx''dt \\ &+ \int_\Omega \int_0^T \int_0^T \frac{|\partial_{t'}^{[k/2]} u(x,t') - \partial_{t''}^{[k/2]} u(x,t'')|^2}{|t'-t''|^{1+\alpha}} dx dt' dt'',\end{aligned} \tag{2.2.1}$$

where $\alpha \in (0,1)$, $[l]$ is the integer part of l,

$$D_x^\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \partial_{x_3}^{\beta_3}, \quad |\beta| = \beta_1 + \beta_2 + \beta_3.$$

We need the Hilbert type spaces because their norms can be expressed in the Fourier-Laplace transforms. We also need the Hilbert type spaces because a proof of global existence of solutions to problem (1.18), (1.20) will be made by the energy method.

In this paper we also need the L_p-Besov spaces. Hence we recall some properties of isotropic Besov spaces which are frequently used in this paper. Next, we define anisotropic Besov spaces and formulate some imbedding theorems which we need.

Let us introduce the differences

$$\Delta_i(h)u(x) = u(x+he_i) - u(x),$$

where $x \in \mathbb{R}^n$ and e_i $i = 1,\dots,n$, are the standard unit vectors. Then we define inductively the m-difference

$$\Delta_i^m(h)u(x) = \Delta_i(h)(\Delta_i^{m-1}(h)u(x)) = \sum_{j=0}^m (-1)^{m-j} c_{jm}(x+jhe_i),$$

where $c_{jm} = \binom{m}{j} = \frac{m!}{j!(m-j)!}$. Moreover, we introduce the difference

$$\Delta(y)f(y) = f(x+y) - f(x), \quad x,y \in \mathbb{R}^n,$$

and inductively

$$\Delta^m(y)f(x) = \Delta(y)(\Delta^{m-1}(y)f(x)).$$

Since

$$\Delta(x-y)f(y) = f(x) - f(y)$$

we have

$$\Delta^m(x-y)f(y)=\sum_{i=1}^m \Delta^m((x-y)\cdot e_i)f(y)=\sum_{i=1}^m \Delta_i^m(h)f(y),$$

where the last equality holds for $(x-y)\cdot e_i=h_i$.

We define the isotropic Besov spaces by introducing the norm (see [18, Ch. 4, Sect. 18])

$$\|u\|_{B_p^l(\mathbb{R}^n)}=\|u\|_{L_p(\mathbb{R}^n)}+\sum_{i=1}^n\bigg(\int_0^{h_0}dh\int_{\mathbb{R}^n}\frac{|\Delta_i^m(h)\partial_{x_i}^k u|^p}{h^{1+(l-k)p}}dx\bigg)^{1/p}, \tag{2.2.2}$$

where $m>l-k$, $m,k\in\mathbb{N}\cup\{0\}$, $l\in\mathbb{R}_+$, $l\neq\mathbb{Z}$, $p\in(1,\infty)$.

It was shown in [19] that the Besov spaces defined by (2.2.2) all coincide and have equivalent norms for all m, k satisfying $m>l-k$.

Next we define the L_p-scale of Sobolev-Slobodetskii spaces by introducing the norm

$$\|u\|_{W_p^l(\mathbb{R}^n)}=\|u\|_{L_p(\mathbb{R}^n)}+\sum_{i=1}^n\bigg(\int_0^{h_0}dh\int_{\mathbb{R}^n}\frac{|\Delta_i(h)\partial_{x_i}^{[l]} u|^p}{h^{1+p(l-[l])}}dx\bigg)^{1/p}, \tag{2.2.3}$$

where $l\notin\mathbb{Z}$, $[l]$ is the integer part of l.

We frequently write $l=k+\alpha$, $k\in\mathbb{N}\cup\{0\}$, $\alpha\in(0,1)$, so $k=[l]$ and $\alpha=l-[l]$.

By the Golovkin theorem (see [19]) the norms of the spaces $B_p^l(\mathbb{R}^n)$ and $W_p^l(\mathbb{R}^n)$ are equivalent.

We also define the norms

$$\|u\|_{\tilde{B}_p^l(\mathbb{R}^n)}=\|u\|_{L_p(\mathbb{R}^n)}+\sum_{|\alpha|=k}\bigg(\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^n}dy\frac{|\Delta^m(x-y)D_y^\alpha u(y)|^p}{|x-y|^{n+p(l-k)}}\bigg)^{1/p} \tag{2.2.4}$$

for any $m>l-k$ and

$$\|u\|_{\tilde{W}_p^l(\mathbb{R}^n)}=\|u\|_{L_p(\mathbb{R}^n)}+\sum_{|\alpha|=[l]}\bigg(\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^n}\frac{|\Delta(x-y)D_y^\alpha u(y)|^p}{|x-y|^{n+p(l-[l])}}\bigg)^{1/p}. \tag{2.2.5}$$

Now we introduce the partial derivatives

$$\langle u\rangle_{\alpha,x,p,\Omega^t}=\bigg(\int_0^t dt'\int_\Omega\int_\Omega\frac{|u(x',t')-u(x'',t')|^p}{|x'-x''|^{3+p\alpha}}dx'dx''\bigg)^{1/p},$$

$$\langle u\rangle_{\frac{\alpha}{2},t,p,\Omega^t}=\bigg(\int_\Omega dx\int_0^t\int_0^t\frac{|u(x,t')-u(x,t'')|^p}{|t'-t''|^{1+p\alpha/2}}dt'dt''\bigg)^{1/p}.$$

Let

$$L_p^k(\Omega)=\{u\colon \sum_{|\alpha|=k}\|D_x^\alpha u\|_{L_p(\Omega)}<\infty\}.$$

We also need the following seminorms

$$\langle u\rangle_{\alpha,x,p,\Omega}=\bigg(\int_\Omega\int_\Omega\frac{|u(x')-u(x'')|^p}{|x'-x''|^{3+p\alpha}}dx'dx''\bigg)^{1/p}$$

and

$$\langle u\rangle_{\frac{\alpha}{2},t,p,(0,t)}=\bigg(\int_0^t\int_0^t\frac{|u(t')-u(t'')|^p}{|t'-t''|^{1+p\alpha/2}}dt'dt''\bigg)^{1/p},$$

where $\Omega\subset\mathbb{R}^3$.

Lemma 2.2.1. *The following imbeddings*

$$\langle D^k u\rangle_{\beta,x,p,\Omega} \le c\|u\|_{H^{l+\alpha}(\Omega)}, \tag{2.2.6}$$

where

$$\frac{3}{2}-\frac{3}{p}+k+\beta\le l+\alpha,\quad \alpha,\beta\in(0,1),\quad p\in[1,\infty],\quad k,l\in\mathbb{N}\cup\{0\},$$

$$\langle \partial_t^k u\rangle_{\frac{\beta}{2},t,p,(0,t)} \le c\|u\|_{H^{1/2+\alpha/2}(0,t)}, \tag{2.2.7}$$

where

$$\frac{1}{2}-\frac{1}{p}+\frac{\beta}{2}+k\le\frac{l}{2}+\frac{\alpha}{2}$$

and

$$\|u\|_{L_p^k(\Omega)}\le c\|u\|_{H^{l+\alpha}(\Omega)}, \tag{2.2.8}$$

where

$$\frac{3}{2}-\frac{3}{p}+k\le l+\alpha,$$

hold.

We recall the following theorems of imbedding freguently used in the paper

Lemma 2.2.2. *1. Let $l,l_1\in\mathbb{R}_+$, $p,p_1\in(1,\infty)$, $p_1\ge p$. Let $\Omega\subset\mathbb{R}^3$. If $3/p-3/p_1+l_1\le l$ then*

$$W_p^l(\Omega)\subset W_{p_1}^{l_1}(\Omega).$$

2. If $\frac{3}{p}-\frac{3}{q}+\alpha\le l$, $\alpha\in\mathbb{N}\cup\{0\}$, $q\in[1,\infty]$. Then

$$\partial_x^\alpha W_p^l(\Omega)\subset L_q(\Omega).$$

Consider anisotropic Sobolev-Slobodetskii spaces $W_{p,q}^{l,l/2}(\Omega\times(0,T))$ with the norm

$$\|u\|_{W_{p,q}^{l,l/2}(\Omega\times(0,T))}=\|u\|_{L_{p,q}(\Omega^T)}+\langle D_x^{[l]}u\rangle_{l-[l],x,p,q,\Omega^T} + \langle\partial_t^{[\frac{l}{2}u]}u\rangle_{\frac{1}{2}-[\frac{l}{2}],t,p,q,\Omega^T},$$

where $[l]$ is the integer part of $l,p,q\in[1,\infty]$,

$$\langle u\rangle_{\alpha,x,p,q,\Omega^T}=\left[\int_0^T dt\left(\int_\Omega\int_\Omega\frac{|u(x',t)-u(x'',t)|^p}{|x'-x''|^{3+p\alpha}}dx'dx''\right)^{q/p}\right]^{1/q},$$

where $\alpha\in(0,1)$ and

$$\langle u\rangle_{\alpha,t,p,q,\Omega^T}=\left[\int_\Omega dx\left(\int_0^T\int_0^T\frac{|u(x,t')-u(x,t'')|^q}{|t'-t''|^{1+q\alpha}}dt'dt''\right)^{p/q}\right]^{1/p}.$$

Lemma 2.2.3. *Let $l,l'\in\mathbb{R}_+$, $p,q,p',q'\in[1,\infty]$, $p'\ge p, q'\ge q$, $\Omega\subset\mathbb{R}^3$. Let*

$$\frac{3}{p}+\frac{2}{q}-\frac{3}{p'}-\frac{2}{q'}+l'\le l.$$

Then the imbedding

$$W_{p,q}^{l,l/2}(\Omega\times(0,T))\subset W_{p',q'}^{l',l'/2}(\Omega\times(0,T))$$

holds.

Now, we recall the trace theorems from [20].

Lemma 2.2.4. (Trace theorem) *Let $S=\partial\Omega$. Let $u\in W_{p,q}^{l,l/2}(\Omega\times(0,T))$, $l\in\mathbb{R}_+$, $(p,q)\in(1,\infty)$. Let $\varphi=u|_{S^T}$ be the trace of u on S^T. Then $\varphi\in W_{p,q}^{l-1/p,l/2-1/2q}(S\times(0,T))$ and*

$$\|\varphi\|_{W_{p,q}^{l-1/p,l/2-1/2q}(S\times(0,T))}\le c\|u\|_{W_{p,q}^{l,l/2}(\Omega\times(0,T))},$$

where c does not depend on u.

Lemma 2.2.5. (Inverse trace theorem) *Let $\varphi\in W_{p,q}^{l-1/p,l/2-1/2q}(S\times(0,T))$, $l\in\mathbb{R}_+$, $(p,q)\in(1,\infty)$. Then there exists a function $u\in W_{p,q}^{l.l/2}(\Omega\times(0,T))$ such that $u|_{S^T}=\varphi$ and there exists a constant c independent of φ such that*

$$\|u\|_{W_{p,q}^{l.l/2}(\Omega\times(0,T))}\le c\|\varphi\|_{W_{p,q}^{l-1/p,l/2-1/2q}(S\times(0,T))}.$$

Lemma 2.2.6. (Time trace theorem) *Let $u\in W_{p,q}^{l,l/2}(\Omega\times(0,T))$, $l\in\mathbb{R}_+$, $p,q\in(1,\infty)$, $t_0\in(0,T)$. Then the time trace $\varphi=u|_{t=t_0}$ belongs to $W_p^{l-2/q}(\Omega)$ and there exists a constant c independent of u such that*

$$\|\varphi\|_{W_p^{l-2/q}(\Omega)}\le c\|u\|_{W_{p,q}^{l,l/2}(\Omega\times(0,T))}.$$

Lemma 2.2.7. (Inverse time trace theorem) *Let $\varphi\in W_p^{l-2/q}(\Omega)$, $l\in\mathbb{R}_+$, $p,q\in(1,\infty)$. Then there exists a function $u\in W_{p,q}^{l,l/2}(\Omega\times(0,T))$ such that*

$$u|_{t=t_0}=\varphi,\quad t_0\in(0,T),$$

and

$$\|u\|_{W_{p,q}^{l,l/2}(\Omega\times(0,T))}\le c\|\varphi\|_{W_p^{l-2/q}(\Omega)}.$$

Finally, we introduce the energy type space

$$\|u\|_{V_2^{2+\alpha}(\Omega^t)}=\sup_t\|u(t)\|_{H^{1+\alpha}(\Omega)}+\|u\|_{H^{2+\alpha,1+\alpha/2}(\Omega^t)}.$$

2.3. The Stokes System

We consider the following Stokes problem in a bounded domain Ω in $\mathbb{R}^3$ with boundary S,

$$\begin{aligned}
&w_{,t}-\operatorname{div}\mathbb{T}(w,p)=f && \text{in } \Omega^T=\Omega\times(0,T),\\
&\operatorname{div}w=0 && \text{in } \Omega^T,\\
&w|_S=b && \text{on } S^T=S\times(0,T),\\
&w|_{t=0}=w_0 && \text{in } \Omega.
\end{aligned}\tag{2.3.1}$$

Lemma 2.3.1. (see [21])

(a) Assume that $f\in H^{\alpha,\alpha/2}(\Omega^T)$, $b\in H^{3/2+\alpha,3/4+\alpha/2}(S^T)$, $w_0\in H^{1+\alpha}(\Omega)$, $\alpha\in(0,1)$. Then there exists a solution to problem (2.3.1) such that $w\in H^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\nabla p\in H^{\alpha,\alpha/2}(\Omega^T)$ and there exists a function $c(T,S)$ such that

$$\begin{aligned}
&\|w\|_{H^{2+\alpha,1+\alpha/2}(\Omega^t)}+\|\nabla p\|_{H^{\alpha,\alpha/2}(S^t)}\\
&\quad\le c(T,S)(\|f\|_{H^{\alpha,\alpha/2}(\Omega^t)}+\|b\|_{H^{3/2+\alpha,3/4+\alpha/2}(S^t)}\\
&\qquad+\|w_0\|_{H^{1+\alpha}(\Omega)}),
\end{aligned}\tag{2.3.2}$$

where $t\le T$.

(b) Assume that $f\in L_2(\Omega^T)$, $b\in H^{3/2,3/4}(S^T)$, $w_0\in H^1(\Omega)$. Then there exists a solution to problem (2.3.1) such that $w\in H^{2,1}(\Omega^T)$, $\nabla p\in L_2(\Omega^T)$ and there exists a function $c(T,S)$ such that

$$\begin{aligned}
&\|w\|_{H^{2,1}(\Omega^t)}+\|\nabla p\|_{L_2(\Omega^t)}\le c(T,S)(\|f\|_{L_2(\Omega^t)}\\
&\quad+\|b\|_{H^{3/2,3/4}(S^t)}+\|w_0\|_{H^1(\Omega)}),
\end{aligned}\tag{2.3.3}$$

where $t\le T$.

Consider the Neumann problem to the Stokes system

$$\begin{aligned} &v_{,t} - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^T, \\ &\operatorname{div} v = g && \text{in } \Omega^T, \\ &\bar{n}\cdot\mathbb{T}(v,p) = d && \text{on } S^T, \\ &v|_{t=0} = v_0 && \text{in } \Omega \end{aligned} \tag{2.3.4}$$

To apply [21] we introduce a function φ satisfying the Dirichlet problem to the Laplace equation

$$\begin{aligned} &\Delta\varphi = g \\ &\varphi|_S = 0 \end{aligned} \tag{2.3.5}$$

Then we introduce the divergence free function

$$w = v - \nabla\varphi, \tag{2.3.6}$$

where w is a solution to the following problem

$$\begin{aligned} &w_t + \nabla\varphi_t - \nu\Delta w - 2\nu\nabla g + \nabla p = f && \text{in } \Omega^T, \\ &\operatorname{div} w = 0 && \text{in } \Omega^T, \\ &\bar{n}\cdot\mathbb{T}(w,p) = d - \bar{n}\cdot\mathbb{D}(\nabla\varphi) \equiv h && \text{on } S^T, \\ &w|_{t=0} = v_0 - \nabla\varphi(0) \equiv w_0 && \text{in } \Omega. \end{aligned} \tag{2.3.7}$$

The existence of solutions to problem (2.3.7) is described by the following lemma

Lemma 2.3.2. *Let T be a positive arbitrary finite number, $S \in H^{3/2+\alpha}$, $\alpha \in (1/2,1)$. Let $f, \nabla g, \nabla\varphi_t \in H^{\alpha,\alpha/2}(\Omega^T)$, $w_0 \in H^{1+\alpha}(\Omega)$, $h \in H^{1/2+\alpha,1/4+\alpha/2}(S^T)$. Assume the compatibility conditions*

$$\operatorname{div} w_0 = 0, \quad \bar{\tau}_\beta\cdot\mathbb{T}(w_0)\cdot\bar{n}|_S = h\cdot\bar{\tau}_\beta|_{t=0}, \quad \beta = 1,2,$$

where $\bar{n}$ and $\bar{\tau}_\beta$ are normal and tangent vectors to S.
Then there exists a unique solution to problem (2.3.7) such that $w \in H^{2+\alpha,1+\alpha/2}(\Omega^t)$, $p \in H^{\alpha,\alpha/2}(\Omega^t)$, $\nabla p \in H^{\alpha,\alpha/2}(\Omega^t)$, $p|_{S^t} \in H^{1/2+\alpha,1/4+\alpha/2}(S^t)$ and the inequality holds

$$\begin{aligned} &\|w\|_{H^{2+\alpha,1+\alpha/2}(\Omega^t)} + \|\nabla p\|_{H^{\alpha,\alpha/2}(\Omega^t)} + \|p\|_{H^{\alpha,\alpha/2}(\Omega^t)} \\ &\quad + \|p\|_{H^{1/2+\alpha,1/4+\alpha/2}(S^t)} \le c(t)(\|\nabla\varphi_t\|_{H^{\alpha,\alpha/2}(\Omega^t)} \\ &\quad + \|\nabla g\|_{H^{\alpha,\alpha/2}(\Omega^t)} + \|f\|_{H^{\alpha,\alpha/2}(\Omega^t)} + \|h\|_{H^{1/2+\alpha,1/4+\alpha/2}(S^t)} \\ &\quad + \|w_0\|_{H^{1+\alpha}(\Omega)}) \equiv H(t), \end{aligned} \tag{2.3.8}$$

where $t \le T$.

In view of (2.3.6) we can write (2.3.8) in the form

$$\begin{aligned} &\|v\|_{H^{2+\alpha,1+\alpha/2}(\Omega^t)} + \|\nabla p\|_{H^{\alpha,\alpha/2}(\Omega^t)} + \|p\|_{H^{\alpha,\alpha/2}(\Omega^t)} \\ &\quad + \|p\|_{H^{1/2+\alpha,1/4+\alpha/2}(S^t)} \le \|\nabla\varphi\|_{H^{2+\alpha,1+\alpha/2}(\Omega^t)} + H(t), \end{aligned} \tag{2.3.9}$$

where $H(t)$ is defined in (2.3.8).

Proofs of Lemmas 2.3.1 and 2.3.2 can be found in [21], where definitions of Besov spaces introduced in [22] were used.

Existence of solutions to the Stokes system can be also found in [23–25].

2.4. Transformation Between Eulerian and Lagrangian Coordinates

Let $v = v(x,t)$ be given. Lagrangian coordinates are the Cauchy data for the problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi,$$

where $x = (x_1, x_2, x_3)$ are Cartesian coordinates.

Integrating the above problem with respect to time yields

$$x = \xi + \int_0^t v(x,t')dt' = x_v(\xi,t).$$

Then we define

$$u(\xi,t) = v(x_v(\xi,t),t).$$

Then the transformation between Cartesian coordinate x and Lagrangian coordinate ξ is described by the relation

$$x = \xi + \int_0^t u(\xi,t')dt' \equiv x_u(\xi,t). \tag{2.4.1}$$

The Jacobian of this transformation is the matrix

$$A = \{x_{i,\xi_j}\} = \{a_{ij}\} = \left\{\delta_{ij} + \int_0^t u_{i,\xi_j}(\xi,t')dt'\right\}.$$

We have $A^{-1} = \{\xi_{j,x_i}\} = \{a^{ji}\}$, $\det A = \exp\int_0^t \operatorname{div}_u u dt' = 1$, $\mathcal{A} = (A^T)^{-1}$ is the matrix of cofactors. Denoting $\mathcal{A} = \{A_{ij}\}$ we have $a^{mj} = A_{jm}$. Since incompressible motions are considered, we have $\sum_k A_{ik,\xi_k}(\xi,t) = 0$ and $\nabla_u = \mathcal{A}\cdot\nabla_\xi = \nabla_\xi\cdot\mathcal{A}^T$.

Assume that S_t is given, at least locally, by the equation $F(x) = 0$ and S_0 by $F(\xi) = 0$.

Then the normal vectors to S_t and S_0 are given, respectively, by

$$\bar n_t = \frac{\nabla_x F(x)}{|\nabla_x F(x)|}, \quad \bar n_0 = \frac{\nabla_\xi F(\xi)}{|\nabla_\xi F(\xi)|}.$$

Then

$$\bar n_u = \frac{\nabla_x F(x_u(\xi,t))}{|\nabla_x F(x_u(\xi,t))|}.$$

Since $\xi_x = x_\xi^{-1}$ we obtain

$$|I - \xi_x| \le \left\|\int_0^t u_{,\xi}(\xi,t')dt'\right\|_{L_\infty(\Omega)} \phi\left(\left\|\int_0^t u_{,\xi}(\xi,t')dt'\right\|_{L_\infty(\Omega)}\right),$$

where ϕ is an increasing positive function such that $\phi(0) \neq 0$. ϕ will play a role of the generic function because it can change its form from formula to formula.

By imbedding we have

$$\begin{aligned}\Big\|\int_0^t u_{,\xi}(\xi,t')dt'\Big\|_{L_\infty(\Omega)} &\le \int_0^t \|u_{,\xi}(\cdot,t')\|_{L_\infty(\Omega)}dt'\\ &\le c\int_0^t \|u_{,\xi}(\cdot,t')\|_{H^{1+\alpha}(\Omega)}dt' \le ct^{1/2}\bigg(\int_0^t \|u(\cdot,t')\|^2_{H^{2+\alpha}(\Omega)}dt'\bigg)^{1/2}\\ &\equiv c\delta_u(t),\end{aligned} \tag{2.4.2}$$

where $\alpha > 1/2$.

In view of definition of $\delta_u(t)$ we have

$$\|I-\xi_x\|_{L_\infty(\Omega)} \le \delta_u(t)\phi(\delta_u(t)). \tag{2.4.3}$$

Continuing, we have

$$|\xi_x| \le |x_\xi^{-1}| \le \phi(\delta_u(t)), \tag{2.4.4}$$

$$|\xi_{xx}| \le |(x_\xi^{-1})_{,\xi}\xi_x| \le \bigg|\int_0^t u_{\xi\xi}(\xi,t')dt'\bigg|\phi(\delta_u(t)). \tag{2.4.5}$$

2.5. Problem for the Magnetic Field

In this paper we restrict our considerations to transmission condition described in the case 2 of Remark 1.3. Then problem (1.20) takes the following form

$$\begin{aligned}
&\mu_1\overset{1}{H}_{,t}+\frac{1}{\sigma_1}\text{rot}^2\overset{1}{H}=\frac{\mu_1}{\sigma_1}\text{rot}\,(\overset{1}{v}\times\overset{1}{H}) && \text{in } \bigcup_t \overset{1}{\Omega}_t,\\
&\text{div}\,\overset{1}{H}=0 && \text{in } \bigcup_t \overset{1}{\Omega}_t,\\
&\mu_2\overset{2}{H}_{,t}+\frac{1}{\sigma_2}\text{rot}^2\overset{2}{H}=0 && \text{in } \bigcup_t \overset{2}{\Omega}_t,\\
&\text{div}\,\overset{2}{H}=0 && \text{in } \bigcup_t \overset{2}{\Omega}_t,\\
&\overset{2}{H}\cdot\bar\tau_\alpha=H_{*\alpha},\quad \text{div}\,\overset{2}{H}|_B=0,\quad \tau_\alpha,\quad \alpha=1,2\\
&\quad\text{is a tangent vector to } B && \text{on } B^t\\
&\bigg(\frac{1}{\sigma_1}\text{rot}\,\overset{1}{H}-\mu_1 v\times\overset{1}{H}\bigg)\cdot\bar\tau_\alpha=\frac{1}{\sigma_2}\text{rot}\,\overset{2}{H}\cdot\bar\tau_\alpha,\quad \alpha=1,2 && \text{on } \bigcup_t S_t,\\
&\overset{1}{H}_\beta=\overset{2}{H}_\beta,\quad \beta=1,2,3, && \text{on } \bigcup_t S_t,\\
&\overset{1}{H}|_{t=0}=\overset{1}{H}(0),\quad \text{div}\,\overset{1}{H}(0)=0 && \text{in } \overset{1}{\Omega}_0,\\
&\overset{2}{H}|_{t=0}=\overset{2}{H}(0),\quad \text{div}\,\overset{2}{H}(0)=0 && \text{in } \overset{2}{\Omega}_0,\\
&\overset{i}{\Omega}_t|_{t=0}=\overset{i}{\Omega}_0,\quad i=1,2,\quad S_t|_{t=0}=S_0.
\end{aligned} \tag{2.5.1}$$

Recall that problem (2.5.1) is formulated in Cartesian coordinates.

To examine problem (2.5.1) we have to consider first the following problem with constant coefficients formulated in Lagrangian coordinates

$$
\begin{aligned}
&\mu_1\overset{1}{\bar H}_{,t}+\frac{1}{\sigma_1}\mathrm{rot}\,_x^2\overset{1}{\bar H}=\overset{1}{M}, && \text{in } \overset{1}{\Omega}{}_0^T,\\
&\mathrm{div}\,_x\overset{1}{\bar H}=\overset{1}{N}, && \text{in } \overset{1}{\Omega}{}_0^T,\\
&\mu_2\overset{2}{\bar H}_{,t}+\frac{1}{\sigma_2}\mathrm{rot}\,_x^2\overset{2}{\bar H}=\overset{2}{M}, && \text{in } \overset{2}{\Omega}{}_0^T,\\
&\mathrm{div}\,_x\overset{2}{\bar H}=\overset{2}{N}, && \text{in } \overset{2}{\Omega}{}_0^T,\\
&\left(\frac{1}{\sigma_1}\mathrm{rot}\,_x\overset{1}{\bar H}-\frac{1}{\sigma_2}\mathrm{rot}\,_x\overset{2}{\bar H}\right)\cdot\bar\tau_\alpha=K_\alpha,\quad \alpha=1,2 && \text{on } S_0^T,\\
&(\overset{1}{\bar H}-\overset{2}{\bar H})\cdot\bar n\times\bar\tau_\alpha=L_\alpha,\quad \alpha=1,2 && \text{on } S_0^T,\\
&\overset{2}{\bar H}\cdot\bar\tau_\alpha|_B=H_{*\alpha},\quad \alpha=1,2,\quad \mathrm{div}\,_x\overset{2}{\bar H}|_B=0 && \text{on } B^T,\\
&\overset{i}{\bar H}|_{t=0}=\overset{i}{H}(0),\quad i=1,2 && \text{in } \overset{i}{\Omega}_0.
\end{aligned}
\tag{2.5.2}
$$

To prove the existence of solutions to problem (2.5.2) we exploit the technique of regularizer described in [26, Ch. 4, Sect. 7] and also utilized in [2, Sect. 5 and Sect. 10].

Introduce function $\overset{i}{\Phi}$, $i=1,2$, as a solution to the problem

$$
\begin{aligned}
&\Delta\overset{i}{\Phi}=\overset{i}{N}\quad \text{in } \overset{i}{\Omega}_0,\\
&\overset{i}{\Phi}|_{S_0}=0,\quad \delta_{2i}\overset{i}{\Phi}|_B=0,\quad i=1,2,
\end{aligned}
\tag{2.5.3}
$$

where δ_{2i} is the Kronecker delta.

Lemma 2.5.1. *Let $\alpha\in(0,1)$ and let $T>0$ be given. Assume that $\overset{i}{M}\in H^{\alpha,\alpha/2}(\overset{i}{\Omega}{}_0^t)$, $\nabla\overset{i}{\bar\Phi}\in H^{2+\alpha,1+\alpha/2}(\overset{i}{\Omega}{}_0^t)$, $\nabla\overset{i}{N}\in H^{\alpha,\alpha/2}(\overset{i}{\Omega}{}_0^t)$, $\overset{i}{H}(0)\in H^{1+\alpha}(\overset{i}{\Omega}_0)$, $i=1,2$. Assume that $H_{*j}\in H^{2+\alpha-1/2,1+\alpha/2-1/4}(B^t)$, $B\in H^{1/2+\alpha,1/4+\alpha/2}(B^t)$, $K_j\in H^{1/2+\alpha,1/4+\alpha/2}(S_0^t)$, $L_j\in H^{3/2+\alpha,3/4+\alpha/2}(S_0^t)$, $j=1,2$, $t\le T$. Then there exists a unique solution to problem (2.5.2) such that $\overset{i}{\bar H}\in V_2^{2+\alpha}(\overset{i}{\Omega}{}_0^t)$, $i=1,2$, and*

$$
\begin{aligned}
\sum_{i=1}^2\|\overset{i}{\bar H}\|_{V_2^{2+\alpha}(\overset{i}{\Omega}{}_0^t)}&\le c\sum_{i=1}^2(\|\overset{i}{M}\|_{H^{\alpha,\alpha/2}(\overset{i}{\Omega}{}_0^t)}+\|\nabla\overset{i}{\Phi}\|_{H^{2+\alpha,1+\alpha/2}(\overset{i}{\Omega}{}_0^t)}\\
&\quad+\|\nabla\overset{i}{N}\|_{H^{\alpha,\alpha/2}(\overset{i}{\Omega}{}_0^t)}+\|\overset{i}{H}(0)\|_{H^{1+\alpha}(\overset{i}{\Omega}_0)})\\
&\quad+c\sum_{j=1}^2\|H_{*j}\|_{H^{3/2+\alpha,3/4+\alpha/2}(B^t)}+\|B\|_{H^{1/2+\alpha,1/4+\alpha/2}(B^t)}\\
&\quad+c\sum_{j=1}^2(\|K_j\|_{H^{1/2+\alpha,1/4+\alpha/2}(S_0^t)}+\|L_j\|_{H^{3/2+\alpha,3/4+\alpha/2}(S_0^t)}),
\end{aligned}
\tag{2.5.4}
$$

where $t\le T$.

Proof. The proof is divided into the following steps:

1. First we introduce a partition of unity connected with the four kinds of subdomains:

   1_1 a neighborhood of an interior point of $\overset{1}{\Omega}_0$ located in a positive distance from S_0,

1_2 a neighborhood of an interior point of $\overset{2}{\Omega}_0$ located in a positive distance from S_0 and B,
1_3 a neighborhood of a point of S_0,
1_4 a neighborhood of a point of B.
There is constructed a partition of unity with supports corresponding to mentioned above subdomains.

2. Using the partition of unity problem (2.5.2) is localized to the above neighborhoods. The local problems in subdomains 1_1 and 1_2 can be easily solved.
The local problem in subdomain 1_3 is solved in Sect. 7 and in 1_4 in Sect. 8.
3. To solve problem (2.5.2) we collect results of all above defined local problems by using the key idea of resularizer (see [26, Ch. 4]). Using the proofs of existence of solutions to problem (2.5.2) in Sects. 5 and 10 from [2] we conclude the proof of the lemma.

□

The result similar to Lemma 2.5.1 is also proved in [14].

2.6. Spaces Defined by the Fourier–Laplace Transforms

In this subsection we follow [25]. Assume that $u \in H^{\alpha,\alpha/2}(\mathbb{R}^3 \times \mathbb{R}_+)$. Assume that u can be extended by zero for $t < 0$ and the extended $u \in H^{\alpha,\alpha/2}(\mathbb{R}^3 \times \mathbb{R})$.

We define the Fourier-Laplace transform for functions vanishing sufficiently fast at infinity by

$$\tilde{u}(\xi, s) = \int_0^\infty e^{-st} ds \int_{\mathbb{R}^3} u(x,t) e^{-ix\cdot\xi} dx,$$

where Re $s > 0$.

For any $u \in H^{\alpha,\alpha/2}(\mathbb{R}^3 \times \mathbb{R})$ the Fourier-Laplace transform is a holomorphic function for Re $s > \gamma$, $\gamma > 0$. We introduce the norm

$$\|u\|_{\tilde{H}_\gamma^{\alpha,\alpha/2}(\mathbb{R}^3\times\mathbb{R})} = \int_{\mathbb{R}^3} d\xi \int_{-\infty}^{\infty} |\tilde{u}(\xi, \gamma + i\xi_0)|^2 (|s| + \xi^2)^\alpha d\xi_0, \tag{2.6.1}$$

where $s = \gamma + i\xi_0$.

Lemma 2.6.1 (see Lemma 2.1 from [25]). *There exist constants c_1, c_2 such that*

$$\begin{aligned} c_1\|u\|_{\tilde{H}_\gamma^{\alpha,\alpha/2}(\mathbb{R}^3\times\mathbb{R})} &\le \|u\|_{H^{\alpha,\alpha/2}(\mathbb{R}^3\times\mathbb{R})} \\ &\le c_2\|u\|_{\tilde{H}_\gamma^{\alpha,\alpha/2}(\mathbb{R}^3\times\mathbb{R})}. \end{aligned}$$

Finally, we recall

Lemma 2.6.2 (see Lemma 3.1 from [25]). *Let $e(x_3) = e^{-\tau x_3}$, $\tau = \sigma s + \xi^2$, $s = \gamma + i\xi_0$, $\gamma > 0$. Then*

$$\int_0^\infty \left| \frac{d^j}{dx_3^j} e(x_3) \right|^2 dx_3 \le \frac{1}{\sqrt{2}} |\tau|^{2j-1},$$

$$\int_0^\infty \int_0^\infty \left| \frac{d^j e(x_3+z)}{dx_3^j} - \frac{d^j e(x_3)}{dx_3^j} \right|^2 \frac{dx_3 dz}{z^{1+2\varkappa}} \le c|\tau|^{2(j+\varkappa)-1}.$$

3. Method of Successive Approximations

Let $v = v(x,t)$ be given, where $x \in \overset{1}{\Omega}_t$.

Definition 3.1. The Lagrangian coordinates in $\overset{1}{\Omega}_0$ are initial data to the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi \in \overset{1}{\Omega}_0 \tag{3.1}$$

Hence the domain $\overset{1}{\Omega}_t$ is defined by

$$\overset{1}{\Omega}_t = \left\{ x \in \mathbb{R}^3 \colon x = x(\xi,t) = \xi + \int_0^t \bar{v}(\xi,t')dt', \xi \in \overset{1}{\Omega}_0 \right\},$$

where $\bar{v}(\xi,t) = v(x(\xi,t),t)$.

In free boundary problems in hydrodynamics the free boundary is built from the same fluid particles as at time $t = 0$ because $v|_{S_t}$ is tangent to S_t. Then

$$S_t = \{x \in \mathbb{R}^3 \colon x = x(\xi,t), \xi \in S_0\}.$$

To formulate problem (2.5.1) in Lagrangian coordinates we have to introduce them in $\overset{2}{\Omega}_0$. Since there is no velocity in $\overset{2}{\Omega}_t$, we have to introduce it artificially.

Definition 3.2. Let $\overset{1}{v} = v$ in $\overset{1}{\Omega}_t$ and construct $\overset{2}{v}$ in $\overset{2}{\Omega}_t$ as a solution to the nonstationary Stokes problem

$$\begin{aligned}
&\overset{2}{v}_{,t} - \operatorname{div} \mathbb{T}(\overset{2}{v}, q) = 0 && \text{in } \overset{2}{\Omega}_t, \\
&\operatorname{div} \overset{2}{v} = 0 && \text{in } \overset{2}{\Omega}_t, \\
&\overset{2}{v}|_{S_t} = \overset{1}{v}|_{S_t}, \quad \overset{2}{v}|_B = 0, && \\
&\overset{2}{v}|_{t=0} = \overset{2}{v}(0) && \text{in } \overset{2}{\Omega}_0,
\end{aligned} \tag{3.2}$$

where q plays the role of pressure but it is not important for any estimate for $\overset{2}{v}$. It is introduced to have $\overset{2}{v}$ divergence free.

Finally, we construct $\overset{2}{v}(0)$ as a solution to the stationary Stokes system

$$\begin{aligned}
&-\Delta \overset{2}{v}(0) + \nabla q(0) = 0 && \text{in } \overset{2}{\Omega}_0, \\
&\operatorname{div} \overset{2}{v}(0) = 0 && \text{in } \overset{2}{\Omega}_0, \\
&\overset{2}{v}(0)|_{S_0} = \overset{1}{v}(0)|_{S_0}, \quad \overset{2}{v}(0)|_B = 0. &&
\end{aligned} \tag{3.3}$$

Having $\overset{2}{v}$ constructed by problems (3.2) and (3.3), we can introduce Lagrangian coordinates $\overset{1}{\xi}$, $\overset{2}{\xi}$ by the Cauchy data to the problems

$$\frac{d\overset{i}{x}}{dt} = \overset{i}{v}(x,t), \quad \overset{i}{x}|_{t=0} = \overset{i}{\xi} \in \overset{i}{\Omega}_0, \quad i = 1,2. \tag{3.4}$$

Then

$$\begin{aligned}\overset{i}{\Omega}_t &= \bigg\{\overset{i}{x}\in\mathbb{R}^3\colon \overset{i}{x}=\overset{i}{x}(\overset{i}{\xi},t)=\overset{i}{\xi}+\int_0^t \overset{i}{v}(\overset{i}{x}(\overset{i}{\xi},t'),t')dt' \\ &= \overset{i}{\xi}+\int_0^t \overset{i}{\bar v}(\overset{i}{\xi},t')dt', \overset{i}{\xi}\in\overset{i}{\Omega}_0\bigg\}, \quad i=1,2,\end{aligned} \tag{3.5}$$

where $\overset{i}{\bar v}(\overset{i}{\xi},t)=\overset{i}{v}(\overset{i}{x}(\overset{i}{\xi},t),t)$, $\overset{i}{\xi}\in\overset{i}{\Omega}_0$, $i=1,2$.

Expressing problems (1.18) and (2.5.1) in Lagrangian coordinates yields

$$\begin{aligned}&\bar v_{,t}-\operatorname{div}_{\bar v}\mathbb{T}_{\bar v}(\bar v,\bar p)=\bar f+\mu_1\operatorname{div}_{\bar v}\mathbb{T}(\overset{1}{\bar H}) && \text{in } \overset{1}{\Omega}{}_0^T,\\ &\operatorname{div}_{\bar v}\bar v=0 && \text{in } \overset{1}{\Omega}{}_0^T,\\ &\bar n_{\bar v}\cdot\mathbb{T}_{\bar v}(\bar v,\bar p)=-\mu_1\bar n_{\bar v}\cdot\mathbb{T}(\overset{1}{\bar H}) && \text{on } S_0^T,\\ &\bar v|_{t=0}=v_0 && \text{in } \overset{1}{\Omega}_0\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\mu_1\overset{1}{\bar H}_{,t}+\frac{1}{\sigma_1}\operatorname{rot}^2_{\overset{1}{\bar v}}\overset{1}{\bar H}=\mu_1\operatorname{rot}_{\overset{1}{\bar v}}(\overset{1}{\bar v}\times\overset{1}{\bar H})+\mu_1\overset{1}{\bar v}\cdot\nabla_{\overset{1}{\bar v}}\overset{1}{\bar H} && \text{in } \overset{1}{\Omega}{}_0^T,\\ &\operatorname{div}_{\overset{1}{\bar v}}\overset{1}{\bar H}=0 && \text{in } \overset{1}{\Omega}{}_0^T,\\ &\mu_2\overset{2}{\bar H}_{,t}+\frac{1}{\sigma_1}\operatorname{rot}^2_{\overset{2}{\bar v}}\overset{2}{\bar H}=\mu_2\overset{2}{\bar v}\cdot\nabla_{\overset{2}{\bar v}}\overset{2}{\bar H} && \text{in } \overset{1}{\Omega}{}_0^T,\\ &\operatorname{div}_{\overset{2}{\bar v}}\overset{2}{\bar H}=0 && \text{in } \overset{1}{\Omega}{}_0^T,\\ &\bigg(\frac{1}{\sigma_1}\operatorname{rot}_{\overset{1}{\bar v}}\overset{1}{\bar H}-\frac{1}{\sigma_2}\operatorname{rot}_{\overset{2}{\bar v}}\overset{2}{\bar H}\bigg)\bar\tau_{\bar v\alpha}=\mu_1\bar v\times\overset{1}{\bar H}\cdot\bar\tau_{\bar v\alpha},\quad \alpha=1,2 && \text{on } S_0^T,\\ &(\overset{1}{\bar H}-\overset{2}{\bar H})\cdot\bar n_{\bar v}\times\bar\tau_{\bar v\alpha}=0,\quad \alpha=1,2 && \text{on } S_0^T,\\ &\overset{2}{\bar H}\cdot\bar\tau_\alpha|_B=H_{*\alpha},\quad \alpha=1,2,\quad \operatorname{div}_{\overset{2}{\bar v}}\overset{2}{\bar H}|_B=0 && \text{on } B^T,\\ &\overset{i}{\bar H}|_{t=0}=\overset{i}{H}(0),\quad i=1,2 && \text{in } \Omega_0,\end{aligned} \tag{3.7}$$

where $\bar v=\overset{1}{\bar v}=\overset{2}{\bar v}$ on S_0 and $\nabla_{\bar v}=\frac{\partial\xi_k}{\partial x}|_{x=x(\xi,t)}\partial_{\xi_k}$.

Moreover, any operator with subscript $\bar v$ means that it contains the transformed gradient $\nabla_{\bar v}$ and any operator with subscript ξ contains derivatives with respect to ξ.

To prove existence of local solutions to problem (3.6), (3.7) we apply the following method of successive approximations

$$
\begin{aligned}
&\bar v_{n+1,t}-\operatorname{div}_\xi\mathbb{T}_\xi(\bar v_{n+1},\bar p_{n+1})=-(\operatorname{div}_\xi\mathbb{T}_\xi(\bar v_n,\bar p_n)\\
&\quad-\operatorname{div}_{\bar v_n}\mathbb{T}_{\bar v_n}(\bar v_n,\bar p_n))+\mu_1\operatorname{div}_{\bar v_n}\mathbb{T}(\overset{1}{\bar H}_n)+\bar f\equiv F_n &&\text{in }\ \overset{1}{\Omega}{}_0^T,\\
&\operatorname{div}_\xi\bar v_{n+1}=\operatorname{div}_\xi\bar v_n-\operatorname{div}_{\bar v_n}\bar v_n\equiv g_n &&\text{in }\ \overset{1}{\Omega}{}_0^T,\\
&\bar n_\xi\cdot\mathbb{T}_\xi(\bar v_{n+1},\bar p_{n+1})=(\bar n_\xi\cdot\mathbb{T}_\xi(\bar v_n,\bar p_n)-\bar n_{\bar v_n}\cdot\mathbb{T}_{\bar v_n}(\bar v_n,\bar p_n))\\
&\quad-\mu_1\bar n_{\bar v_n}\cdot\mathbb{T}(\overset{1}{\bar H}_n)\equiv G_n &&\text{on }\ S_0^T,\\
&\bar v_{n+1}|_{t=0}=v(0) &&\text{in }\ \overset{1}{\Omega}_0,
\end{aligned}
\tag{3.8}
$$

where $\bar v_n=\overset{1}{\bar v}_n$, and

$$
\begin{aligned}
&\mu_1\overset{1}{\bar H}_{n+1,t}+\frac{1}{\sigma_1}\operatorname{rot}^2_\xi\overset{1}{\bar H}_{n+1}=\frac{1}{\sigma_1}(\operatorname{rot}^2_\xi\overset{1}{\bar H}_n-\operatorname{rot}^2_{\overset{1}{\bar v}_n}\overset{1}{\bar H}_n)\\
&\quad+\mu_1\operatorname{rot}_{\overset{1}{\bar v}_n}(\overset{1}{\bar v}_n\times\overset{1}{\bar H}_n)+\mu_1\overset{1}{\bar v}_n\cdot\nabla_{\overset{1}{\bar v}_n}\overset{1}{\bar H}_n\equiv\overset{1}{M}_n &&\text{in }\ \overset{1}{\Omega}{}_0^T,\\
&\operatorname{div}_\xi\overset{1}{\bar H}_{n+1}=\operatorname{div}_\xi\overset{1}{\bar H}_n-\operatorname{div}_{\overset{1}{\bar v}_n}\overset{1}{\bar H}_n\equiv\overset{1}{N}_n &&\text{in }\ \overset{1}{\Omega}{}_0^T,\\
&\mu_2\overset{2}{\bar H}_{n+1,t}+\frac{1}{\sigma_2}\operatorname{rot}^2_\xi\overset{2}{\bar H}_{n+1}=\frac{1}{\sigma_2}(\operatorname{rot}^2_\xi\overset{2}{\bar H}_n-\operatorname{rot}_{\overset{2}{\bar v}_n}\overset{2}{\bar H}_n)\\
&\quad+\mu_2\overset{2}{\bar v}_n\cdot\nabla_{\overset{2}{\bar v}_n}\overset{2}{\bar H}_n\equiv\overset{2}{M}_n &&\text{in }\ \overset{2}{\Omega}{}_0^T,\\
&\operatorname{div}_\xi\overset{2}{\bar H}_{n+1}=\operatorname{div}_\xi\overset{2}{\bar H}_n-\operatorname{div}_{\overset{2}{\bar v}_n}\overset{2}{\bar H}_n\equiv\overset{2}{N}_n &&\text{in }\ \overset{2}{\Omega}{}_0^T,\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi\overset{1}{\bar H}_{n+1}-\frac{1}{\sigma_2}\operatorname{rot}_\xi\overset{2}{\bar H}_{n+1}\right)\cdot\bar\tau_\alpha\\
&\quad=\frac{1}{\sigma_1}(\operatorname{rot}_\xi\overset{1}{\bar H}_n-\operatorname{rot}_{\bar v_n}\overset{1}{\bar H}_n)\cdot\bar\tau_\alpha-\frac{1}{\sigma_2}(\operatorname{rot}_\xi\overset{2}{\bar H}_n-\operatorname{rot}_{\bar v_n}\overset{2}{\bar H}_n)\cdot\bar\tau_\alpha\\
&\qquad+\frac{1}{\sigma_1}\operatorname{rot}_{\bar v_n}\overset{1}{\bar H}_n\cdot(\bar\tau_\alpha-\bar\tau_{\bar v_n\alpha})-\frac{1}{\sigma_2}\operatorname{rot}_{\bar v_n}\overset{2}{\bar H}_n\cdot(\bar\tau_\alpha-\bar\tau_{\bar v_n\alpha})\\
&\qquad+\mu_1\bar v_n\times\overset{1}{\bar H}_n\cdot\bar\tau_{\bar v_n\alpha}\equiv K_{\alpha n},\quad\alpha=1,2 &&\text{on }\ S_0^T,\\
&(\overset{1}{\bar H}_{n+1}-\overset{2}{\bar H}_{n+1})\cdot\bar n\times\bar\tau_\alpha\\
&\quad=(\overset{1}{\bar H}_n-\overset{2}{\bar H}_n)\cdot(\bar n\times\bar\tau_\alpha-\bar n_{\bar v_n}\times\bar\tau_{\bar v_n\alpha})\equiv L_{\alpha n}\quad\alpha=1,2 &&\text{on }\ S_0^T,\\
&\overset{2}{\bar H}_{n+1}\cdot\bar\tau'_\alpha|_B=H_{*\alpha}\quad\alpha=1,2,\quad\operatorname{div}_{\overset{2}{\bar v}_n}\overset{2}{\bar H}_{n+1}|_B=0\\
&\overset{i}{\bar H}_{n+1}|_{t=0}=\overset{i}{H}(0),\quad i=1,2.
\end{aligned}
\tag{3.9}
$$

In problems (3.8) and (3.9) $\overset{1}{\bar v}_n$, $\overset{2}{\bar v}_n$, $\overset{1}{\bar H}_n$, $\overset{2}{\bar H}_n$ are treated as given.

Moreover, $\bar\tau_\alpha$, $\alpha=1,2$, are tangent to S_0, $\bar n$ is normal and $\bar\tau'_\alpha$, $\alpha=1,2$, are tangent to B.

4. Estimates for Solutions to Problem (3.8)

Let φ_n be a solution to the problem

$$\begin{aligned} \Delta\varphi_n &= g_n \quad \text{in } \overset{1}{\Omega}_0, \\ \varphi_n|_{S_0} &= 0. \end{aligned} \tag{4.1}$$

There exists the Green function to problem (4.1) such that

$$\varphi_n(x,t) = \int\limits_{\overset{1}{\Omega}_0} G(x,y) g_n(y,t) dy. \tag{4.2}$$

Lemma 2.3.2 yields

Lemma 4.1. *Assume that* $F_n \in H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)$, $\nabla g_n \in H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)$, $G_n \in H^{1/2+\alpha,1/4+\alpha/2} S_0^t)$, $v(0) \in H^{1+\alpha}(\overset{1}{\Omega}_0)$, $\nabla\varphi_n \in H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)$, $\nabla\varphi_t \in H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)$. *Then there exists a unique solution to problem (3.8) such that* $\bar v_{n+1} \in V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)$, $\nabla\bar p_{n+1}, \bar p_{n+1} \in H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)$, $\bar p_{n+1} \in H^{1/2+\alpha,1/4+\alpha/2}(S_0^t)$ *and the estimate holds*

$$\begin{aligned} &\|\bar v_{n+1}\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)} + \|\nabla\bar p_{n+1}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} + \|\bar p_{n+1}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} \\ &\quad + \|\bar p_{n+1}\|_{H^{1/2+\alpha,1/4+\alpha/2}(S_0^t)} \le c(\|F_n\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} \\ &\quad + \|\nabla g_n\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} + \|\nabla\varphi_n\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)} + \|\nabla\varphi_{nt}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} \\ &\quad + \|G_n\|_{H^{1/2+\alpha,1/4+\alpha/2}(S_0^t)} + \|v(0)\|_{H^{1+\alpha}(\overset{1}{\Omega}_0)}). \end{aligned} \tag{4.3}$$

The expressions on the r.h.s. of (4.3) depend on $\bar v_n$, $\bar p_n$ and $\overset{1}{\bar H}_n$. Therefore we estimate them using the dependence.

Lemma 4.2. *Assume that* $\bar v_n, \overset{1}{\bar H}_n \in H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)$, $\bar p_{n,\xi} \in H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)$.
Assume that $\alpha > 5/8$, $a > 0$.
Then

$$\begin{aligned} \|F_n\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} &\le t^a \phi(\delta_{\bar v_n}(t))(\|\bar v_n\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)} + 1)(\|\bar v_n\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)} \\ &\quad + \|\bar p_n\|_{L_2(0,t;H^1(\overset{1}{\Omega}_0))} + \|\bar p_{n,\xi}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} + \|\overset{1}{\bar H}_n\|^2_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}), \end{aligned} \tag{4.4}$$

where $\delta_{\bar v_n}(t)$ *is introduced in* (2.4.2).

Proof. We have

$$\begin{aligned} F_n &= -(\operatorname{div}_\xi \mathbb{T}(\bar v_n, \bar p_n) - \operatorname{div}_{\bar v_n} \mathbb{T}_{\bar v_n}(\bar v_n, \bar p_n)) \\ &\quad + \mu_1 \operatorname{div}_{\bar v_n} \mathbb{T}(\overset{1}{\bar H}_n) + \bar f \equiv \overset{1}{F}_n + \overset{2}{F}_n + \bar f. \end{aligned} \tag{4.5}$$

In this proof we drop the index n and introduce the simplified notation

$$v = \bar v_n, \quad p = \bar p_n, \quad H = \overset{1}{\bar H}_n. \tag{4.6}$$

First we consider

$$\begin{aligned} \|\overset{1}{F}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} &\le c\|(I - \xi_x^2) v_{\xi\xi} + \xi_x \xi_{xx} x_\xi v_\xi\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} \\ &\quad + c\|(I - \xi_x) p_\xi\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)}. \end{aligned} \tag{4.7}$$

The norm of space $H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)$ can be expressed in the form

$$\|\overset{1}{F}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} = \|\overset{1}{F}\|_{L_2(\overset{1}{\Omega}{}_0^t)} + \langle\overset{1}{F}\rangle_{\alpha,x,2,\overset{1}{\Omega}{}_0^t} + \langle\overset{1}{F}\rangle_{\alpha/2,t,2,\overset{1}{\Omega}{}_0^t}. \tag{4.8}$$

Using Sect. 2.4 we have

$$\begin{aligned}
\|\overset{1}{F}\|_{L_2(\overset{1}{\Omega}{}_0^t)} &\le c\|(I-\xi_x^2)v_{\xi\xi}\|_{L_2(\overset{1}{\Omega}{}_0^t)} + c\|\xi_x\xi_{xx}x_\xi v_\xi\|_{L_2(\overset{1}{\Omega}{}_0^t)} \\
&\quad + c\|(I-\xi_x)p_\xi\|_{L_2(\overset{1}{\Omega}{}_0^t)} \le c\phi(\delta_v(t))\bigg\|\int_0^t v_\xi(\tau)d\tau v_{\xi\xi}\bigg\|_{L_2(\overset{1}{\Omega}{}_0^t)} \\
&\quad + c\phi(\delta_v(t))\bigg\|\int_0^t v_{\xi\xi}d\tau v_\xi\bigg\|_{L_2(\overset{1}{\Omega}{}_0^t)} + c\phi(\delta_\sigma(t))\bigg\|\int_0^t v_\xi d\tau p_\xi\bigg\|_{L_2(\overset{1}{\Omega}{}_0^t)} \\
&\le ct^{1/2}\phi(\delta_v(t))\bigg[\bigg(\int_0^t \|v_\xi(\cdot,\tau)\|^2_{L_\infty(\overset{1}{\Omega}_0)}d\tau\bigg)^{1/2}\|v_{\xi\xi}\|_{L_2(\overset{1}{\Omega}{}_0^t)} \\
&\quad + \bigg(\int_0^t \|v_\xi(\cdot,\tau)\|^2_{L_4(\overset{1}{\Omega}_0)}d\tau\bigg)^{1/2}\|v_\xi\|_{L_2(0,t;L_4(\overset{1}{\Omega}_0))} \\
&\quad + \bigg(\int_0^t \|v_\xi(\cdot,\tau)\|^2_{L_\infty(\overset{1}{\Omega}_0)}d\tau\bigg)^{1/2}\|p_\xi\|_{L_2(\overset{1}{\Omega}{}_0^t)}\bigg] \\
&\le ct^{1/2}\phi(\delta_v(t))(\|v\|_{L_2(0,t;H^2(\overset{1}{\Omega}_0))} + \|p\|_{L_2(0,t;H^1(\overset{1}{\Omega}_0))}).
\end{aligned} \tag{4.9}$$

The second term on the r.h.s. of (4.8) is bounded by

$$\begin{aligned}
\langle\overset{1}{F}\rangle_{\alpha,x,2,\overset{1}{\Omega}{}_0^t} &\le \langle(I-\xi_x^2)v_{\xi\xi}\rangle_{\alpha,x,2,\overset{1}{\Omega}{}_0^t} + \langle\xi_x\xi_{xx}x_\xi v_\xi\rangle_{\alpha,x,2,\overset{1}{\Omega}{}_0^t} \\
&\quad + \langle(I-\xi_x)p_\xi\rangle_{\alpha,x,2,\overset{1}{\Omega}{}_0^t} \equiv L_1^1 + L_1^2 + L_1^3.
\end{aligned} \tag{4.10}$$

First, we examine

$$\begin{aligned}
L_1^1 &= \bigg(\int_0^t\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0}\bigg(\frac{|(I-\xi_x^2(\xi',t'))v_{\xi\xi}(\xi',t')}{|\xi'-\xi''|^{3+2\alpha}} \\
&\quad - \frac{(I-\xi_x^2(\xi'',t'))v_{\xi\xi}(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}}\bigg)d\xi'd\xi''dt'\bigg)^{1/2} \\
&\le \bigg(\int_0^t\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0}\frac{|\xi_x^2(\xi',t')-\xi_x^2(\xi'',t')|^2|v_{\xi\xi}(\xi',t')|^2}{|\xi'-\xi''|^{3+2\alpha}}d\xi'd\xi''dt'\bigg)^{1/2} \\
&\quad + \bigg(\int_0^t\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0}\frac{|I-\xi_x^2(\xi'',t')|^2|v_{\xi\xi}(\xi',t')-v_{\xi\xi}(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}}d\xi'd\xi''dt'\bigg)^{1/2} \\
&\equiv J_1 + J_2.
\end{aligned}$$

In view of properties of matrix ξ_x, we have

$$J_1 \leq \phi(\delta_v(t))\bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|\int_0^{t'}(v_\xi(\xi',\tau)-v_\xi(\xi'',\tau))d\tau|^2}{|\xi'-\xi''|^{3/2+2\alpha}} \cdot$$

$$\cdot \frac{|v_{\xi\xi}(\xi',t')|^2}{|\xi'-\xi''|^{3/2}} d\xi' d\xi'' dt'\bigg)^{1/2} \equiv J_1^1.$$

The Hölder inequality implies

$$J_1^1 \leq \phi(\delta_v(t))\bigg(\int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|\int_0^t(v_\xi(\xi',\tau)-v_\xi(\xi'',\tau))d\tau|^{2p}}{|\xi'-\xi''|^{3+2p\alpha'}} d\xi' d\xi''\bigg)^{1/2p} \cdot$$

$$\cdot \bigg[\int_0^t \bigg(\int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|v_{\xi\xi}(\xi',t')|^{2p'}}{|\xi'-\xi''|^{(3/2)p'}} d\xi' d\xi''\bigg)^{1/p'} dt'\bigg]^{1/2} \equiv J_1^{11} J_1^{12},$$

where $1/p + 1/p' = 1$, $p' < 2$, $\alpha' = \alpha + \frac{1}{2p}\big(\frac{3}{2}p - 3\big)$.

By the Minkowski and Hölder inequalities we get

$$J_1^{11} \leq t^{1-1/2p}\phi(\delta_v(t))\bigg[\int_0^t \bigg(\int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|v_\xi(\xi',t')-v_\xi(\xi'',t')|^{2p}}{|\xi'-\xi''|^{3+2p\alpha'}} d\xi' d\xi''\bigg)^{1/p} dt'\bigg]^{1/2}$$

$$\leq ct^{1-1/2p}\phi(\delta_v(t))\bigg(\int_0^t \|v\|^2_{H^{2+\alpha}(\overset{1}{\Omega}_0)} dt'\bigg)^{1/2},$$

where the above imbedding holds under the condition

$$\frac{3}{2} - \frac{3}{2p} + \alpha' \leq 1 + \alpha \quad \text{so} \quad \frac{3}{2} + \frac{3}{4} - \frac{3}{p} \leq 1 \tag{4.11}$$

However $p > 2$ the last restriction can hold for p close to 2.

Since $p' < 2$ we obtain

$$J_1^{12} \leq c\bigg[\int_0^t \bigg(\oint_{\overset{1}{\Omega}_0} |v_{\xi\xi}(\xi',t')|^{2p'} d\xi'\bigg)^{1/p'}\bigg]^{1/2} \leq c\|v\|_{L_2(0,t,H^{2+\alpha}(\overset{1}{\Omega}_0))}$$

where the last inequality holds for

$$\frac{3}{2} - \frac{3}{2p'} \leq \alpha \tag{4.12}$$

Conditions (4.11) and (4.12) imply

$$\frac{5}{8} \leq \alpha. \tag{4.13}$$

Summarizing,

$$L_1^1 \leq t^a \phi(\delta_v(t))\|v\|^2_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}, \tag{4.14}$$

where we used that J_2 is also bounded by the above bound.

Next, we examine

$$L_1^2 = \bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|\xi_x(\xi',t')\xi_{xx}(\xi',t')x_\xi(\xi',t')v_\xi(\xi',t')}{|\xi'-\xi''|^{3+2\alpha}} - \frac{\xi_x(\xi'',t')\xi_{xx}(\xi'',t')x_\xi(\xi'',t')v_\xi(\xi'',t')|^2}{|\xi-\xi''|^{3+2\alpha}} d\xi' d\xi'' dt' \bigg)^{1/2}$$

$$\leq \phi(\delta_v(t)) \bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|\int_0^{t'} (v_\xi(\xi',\tau) - v_\xi(\xi'',\tau))d\tau|^2}{|\xi'-\xi''|^{3+2\alpha}} \cdot \bigg| \int_0^{t'} v_{\xi\xi}(\xi',\tau)d\tau \bigg|^2 |v_\xi(\xi',t')|^2 d\xi' d\xi'' dt' \bigg)^{1/2}$$

$$+ t^{1/2}\phi(\delta_v(t)) \bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|\int_0^{t'} (v_{\xi\xi}(\xi',\tau) - v_{\xi\xi}(\xi'',\tau))d\tau|^2}{|\xi'-\xi''|^{3+2\alpha}} \cdot |v_\xi(\xi',t')|^2 d\xi' d\xi'' dt' \bigg)^{1/2} + t^{1/2}\phi(\delta_v(t))$$

$$\cdot \bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|\int_0^{t'} v_{\xi\xi}(\xi'',\tau)d\tau|^2 |v_\xi(\xi',t') - v_\xi(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}} d\xi' d\xi'' dt' \bigg)^{1/2}$$

$$\equiv H_1 + H_2 + H_3.$$

First, similarly as in the estimate of J_1^1 we derive

$$H_1 \leq t^a \phi(\delta_v(t)) \|v\|^2_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}$$

Next

$$H_2 \leq t^{1/2}\phi(\delta_v(t)) \|v_\xi\|_{L_2(0,t;L_\infty(\overset{1}{\Omega}_0))} \|v_{\xi\xi}\|_{L_2(0,t;H^\alpha(\overset{1}{\Omega}_0))}$$

Finally, H_3 is estimated by the same bound as H_1.

Summarizing,

$$L_1^2 \leq t^a \phi(\delta_v(t)) \|v\|^2_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))} \tag{4.15}$$

Finally, we estimate the last term on the r.h.s. of (4.10),

$$L_1^3 = \bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|(I-\xi_x(\xi',t'))p_\xi(\xi',t') - (I-\xi_x(\xi'',t'))p_\xi(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}} d\xi' d\xi'' dt' \bigg)^{1/2}$$

$$\leq \bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|\xi_x(\xi',t') - \xi_x(\xi'',t')|^2 |p_\xi(\xi',t')|^2}{|\xi'-\xi''|^{3+2\alpha}} d\xi d\xi'' dt' \bigg)^{1/2}$$

$$+ \bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|I-\xi_x(\xi'',t')|^2 |p_\xi(\xi',t') - p_\xi(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}} d\xi' d\xi'' dt' \bigg)^{1/2}$$

$$\equiv M_1 + M_2.$$

Using the form of ξ_x, we have

$$M_1 \le \phi(\delta_v(t))\bigg(\int_0^t\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0}\frac{|\int_0^t(v_\xi(\xi',\tau)-v_\xi(\xi'',\tau))d\tau|^2|p_\xi(\xi',t')|^2}{|\xi'-\xi''|^{3+2\alpha}}d\xi'd\xi''dt'\bigg)^{1/2} \equiv M_1^1.$$

M_1^1 has the same structure as J_1^1, where $v_{\xi\xi}$ is replaced by p_ξ. Therefore, instead of (4.14) we derive the estimate

$$M_1^1 \le t^a\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|p_\xi\|_{L_2(0,t;H^\alpha(\overset{1}{\Omega}_0))}.$$

Finally M_2 is estimated by the same bound as M_1^1.

Summarizing,

$$L_1^3 \le t^a\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|p_\xi\|_{L_2(0,t;H^\alpha(\overset{1}{\Omega}_0))} \tag{4.16}$$

Using estimates (4.14), (4.15) and (4.16) in (4.10) yields

$$\langle\overset{1}{F}\rangle_{\alpha,x,2,\overset{1}{\Omega}{}_0^t} \le t^a\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0)}(\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))} + \|p_\xi\|_{L_2(0,t;H^\alpha(\overset{1}{\Omega}_0))}). \tag{4.17}$$

The last term on the r.h.s. of (4.8) can be bounded by

$$\begin{aligned}\langle\overset{1}{F}\rangle_{\alpha/2,t,2,\overset{1}{\Omega}{}_0^t} &= \langle(I-\xi_x^2)v_{\xi\xi}\rangle_{\frac{\alpha}{2},t,2,\overset{1}{\Omega}{}_0^t}\\ &\quad+\langle\xi_x\xi_{xx}x_\xi v_\xi\rangle_{\frac{\alpha}{2},t,2,\overset{1}{\Omega}{}_0^t}+\langle(I-\xi_x)p_\xi\rangle_{\frac{\alpha}{2},t,2,\overset{1}{\Omega}{}_0^t}\\ &\equiv A_1+A_2+A_3.\end{aligned} \tag{4.18}$$

First, we estimate

$$\begin{aligned}A_1 &= \bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t\frac{|(I-\xi_x^2(\xi,t'))v_{\xi\xi}(\xi,t')-(I-\xi_x^2(\xi,t''))v_{\xi\xi}(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2}\\ &\le \bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t\frac{|\xi_x^2(\xi,t')-\xi_x^2(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}|v_{\xi\xi}(\xi,t')|^2dt'dt''d\xi\bigg)^{1/2}\\ &\quad+\bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t\frac{|I-\xi_x^2(\xi,t'')|^2|v_{\xi\xi}(\xi,t')-v_{\xi\xi}(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2}\\ &\equiv A_1^1+A_1^2.\end{aligned}$$

Using the form of ξ_x yields

$$\begin{aligned} A_1^1 &\le \phi(\delta_v(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{|\int_{t'}^{t''} v_\xi(\xi,\tau)d\tau|^2|v_{\xi\xi}(\xi,t')|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2} \\ &\le \phi(\delta_v(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t |t'-t''|^{-\alpha}\int\limits_{t'}^{t''}|v_\xi(\xi,\tau)|^2d\tau|v_{\xi\xi}(\xi,t')|^2dt'dt''d\xi\bigg)^{1/2} \\ &\le t^{1/2-\alpha/2}\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|v\|_{L_2(0,t;H^2(\overset{1}{\Omega}_0))}. \end{aligned}$$

Exploiting the form of ξ_x and the imbedding

$$\|v_\xi\|_{L_\infty(\overset{1}{\Omega}_0)} \le c\|v\|_{H^{2+\alpha}(\overset{1}{\Omega}_0)} \quad \text{for} \ \ \alpha > 1/2$$

we obtain

$$A_1^2 \le t^{1/2}\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|v_{\xi\xi}\|_{L_2(\Omega;H^{\alpha/2}(0,t))}.$$

Summarizing, we have

$$A_1 \le t^a\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|v\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}. \tag{4.19}$$

Expression A_2 has the form

$$\begin{aligned} A_2 &= \bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{|\xi_x(\xi,t')\xi_{xx}(\xi,t')x_\xi(\xi,t')v_\xi(\xi,t')}{|t'-t''|^{1+\alpha}} \\ &\quad - \frac{\xi_x(\xi,t'')\xi_{xx}(\xi,t'')x_\xi(\xi,t'')v_\xi(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}d\xi dt'dt''\bigg)^{1/2} \\ &\le \bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{|\xi_x(\xi,t')-\xi_x(\xi,t'')|^2|\xi_{xx}(\xi,t')|^2|x_\xi(\xi,t')|^2|v_\xi(\xi,t)|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2} \\ &\quad + \bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{|\xi_x(\xi,t'')|^2|\xi_{xx}(\xi,t')-\xi_{xx}(\xi,t'')|^2|x_\xi(\xi,t')|^2|v_\xi(\xi,t')|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2} \\ &\quad + \bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{|\xi_x(\xi,t'')|^2|\xi_{xx}(\xi,t'')|^2|x_\xi(\xi,t')-x_\xi(\xi,t'')|^2|v_\xi(\xi,t')|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2} \\ &\quad + \bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{|\xi_x(\xi,t'')|^2|\xi_{xx}(\xi,t'')|^2|x_\xi(\xi,t'')|^2|v_\xi(\xi,t')-v_\xi(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2} \\ &\equiv \sum_{i=1}^4 A_2^i. \end{aligned}$$

First, we estimate

$$
\begin{aligned}
A_2^1 &\le \phi(\delta_v(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{|\int_{t'}^{t''} v_\xi(\xi,\tau)d\tau|^2 |\int_0^{t'} v_{\xi\xi}(\xi,\tau)d\tau|^2 |v_\xi(\xi,t')|^2}{|t'-t''|^{1+\alpha}} dt'dt''d\xi\bigg)^{1/2} \\
&\le t^{1/2-\alpha/2}\phi(\delta_v(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t |v_{\xi\xi}(\xi,\tau)|^2 d\tau \int\limits_0^t |v_\xi(\xi,\tau)|^2 d\tau\bigg)^{1/2} \\
&\le t^{1/2-\alpha/2}\phi(\delta_v(t))\bigg(\int\limits_0^t \|v(\cdot,\tau)\|^2_{H^{2+\alpha}(\overset{1}{\Omega}_0)} d\tau\bigg)^{1/2}\bigg(\int\limits_0^t \|v_{\xi\xi}(\cdot,\tau)\|^2_{L_2(\overset{1}{\Omega}_0)} d\tau\bigg)^{1/2} \\
&\le t^{1/2-\alpha/2}\phi(\delta_v(t))\|v\|^2_{L_2(0,t:H^{2+\alpha}(\overset{1}{\Omega}_0))}.
\end{aligned}
$$

Next, we estimate

$$
\begin{aligned}
A_2^2 &\le \phi(\delta_v(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{|\int_{t'}^{t''} v_{\xi\xi}(\xi,\tau)d\tau|^2 |v_\xi(\xi,t')|^2}{|t'-t''|^{1+\alpha}} dt'dt''d\xi\bigg)^{1/2} \\
&\le t^{1/2-\alpha/2}\phi(\delta_v(t))\|v_{\xi\xi}\|_{L_2(\overset{1}{\Omega}{}_0^t)}\bigg(\int\limits_0^t \|v_\xi(\cdot,\tau)\|^2_{L_\infty(\overset{1}{\Omega}_0)} d\tau\bigg)^{1/2} \\
&\le t^{1/2-\alpha/2}\phi(\delta_v(t))\|v\|^2_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}.
\end{aligned}
$$

Next $A_2^3 \le A_2^1$ and finally

$$
\begin{aligned}
A_2^4 &\le t^{1/2}\phi(\delta_v(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t\int\limits_0^{t'} |v_{\xi\xi}(\xi,\tau)|^2 d\tau \frac{|v_\xi(\xi,t')-v_\xi(\xi,t'')|^2}{|t'-t''|^{1+\alpha}} dt'dt''d\xi\bigg)^{1/2} \\
&\equiv A_2^{41}.
\end{aligned}
$$

Applying the Hölder inequality with respect to ξ yields

$$
\begin{aligned}
A_2^{41} \le t^{1/2}\phi(\delta_v(t))\bigg(\int\limits_0^t \|v_{\xi\xi}(\cdot,\tau)\|^2_{L_{2p}(\overset{1}{\Omega}_0)} d\tau\bigg)^{1/2} \cdot \\
\cdot\bigg(\int\limits_0^t\int\limits_0^t \frac{\|v_\xi(\cdot,t')-v_\xi(\cdot,t'')\|^2_{L_{2p'}(\overset{1}{\Omega}_0)}}{|t'-t''|^{1+\alpha}} dt'dt''\bigg)^{1/2} \equiv A_2^{42},
\end{aligned}
$$

where $1/p + 1/p' = 1$.

In view of imbeddings

$$
\begin{aligned}
&\|v_{\xi\xi}\|_{L_{2p}(\overset{1}{\Omega}_0)} \le c\|v\|_{H^{2+\alpha}(\overset{1}{\Omega}_0)} \quad \text{for} \ \ \frac{3}{2} - \frac{3}{2p} \le \alpha, \\
&\|v_\xi(\cdot,t')-v_\xi(\cdot,t'')\|_{L_{2p'}(\overset{1}{\Omega}_0)} \le c\|v_\xi(\cdot,t')-v_\xi(\cdot,t'')\|_{H^1(\overset{1}{\Omega}_0)}
\end{aligned}
$$

which holds for $3/2 - 3/2p' \le 1$, we finally obtain the estimate for $\alpha \ge 1/2$

$$
A_2^{42} \le t^{1/2}\phi(\delta(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|v\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}.
$$

Summarizing,

$$A_2 \le t^a \phi(\delta_v(t)) \|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))} \|v\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}. \tag{4.20}$$

Since p_ξ in A_3 plays the same role as $v_{\xi\xi}$ in A_1 we obtain from (4.19) the estimate

$$A_3 \le t^a \phi(\delta_v(t)) \|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))} \|p_\xi\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)}. \tag{4.21}$$

Using estimates (4.19), (4.20) and (4.21) in (4.18) implies

$$\begin{aligned} \langle \overset{1}{F} \rangle_{\frac{\alpha}{2},t,2,\overset{1}{\Omega}{}_0^t} \le t^a \phi(\delta_v(t)) \|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))} \cdot \\ \cdot (\|v\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)} + \|p_\xi\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)}). \end{aligned} \tag{4.22}$$

Exploiting estimates (4.9), (4.17) and (4.22) in (4.8), we obtain

$$\begin{aligned} \|\overset{1}{F}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} \le t^a \phi(\delta_v(t)) \|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))} (\|v\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)} \\ + \|p\|_{L_2(0,t;H^1(\overset{1}{\Omega}_0))} + \|p_\xi\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)}), \end{aligned} \tag{4.23}$$

where $\alpha > 5/8$.

Next, we estimate the norm

$$\|\overset{2}{F}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} = \|\overset{2}{F}\|_{L_2(\overset{1}{\Omega}{}_0^t)} + \langle \overset{2}{F} \rangle_{\alpha,x,2,\overset{1}{\Omega}{}_0^t} + \|\langle \overset{2}{F} \rangle_{\alpha/2,t,2,\overset{1}{\Omega}{}_0^t}. \tag{4.24}$$

It is sufficient to estimate the last two term on the r.h.s. of (4.24) only. First we examine

$$\begin{aligned} \langle \overset{2}{F} \rangle_{\alpha,x,2,\overset{1}{\Omega}{}_0^t} &= \bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \bigg(\frac{|\xi_x(\xi',t') \overset{1}{H}_\xi(\xi',t') \overset{1}{H}(\xi',t')}{|\xi'-\xi''|^{3+2\alpha}} \\ &\quad - \frac{\xi_x(\xi'',t') \overset{1}{H}_\xi(\xi'',t') \overset{1}{H}(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}} \bigg) d\xi' d\xi'' dt' \bigg)^{1/2} \\ &\le \phi(\delta_v(t)) \bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \frac{|\int_0^{t'} (v_\xi(\xi',\tau) - v_\xi(\xi'',\tau)) d\tau|^2|}{|\xi'-\xi''|^{3+2\alpha}} \cdot \\ &\quad \cdot \overset{1}{H}_\xi(\xi',t')|^2 \overset{1}{H}(\xi',t')|^2 d\xi' d\xi'' dt' \bigg)^{1/2} + \phi(\delta_v(t)) \bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} \\ &\quad \cdot \frac{|\overset{1}{H}_\xi(\xi',t') - \overset{1}{H}_\xi(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}} |\overset{1}{H}(\xi',t')|^2 d\xi' d\xi'' dt' \bigg)^{1/2} \\ &\quad + \phi(\delta_v(t)) \bigg(\int_0^t \int_{\overset{1}{\Omega}_0} \int_{\overset{1}{\Omega}_0} |\overset{1}{H}_\xi(\xi'',t')|^2 \cdot \\ &\quad \cdot \frac{|\overset{1}{H}(\xi',t') - \overset{1}{H}(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}} d\xi' d\xi'' dt' \bigg)^{1/2} \\ &\equiv P_1 + P_2 + P_3. \end{aligned} \tag{4.25}$$

Applying the Hölder inequality with respect to ξ', ξ'' in P_1 yields

$$P_1 \le t^{1/2}\phi(\delta_v(t))\bigg(\int_0^t \bigg(\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0} \frac{|v_\xi(\xi',t')-v_\xi(\xi'',t')|^{2p_1}}{|\xi'-\xi''|^{3+2p_1\alpha'}}d\xi' d\xi''\bigg)^{1/p_1}\bigg)^{1/2}\cdot$$
$$\cdot\bigg[\int_0^t \bigg(\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0} \frac{|\overset{1}{H}_\xi(\xi',t')|^{2p_2}}{|\xi'-\xi''|^{\frac{3}{2}p_2}}d\xi' d\xi''\bigg)^{1/p_2}\cdot$$
$$\cdot\bigg(\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0} |\overset{1}{H}(\xi',t')|^{2p_3}d\xi'\bigg)^{1/p_3}\bigg]^{1/2}$$
$$\equiv P_1^1,$$

where $1/p_1+1/p_2+1/p_3=1$, $p_2<2$, $\alpha'=\alpha+\frac{1}{2p_1}\big(\frac{3}{2}p_1-3\big)$.

we use imbeddings

$$\|v_\xi\|_{W^{\alpha'}_{2p_1}(\overset{1}{\Omega}_0)} \le c\|v\|_{H^{2+\alpha}(\overset{1}{\Omega}_0)}, \quad \frac{3}{2}-\frac{3}{2p_1}+\alpha'+1\le 2+\alpha, \tag{4.26}$$

$$\|\overset{1}{H}_\xi\|_{L_{2p_2}(\overset{1}{\Omega}_0)} \le c\|\overset{1}{H}\|_{H^{2+\alpha}(\overset{1}{\Omega}_0)}, \quad \frac{3}{2}-\frac{3}{2p_2}+1\le 2+\alpha, \tag{4.27}$$

and

$$\|\overset{1}{H}\|_{L_{2p_3}(\overset{1}{\Omega}_0)} \le c\|\overset{1}{H}\|_{H^{1+\alpha}(\overset{1}{\Omega}_0)}, \quad \frac{3}{2}-\frac{3}{2p_3}\le 1+\alpha. \tag{4.28}$$

From (4.25)–(4.28) and the form of α' we obtain the restrictions

$$\frac{3}{2}-\frac{3}{p_1}+\frac{3}{4}\le 1, \quad 3-\frac{3}{p_2}\le 2+2\alpha, \quad 3-\frac{3}{p_3}\le 2+2\alpha.$$

Eliminating p_1, p_2, p_3 yields the restriction

$$\frac{3}{8}\le\alpha.$$

Hence

$$P_1\le P_1^1 \le t^{1/2}\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|\overset{1}{H}\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\cdot \\ \cdot\|\overset{1}{H}\|_{L_\infty(0,t;H^{1+\alpha}(\overset{1}{\Omega}_0))} \tag{4.29}$$

Next, we examine P_2,

$$P_2\le\phi(\delta_v(t))$$
$$\cdot\bigg[\int_0^t \bigg(\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0} \frac{|\overset{1}{H}_\xi(\xi',t')-\overset{1}{H}_\xi(\xi'',t')|^{2p_1}}{|\xi'-\xi''|^{3+2p_1\alpha'}}d\xi' d\xi''\bigg)^{1/p_1}\cdot$$
$$\cdot\bigg(\int_{\overset{1}{\Omega}_0} |\overset{1}{H}(\xi',t')|^{2p_2}d\xi'\bigg)^{1/p_2}dt'\bigg]^{1/2}\equiv P_2^1,$$

where $1/p_1+1/p_2=1$, $p_2<2$, $\alpha'=\alpha+\frac{3}{4}-\frac{3}{2p_1}$.

We use the imbeddings

$$\|\overset{1}{H}_\xi\|_{W^{\alpha'}_{2p_1}(\overset{1}{\Omega}_0)} \le c\|\overset{1}{H}\|_{H^{2+\alpha}(\overset{1}{\Omega}_0)}, \quad \frac{3}{2}-\frac{3}{2p_1}+\alpha'\le 1+\alpha, \tag{4.30}$$

$$\|\overset{1}{H}\|_{L_{2p_2}(\overset{1}{\Omega}_0)} \le c\|\overset{1}{H}\|_{H^{1+\alpha}(\overset{1}{\Omega}_0)}, \quad \frac{3}{2} - \frac{3}{2p_2} \le 1+\alpha. \tag{4.31}$$

Since restrictions (4.30), (4.31) can hold together we obtain the estimate

$$\begin{aligned} P_2 \le P_2^1 \le t^{1/2}\phi(\delta_v(t))\|\overset{1}{H}\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\cdot \\ \cdot \|\overset{1}{H}\|_{L_\infty(0,t;H^{1+\alpha}(\overset{1}{\Omega}_0))}. \end{aligned} \tag{4.32}$$

Finally,

$$\begin{aligned} P_3 &\le \phi(\delta_v(t)) \\ &\quad\cdot \bigg(\int_0^t \|\overset{1}{H}_\xi(\cdot,\tau)\|^2_{L_\infty(\overset{1}{\Omega}_0)} d\tau\bigg)^{1/2} \|\overset{1}{H}\|_{L_\infty(0,t;H^\alpha(\overset{1}{\Omega}_0))}. \\ &\le t^a\phi(\delta_v(t)))\bigg(\int_0^t \|\overset{1}{H}_\xi(\cdot\tau)\|^{2\lambda}_{L_\infty(\overset{1}{\Omega}_0)} dt\bigg)^{1/2\lambda} \|\overset{1}{H}\|_{L_\infty(0,t;H^\alpha(\overset{1}{\Omega}_0))} \\ &\le t^a\phi(\delta_v(t))\|\overset{1}{H}\|^2_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}, \end{aligned} \tag{4.33}$$

where $\lambda < 8/3$ and $\alpha > 5/8$.

Using estimates (4.29), (4.32) and (4.33) in (4.25) implies the estimate

$$\begin{aligned} \langle F_2\rangle_{\alpha,x,2,\overset{1}{\Omega}{}_0^t} \le t^a\phi(\delta_v(t))(\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))} + 1)\cdot \\ \cdot \|\overset{1}{H}\|^2_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}. \end{aligned} \tag{4.34}$$

Finally, we examine

$$\begin{aligned} &\langle F_2\rangle_{\frac{\alpha}{2},t,2,\overset{1}{\Omega}{}_0^t} \\ &\quad= \bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t \frac{|\xi_x(\xi,t')\overset{1}{H}_\xi(\xi,t')\overset{1}{H}(\xi,t') - \xi_x(\xi,t'')\overset{1}{H}_\xi(\xi,t'')\overset{1}{H}(\xi,t'')|^2}{|t'-t''|^{1+\alpha}} dt'dt''d\xi\bigg)^{1/2} \\ &\quad\le \bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t \frac{|\xi_x(\xi,t') - \xi_x(\xi,t'')|^2}{|t'-t''|^{1+\alpha}} |\overset{1}{H}_\xi(\xi,t')|^2|\overset{1}{H}(\xi,t')|^2 dt'dt''d\xi\bigg)^{1/2} \\ &\qquad+ \bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t |\xi_x(\xi,t'')|^2 \frac{|\overset{1}{H}_\xi(\xi,t') - \overset{1}{H}_\xi(\xi,t'')|^2}{|t'-t''|^{1+\alpha}} |\overset{1}{H}(\xi,t')|^2 dt'dt''d\xi\bigg)^{1/2} \\ &\qquad+ \bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t |\xi_x(\xi,t'')|^2|\overset{1}{H}_\xi(\xi,t'')|^2 \frac{|\overset{1}{H}(\xi,t') - \overset{1}{H}(\xi,t'')|^2}{|t'-t''|^{1+\alpha}} dt'dt''d\xi\bigg)^{1/2} \\ &\quad\equiv Q_1 + Q_2 + Q_3. \end{aligned} \tag{4.35}$$

First, we estimate

$$Q_1 \leq \phi(\delta_v(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t |t'-t''|^{-\alpha}\int\limits_{t'}^{t''}|v_\xi(\xi,\tau)|^2 d\tau\cdot$$

$$\cdot|\overset{1}{H}_\xi(\xi,t')|^2|\overset{1}{H}(\xi,t')|^2dt'dt''d\xi\bigg) = Q_1^1.$$

Performing integration with respect to t'' gives

$$\begin{aligned}
Q_1 \leq Q_1^1 &\leq t^{1/2-\alpha/2}\phi(\delta_v(t))\bigg(\int\limits_0^t \|v_\xi(\cdot,\tau)\|^2_{L_\infty(\overset{1}{\Omega}_0)}d\tau\bigg)^{1/2}\cdot\\
&\cdot\bigg(\int\limits_0^t\|\overset{1}{H}_\xi(\cdot,t')\|^2_{L_\infty(\overset{1}{\Omega}_0}dt'\bigg)\sup_t\|\overset{1}{H}\|_{L_2(\overset{1}{\Omega}_0)}\\
&\leq t^{1/2-\alpha/2}\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|\overset{1}{H}\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\cdot\\
&\cdot\|\overset{1}{H}\|_{L_\infty(0,t;L_2(\overset{1}{\Omega}_0))}\\
&\leq t^{1/2-\alpha/2}\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|\overset{1}{H}\|^2_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}.
\end{aligned} \tag{4.36}$$

The second term Q_2 is bounded by

$$\begin{aligned}
Q_2 &\leq \phi(\delta_v(t))\|\overset{1}{H}_\xi\|_{L_2(\overset{1}{\Omega}_0;H^{\alpha/2}(0,t))}\cdot\|\overset{1}{H}\|_{L_\infty(0,t;L_\infty(\overset{1}{\Omega}_0))}\\
&\leq t^a\phi(\delta_v(t))\|\overset{1}{H}_\xi\|_{L_2(\overset{1}{\Omega}_0;\overset{1}{H}{}^{\beta/2}(0,t))}\cdot\|\overset{1}{H}\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}\\
&\leq t^a\phi(\delta_v(t))\|\overset{1}{H}\|^2_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)},
\end{aligned} \tag{4.37}$$

where $\beta > \alpha$ and $\beta \leq 1+\alpha$.

Finally

$$\begin{aligned}
Q_3 &\leq t^a\phi(\delta_v(t))\\
&\cdot\sup_t\|\overset{1}{H}_\xi\|_{L_2(\overset{1}{\Omega}_0)}\sup_\xi\|\overset{1}{H}\|_{H^{\beta/2}(0,t)}\\
&\leq t^{1/2}\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|\overset{1}{H}\|^2_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)},
\end{aligned} \tag{4.38}$$

where $\beta > \alpha$, $\beta \leq 1/2+\alpha$.

Using estimates (4.36)–(4.38) in (4.35) yields

$$\langle F_2\rangle_{\frac{\alpha}{2},t,2,\overset{1}{\Omega}{}_0^t} \leq t^a\phi(\delta_v(t))(\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}+1)\|\overset{1}{H}\|^2_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}. \tag{4.39}$$

Using (4.34) and (4.39) in (4.24) we obtain the estimate

$$\|\overset{2}{F}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} \leq t^a\phi(\delta_v(t))(\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}+1)\cdot\|\overset{1}{H}\|^2_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)} \tag{4.40}$$

From (4.23) and (4.40) and recalling the index n we derive (4.4). This concludes the proof of Lemma 4.2. □

From the properties of the space trace operators we have

Lemma 4.3. *Let the assumptions of Lemma 4.2 hold. Then*

$$\begin{aligned}\|G_n\|_{H^{\frac{1}{2}+\alpha,\frac{1}{4}+\frac{\alpha}{2}}(S_0^t)} &\le t^a\phi(\delta_{v_n}(t))(\|\bar v_n\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}\\ &+\|\bar p_n\|_{H^{1/2+\alpha,1/4+\alpha/2}(S_0^t)}+\|\bar{\overset{1}{H}}_n\|^2_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}).\end{aligned}\tag{4.41}$$

In the next lemma we estimate norms of $\nabla\varphi_n$ in the r.h.s. of (4.3).

Lemma 4.4. *Assume that* $\bar v_n\in V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)$, $a>0$.
Then solutions to problem (4.1) *satisfy*

$$\|\nabla\varphi_{n,t}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)}\le\|\nabla\varphi_n\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}\le t^a\phi(\delta_{v_n}(t))\|\bar v_n\|^2_{V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)},\tag{4.42}$$

where g_n *is defined in* $(3.8)_2$.

Proof. In the proof we use the simplified notation

$$\varphi=\varphi_n,\quad v=\bar v_n\tag{4.43}$$

The aim of this proof is to find an estimate for the expression

$$\begin{aligned}\|\nabla\varphi\|_{H^{2+\alpha,1+\alpha/2}(\overset{i}{\Omega}_0)}&=|\nabla\varphi|_{2,\Omega^t}+\|\nabla\varphi\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\\ &+\|\nabla\varphi\|_{L_2(\Omega;H^{1+\alpha/2}(0,t))}\equiv I_1+I_2+I_3,\end{aligned}\tag{4.44}$$

where

$$\varphi(x,t)=\int_{\overset{1}{\Omega}_0}G(x,y)g(y,t)dy,\tag{4.45}$$

where G is the Green function to the Dirichlet problem

$$\begin{aligned}&\Delta\varphi=g\quad\text{in }\ \overset{1}{\Omega}_0,\\ &\varphi|_{S_0}=0.\end{aligned}\tag{4.46}$$

Hence the Green function satisfies the condition

$$G(x,y)|_{y\in S_0}=0.\tag{4.47}$$

Moreover

$$g=\partial_\xi[(1-\xi_x)v].\tag{4.48}$$

In view of (4.45), (4.47), (4.48) and the Calderon–Zygmund estimate we get

$$\begin{aligned}I_1=|\nabla\varphi|_{2,\Omega^t}&=\Big\|\nabla_x\int_{\overset{1}{\Omega}_0}G(x,\xi)_\xi(I-\xi_x)vd\xi\Big\|_{L_2(\Omega^t)}\\ &\le t^{1/2}\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|v\|_{L_2(\overset{1}{\Omega}{}_0^t)}.\end{aligned}\tag{4.49}$$

Using definition (2.2.3) of the Besov space equivalent to definition (2.2.5) and applying the Calderon–Zygmund estimate, we obtain

$$
\begin{aligned}
I_2 &= \|\nabla\varphi\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))} \le c\|\nabla g\|_{L_2(0,t;H^{\alpha}(\overset{1}{\Omega}_0))} \\
&\le c\bigg(\int_0^t\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0}\bigg(\frac{|(I-\xi_x(\xi',t'))v_{\xi\xi}(\xi',t')}{|\xi'-\xi''|^{3+2\alpha}} \\
&\quad - \frac{(I-\xi_x(\xi'',t'))v_{\xi\xi}(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}}\bigg)d\xi' d\xi'' dt'\bigg)^{1/2} \\
&\quad + c\bigg(\int_0^t\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0}\bigg(\frac{|\xi_{xx}(\xi',t')x_\xi(\xi',t')v_\xi(\xi',t')}{|\xi'-\xi''|^{3+2\alpha}} \\
&\quad - \frac{\xi_{xx}(\xi'',t')x_\xi(\xi'',t')v_\xi(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}}\bigg)d\xi' d\xi'' dt'\bigg)^{1/2} \equiv J_1 + J_2.
\end{aligned}
$$

Considering J_1, we have

$$
\begin{aligned}
J_1 &\le \phi(\delta_v(t))\bigg(\int_0^t\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0}\frac{|\int_0^{t'}(v_\xi(\xi',\tau)-v_\xi(\xi'',\tau))d\tau|^2|v_{\xi\xi}(\xi',t')|^2}{|\xi'-\xi''|^{3+2\alpha}}d\xi' d\xi'' dt'\bigg)^{1/2} \\
&\quad + \phi(\delta_v(t))\bigg(\int_0^t\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0}\bigg|\int_0^{t'}v_\xi(\xi'',\tau)d\tau\bigg|^2\frac{|v_{\xi\xi}(\xi',t')-v_{\xi\xi}(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}}d\xi' d\xi'' dt'\bigg)^{1/2} \\
&\equiv J_1^1 + J_1^2,
\end{aligned}
$$

where

$$
J_1^2 \le t^{1/2}\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}.
$$

To estimate J_1^1 we use the Hölder inequality with respect to ξ', ξ'' and the Minkowski inequality.

Then we obtain

$$
\begin{aligned}
J_1^1 &\le t^{1-1/2p}\phi(\delta_v(t))\bigg[\int_0^t\bigg(\int|v_{\xi\xi}(\xi',t')|^{2p'}d\xi'\bigg)^{1/p'}\bigg]^{1/2}\cdot \\
&\quad\cdot\bigg[\int_0^t\bigg(\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0}\frac{|v_\xi(\xi',\tau)-v_\xi(\xi'',\tau)|^{2p}}{|\xi'-\xi''|^{3+2\alpha'}}d\xi' d\xi'' dt'\bigg)^{1/p}d\tau\bigg]^{1/2} \\
&\le t^{1-1/2p}\phi(\delta_v(t))\|v\|^2_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))},
\end{aligned}
$$

where $1/p + 1/p' = 1$, $p' < 2$, $\alpha' = \alpha + \frac{1}{2p}\left(\frac{3}{2}p - 3\right)$.

Next, we examine

$$\begin{aligned} J_2 &\le t^{1/2}\phi(\delta_v(t))\bigg(\int_0^t\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0}\frac{\int_0^{t'}|v_{\xi\xi}(\xi',\tau)-v_{\xi\xi}(\xi'',\tau)|^2d\tau}{|\xi'-\xi''|^{3+2\alpha}}|v_\xi(\xi',t')|^2d\xi'd\xi''dt'\bigg)^{1/2}\\ &\quad+t^{1/2}\phi(\delta_v(t))\bigg(\int_0^t\int_{\overset{1}{\Omega}_0}\int_{\overset{1}{\Omega}_0}\int_0^t|v_{\xi\xi}(\xi'',\tau)|^2d\tau\frac{|v_\xi(\xi',t')-v_\xi(\xi'',t')|^2}{|\xi'-\xi''|^{3+2\alpha}}d\xi'd\xi''dt'\bigg)^{1/2}\\ &\equiv J_2^1+J_2^2, \end{aligned}$$

where

$$J_2^1\le t^{1/2}\phi(\delta_v(t))\|v\|^2_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}.$$

Finally, J_2^2 can be treated in the same way as J_1^1. Then we obtain

$$J_2^2\le t^{1-1/2p}\phi(\delta_v(t))\|v\|^2_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}.$$

Summarizing,

$$I_2\le t^\alpha\phi(\delta_v(t))\|v\|^2_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))} \tag{4.50}$$

Finally, we examine I_3. The Calderon–Zygmund theorem implies

$$\begin{aligned} I_3 &\le c\|(I-\xi_x)v\|_{L_2(\overset{1}{\Omega}_0;H^{1+\alpha/2}(0,t))}=c\bigg(\int_{\overset{1}{\Omega}_0}\|(I-\xi_x)v\|^2_{H^{1+\alpha}(0,t)}d\xi\bigg)^{1/2}\\ &=c\bigg(\int_{\overset{1}{\Omega}_0}\|(I-\xi_x)v_t-\xi_{xt}v\|^2_{H^\alpha(0,t)}d\xi\bigg)^{1/2}\\ &\le\bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t\frac{|(I-\xi_x(\xi,t'))v_t(\xi,t')-(I-\xi_x(\xi,t''))v_t(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2}\\ &\quad+\bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t\frac{|\xi_{xt}(\xi,t')v(\xi,t')-\xi_{xt}(\xi,t'')v(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2}\\ &\equiv L_1+L_2, \end{aligned}$$

where

$$\begin{aligned} L_1 &\le\bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t|1-\xi_x(\xi,t')|^2\frac{|v_t(\xi,t')-v_t(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2}\\ &\quad+\bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t\frac{|\xi_x(\xi,t')-\xi_x(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}|v_t(\xi,t'')|^2dt'dt''d\xi\bigg)^{1/2}\\ &\equiv L_1^1+L_1^2. \end{aligned}$$

Estimating L_1^1 yields

$$L_1^1\le t^{1/2}\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0))}\|v_t\|_{L_2(\Omega;H^{\alpha/2}(0,t))}. \tag{4.51}$$

Next, L_1^2 is estimated by

$$\begin{aligned} L_1^2 &\le \phi(\delta(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{|\int_{t'}^{t''} v_\xi(\xi,\tau)d\tau|^2}{|t'-t''|^{1+\alpha}}|v_t(\xi,t'')|^2dt'dt''d\xi\bigg)^{1/2} \\ &\le \phi(\delta_v(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t |t'-t''|^{-\alpha}\int\limits_{t'}^{t''} v_\xi^2(\xi,\tau)d\tau|v_t(\xi,t'')|^2dt'dt''d\xi\bigg)^{1/2} \\ &\le t^{1/2-\alpha/2}\phi(\delta_v(t))\|v\|_{L_2(0,t;H^{2+\alpha}(\overset{1}{\Omega}_0)}\|v_t\|_{L_2(\overset{1}{\Omega}{}_0^t)}. \end{aligned} \tag{4.52}$$

Finally, (4.51) and (4.52) imply

$$L_1 \le t^a\phi(\delta_v(t))\|v\|^2_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}. \tag{4.53}$$

Finally, we estimate L_2,

$$\begin{aligned} L_2 \le &\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{|\xi_{xt}(\xi,t')-\xi_{xt}(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}|v(\xi,t')|^2dt'dt''d\xi\bigg)^{1/2} \\ &+\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t |\xi_{xt}(\xi,t'')|^2\frac{|v(\xi,t')-v(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2} \equiv L_2^1+L_2^2. \end{aligned}$$

First, we consider

$$L_2^1 \le \phi(\delta_v(t))\bigg(\int\limits_{\overset{1}{\Omega}_0}\int\limits_0^t\int\limits_0^t \frac{|v_\xi(\xi,t')-v_\xi(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}|v(\xi,t'')|^2dt'dt''d\xi\bigg)^{1/2} \equiv L_2^{11}.$$

By the Hölder inequality we get

$$\begin{aligned} L_2^{11} \le \phi(\delta_v(t))&\bigg[\int\limits_{\overset{1}{\Omega}_0}\bigg(\int\limits_0^t\int\limits_0^t \frac{|v_\xi(\xi,t')-v_\xi(\xi,t'')|^{2p}}{|t'-t''|^{1+2p\alpha'}}dt'dt''\bigg)^{q/p}d\xi\bigg]^{1/2q}\cdot \\ &\cdot\bigg[\int\limits_{\Omega}\bigg(\int\limits_0^t\int\limits_0^t \frac{|v(\xi,t')|^{2p'}}{|t'-t''|^{\frac{1}{2}p'}}dt'dt''\bigg)^{q'/p'}d\xi\bigg]^{1/2q'} \equiv L_2^{12}, \end{aligned}$$

where $1/p+1/p'=1$, $1/q+1/q'=1$, $p'<2$, $\alpha'=\frac{\alpha}{2}+\frac{1}{2p}\big(\frac{1}{2}p-1\big)$.

Performing integration with respect to t'' in the second factor yields

$$\begin{aligned} L_2^1 \le L_2^{12} &\le t^{1/2p'-1/4}\phi(\delta_v(t))\|v_\xi\|_{L_{2q}(\Omega;W_{2p}^{\alpha'}(0,t))}\|v\|_{L_{2q'}(\Omega;L_{2p'}(0,t))} \\ &\le t^{1/2p'-1/4}\phi(\delta_v(t))\|v\|^2_{V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)}, \end{aligned} \tag{4.54}$$

where we need the following restrictions

$$\begin{aligned} &\frac{5}{2}-\frac{3}{2q}-\frac{2}{2p}+\alpha+\frac{1}{2}-\frac{1}{p}+1 \le 2+\alpha \quad \text{so} \quad 3-\frac{2}{p}-\frac{3}{2q} \le 1, \\ &\frac{5}{2}-\frac{3}{2q'}-\frac{1}{p'} \le 2+\alpha. \end{aligned}$$

We see that the above inequalities do not imply any restriction.

Finally, we estimate

$$L_2^2 \le \phi(\delta_v(t))\bigg(\int_{\overset{1}{\Omega}_0}\int_0^t\int_0^t |v_\xi(\xi,t'')|^2 \frac{|v(\xi,t')-v(\xi,t'')|^2}{|t'-t''|^{1+\alpha}}dt'dt''d\xi\bigg)^{1/2}$$

$$\le \phi(\delta_v(t))\bigg[\int_{\overset{1}{\Omega}_0}\bigg(\int_0^t\int_0^t \frac{|v_\xi(\xi,t'')|^{2p'}}{|t'-t''|^{\frac{1}{2}p'}}dt'dt''\bigg)^{q'/p'}d\xi\bigg]^{1/2q'}\cdot$$

$$\cdot\bigg[\int_{\overset{1}{\Omega}_0}\bigg(\int_0^t\int_0^t \frac{|v(\xi,t')-v(\xi,t'')|^{2p}}{|t'-t''|^{1+2p\alpha'}}dt'dt''\bigg)^{q/p}d\xi\bigg]^{1/2q} \equiv L_2^2,$$

where $1/p+1/p'=1$, $1/q+1/q'=1$, $p'<2$, $\alpha'=\frac{\alpha}{2}+\frac{1}{4}-\frac{1}{2p}$.

If the following restrictions hold

$$\frac{5}{2}-\frac{3}{2q'}-\frac{1}{p'}\le 1+\alpha,$$
$$3-\frac{3}{2q}-\frac{2}{p}\le 2$$

we obtain the estimate

$$L_2^2 \le t^{1/2p'-1/4}\phi(\delta_v(t))\|v\|^2_{V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)}. \tag{4.55}$$

Estimates (4.54) and (4.55) imply

$$L_2 \le t^a\phi(\delta_v(t))\|v\|^2_{V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)}. \tag{4.56}$$

From (4.53) and (4.56) we derive the bound

$$I_3 \le t^a\phi(\delta_v(t))\|v\|^2_{V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)}. \tag{4.57}$$

Using estimates (4.49), (4.50) and (4.57) in (4.44), we obtain

$$\|\nabla\varphi\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)} \le t^a\phi(\delta_{\bar v}(t))\|v\|^2_{V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)}. \tag{4.58}$$

Recalling notation (4.43) in (4.58) gives (4.42). This ends the proof. □

Corollary 4.5. *Using estimates* (4.4), (4.41) *and* (4.42) *in* (4.3) *yields*

$$\begin{aligned}&\|\bar v_{n+1}\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)}+\|\nabla\bar p_{n+1}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)}+\|\bar p_{n+1}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)}\\&\quad+\|\bar p_{n+1}\|_{H^{1/2+\alpha,1/4+\alpha/2}(S_0^t)}\le t^a\phi(\delta_{\bar v_n}(t))\cdot\|\bar v_n\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}[\|\bar v_n\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)}\\&\quad+\|\bar p_n\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)}+\|\bar p_{n,\xi}\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)}+\|\bar p_n\|_{H^{1/2+\alpha,1/4+\alpha/2}(S_0^t)}\\&\quad+\|\overset{1}{\bar H}_n\|^2_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}]+c(\|f\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)}+\|v(0)\|_{H^{1+\alpha}(\overset{1}{\Omega}_0)}).\end{aligned} \tag{4.59}$$

5. Estimates for Solutions to Problem (3.9)

Lemma 5.1. *Assume that* $\overset{i}{v}_n, \overset{i}{\bar H}_n\in V_2^{2+\alpha}(\overset{i}{\Omega}{}_0^t)$, $i=1,2$ *and* $a>0$. *Assume also that* $H_{*i}\in H^{3/2+\alpha,3/4+\alpha/2}$ (B^t), $i=1,2$, $\overset{i}{H}(0)\in H^{1+\alpha}(\overset{i}{\Omega}_0)$, $i=1,2$.

Then

$$\begin{aligned}\sum_{i=1}^{2}\|\overset{i}{\bar{H}}_{n+1}\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}^t_0)} &\le t^a\phi(\delta_{\overset{1}{\bar{v}}_n}(t),\delta_{\overset{2}{\bar{v}}_n}(t))\cdot\\ &\cdot\bigg(\sum_{i=1}^{2}(\|\overset{i}{\bar{H}}_n\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}^t_0)}+\|\overset{i}{\bar{v}}_n\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}^t_0)}\|\overset{i}{\bar{H}}_n\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}^t_0)})\\ &+c\sum_{i=1}^{2}\|H_{*i}\|_{H^{\frac{3}{2}+\alpha,\frac{3}{4}+\frac{\alpha}{2}}(B^t)}+c\sum_{i=1}^{2}\|\overset{i}{H}(0)\|_{H^{1+\alpha}(\overset{i}{\Omega}_0)}.\end{aligned}\tag{5.1}$$

Proof. Applying Lemma 2.5.1 to problem (3.9) yields the existence of solutions such that $\overset{i}{\bar{H}}_{n+1}\in V_2^{2+\alpha}(\overset{i}{\Omega}{}^t_0)$ satisfying boundary conditions $(3.9)_{5,6,7}$ and initial conditions $(3.9)_8$. Moreover, Lemma 2.5.1 implies the estimate

$$\begin{aligned}\sum_{i=1}^{2}\|\overset{i}{\bar{H}}_{n+1}\|_{V_2^{2+\alpha}(\overset{i}{\Omega}{}^t_0)} &\le c\sum_{i=1}^{2}[\|\overset{i}{M}_n\|_{H_2^{\alpha,\alpha/2}(\overset{i}{\Omega}{}^t_0)}\\ &+\|\nabla\overset{i}{N}_n\|_{H^{\alpha,\alpha/2}(\overset{i}{\Omega}{}^t_0)}+\|K_{in}\|_{H^{\frac{1}{2}+\alpha,\frac{1}{4}+\frac{\alpha}{2}}(S^t_0)}\\ &+\|L_{in}\|_{H^{3/2+\alpha,3/4+\alpha/2}(S^t_0)}+\|\nabla\overset{i}{\bar{\Phi}}_n\|_{H^{2+\alpha,1+\alpha/2}(\overset{i}{\Omega}{}^t_0)}\\ &+\|H_{*i}\|_{H^{3/2+\alpha,3/4+\alpha/2}(B^t)}\\ &+\|\overset{i}{H}(0)\|_{H^{1+\alpha}(\overset{i}{\Omega}_0)}]+\|B_0\|_{H^{1/2+\alpha,1/4+\alpha/2}(B^t)},\end{aligned}\tag{5.2}$$

where $\overset{i}{\bar{\Phi}}_n$ is a solution to (2.5.3) with $\overset{i}{N}_n$ which replaces $\overset{i}{N}$, $i=1,2$.

Recalling the forms of quantities $\overset{i}{M}_n$, $\overset{i}{N}_n$, $i=1,2$, and repeating the estimations appeared in the proofs of Lemmas 4.2 and 4.4, we obtain

$$\begin{aligned}&\|\overset{1}{M}_n\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}^t_0)}+\|\nabla\overset{1}{N}_n\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}^t_0)}\\ &\quad\le t^a\phi(\delta_{\overset{1}{\bar{v}}_n}(t))(\|\overset{1}{\bar{v}}_n\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}^t_0)}+1)\|\overset{1}{\bar{H}}_n\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}^t_0)}\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}&\|\overset{2}{M}_n\|_{H^{\alpha,\alpha/2}(\overset{2}{\Omega}{}^t_0)}+\|\nabla\overset{2}{N}_n\|_{H^{\alpha,\alpha/2}(\overset{2}{\Omega}{}^t_0)}\\ &\quad\le t^a\phi(\delta_{\overset{2}{\bar{v}}_n}(t))(\|\overset{2}{\bar{v}}_n\|_{V_2^{2+\alpha}(\overset{2}{\Omega}{}^t_0)}+1)\|\overset{2}{\bar{H}}_n\|_{V_2^{2+\alpha}(\overset{2}{\Omega}{}^t_0)}.\end{aligned}\tag{5.4}$$

Repeating the proof of (4.58) we obtain

$$\|\nabla\overset{i}{\Phi}\|_{H^{2+\alpha,1+\alpha/2}(\overset{i}{\Omega}{}^t_0)}\le t^a\phi(\delta_{\overset{i}{\bar{v}}_n}(t))\|\overset{i}{\bar{v}}_n\|_{V_2^{2+\alpha}(\overset{i}{\Omega}{}^t_0)},\quad i=1,2.$$

Moreover, Sect. 7 implies the estimates

$$\begin{aligned}\|K_{in}\|_{H^{1/2+\alpha,1/4+\alpha/2}(S^t_0)}+\|L_{in}\|_{H^{3/2+\alpha,3/4+\alpha/2}(S^t_0)} &\le t^a\phi(\delta_{\overset{1}{\bar{v}}_n}(t))\|\overset{1}{\bar{v}}_n\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}^t_0)}\cdot\\ \cdot(\|\overset{1}{\bar{H}}_n\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}^t_0)}+\|\overset{2}{\bar{H}}_n\|_{V_2^{2+\alpha}(\overset{2}{\Omega}{}^t_0)}).\end{aligned}$$

The above estimates imply (5.1). This concludes the proof. □

A solution $\overset{2}{\bar{v}}$ to problem (3.2) appeared in problem (3.7) can be estimated by appropriate norms of $\overset{1}{\bar{v}}$. Expressing (3.2) in Lagrangian coordinates we obtain

$$\begin{aligned}
&\overset{2}{\bar{v}}_t - \operatorname{div}_{\overset{2}{\bar{v}}}\mathbb{T}_{\overset{2}{\bar{v}}}(\overset{2}{\bar{v}},\bar{q}) = \overset{2}{\bar{v}}\cdot\nabla_{\overset{2}{\bar{v}}}\overset{2}{\bar{v}} \quad &&\text{in } \overset{2}{\Omega}{}_0^T,\\
&\operatorname{div}_{\overset{2}{\bar{v}}}\overset{2}{\bar{v}} = 0 &&\text{in } \overset{2}{\Omega}{}_0^T,\\
&\overset{2}{\bar{v}}|_{S_0} = \overset{1}{\bar{v}}|_{S_0}, \quad \overset{2}{\bar{v}}|_B = 0,\\
&\overset{2}{\bar{v}}|_{t=0} = \overset{2}{v}(0),
\end{aligned}\tag{5.5}$$

where $\overset{2}{v}(0)$ is a solution to problem (3.3)

$$\begin{aligned}
&-\Delta_\xi\overset{2}{v}(0) + \nabla_\xi q(0) = 0\\
&\operatorname{div}_\xi\overset{2}{v} = 0\\
&\overset{2}{v}(0)|_{S_0} = \overset{1}{v}(0)|_{S_0}, \quad \overset{2}{v}(0)|_B = 0
\end{aligned}\tag{5.6}$$

We prove existence of solutions to problem (5.5) by the following method of successive approximations

$$\begin{aligned}
&\overset{2}{\bar{v}}_{n+1,t} - \operatorname{div}_\xi\mathbb{T}_\xi(\overset{2}{\bar{v}}_{n+1},\bar{q}_{n+1}) = -(\operatorname{div}_\xi\mathbb{T}_\xi(\overset{2}{\bar{v}}_n,\bar{q}_n)\\
&\quad - \operatorname{div}_{\overset{2}{\bar{v}}_n}\mathbb{T}_{\overset{2}{\bar{v}}_n}(\overset{2}{\bar{v}}_n,\bar{q}_n) + \overset{2}{\bar{v}}_n\cdot\nabla_{\overset{2}{\bar{v}}_n}\overset{2}{\bar{v}}_n \quad &&\text{in } \overset{2}{\Omega}{}_0^T,\\
&\operatorname{div}_\xi\overset{2}{\bar{v}}_{n+1} = \operatorname{div}_\xi\overset{2}{\bar{v}}_n - \operatorname{div}_{\overset{2}{\bar{v}}_n}\overset{2}{\bar{v}}_n &&\text{in } \overset{2}{\Omega}{}_0^t,\\
&\overset{2}{\bar{v}}_{n+1}|_{S_0} = \overset{1}{\bar{v}}|_{S_0}, &&\text{on } S_0^T,\\
&\overset{2}{\bar{v}}_{n+1}|_B = 0 &&\text{on } B^T,\\
&\overset{2}{\bar{v}}_{n+1}|_{t=0} = \overset{2}{v}(0) &&\text{in } \overset{2}{\Omega}_0.
\end{aligned}\tag{5.7}$$

Lemma 5.2. *Let $\overset{1}{\bar{v}} \in H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^T)$, $\overset{2}{v}(0) \in H^{1+\alpha}(\overset{2}{\Omega}_0)$. Let $\alpha > 5/8$. Let $D = c(\|\overset{1}{\bar{v}}\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)} + \|\overset{2}{v}(0)\|_{H^{1+\alpha}(\overset{2}{\Omega}_0)})$. For T sufficiently small there exists a solution to problem (5.5) such that $\overset{2}{\bar{v}} \in H^{2+\alpha,1+\alpha/2}(\overset{2}{\Omega}{}_0^t)$, $t \le T$ and the estimate*

$$\|\overset{2}{\bar{v}}\|_{H^{2+\alpha,1+\alpha/2}(\overset{2}{\Omega}{}_0^t)} + \|\bar{q}_\xi\|_{H^{\alpha,\alpha/2}(\overset{2}{\Omega}{}_0^t)} \le \gamma D, \tag{5.8}$$

where $\gamma > 1$ holds.

Proof. We use the method of successive approximations described by (5.7). Using the techniques from the proofs of Lemmas 4.2 and 4.4 we obtain

$$\begin{aligned}
&\|\overset{2}{\bar{v}}_{n+1}\|_{V_2^{2+\alpha}(\overset{2}{\Omega}{}_0^t)} + \|\bar{q}_{n+1,\xi}\|_{H^{\alpha,\alpha/2}(\overset{2}{\Omega}{}_0^t)}\\
&\quad \le t^a\phi(\delta_{\overset{2}{\bar{v}}_n}(t))(\|\overset{2}{\bar{v}}_n\|_{V_2^{2+\alpha}(\overset{2}{\Omega}{}_0^t)} + \|\bar{q}_{n,\xi}\|_{H^{\alpha,\alpha/2}(\overset{2}{\Omega}{}_0^t)})\\
&\qquad + c\|\overset{1}{\bar{v}}\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)} + c\|\overset{2}{v}(0)\|_{H^{1+\alpha}(\overset{2}{\Omega}_0)},
\end{aligned}\tag{5.9}$$

where $a > 0$.

Let $\gamma > 1$. Assume that

$$X_n(t) \equiv \|\overset{2}{\bar{v}}_n\|_{V_2^{2+\alpha}(\overset{2}{\Omega}{}_0^t)} + \|\bar{q}_{n,\xi}\|_{H^{\alpha,\alpha/2}(\overset{2}{\Omega}{}_0^t)} \le \gamma D, \tag{5.10}$$

where

$$D = c(\|\overset{1}{\bar{v}}\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)} + \|\overset{2}{v}(0)\|_{H^{1+\alpha}(\overset{2}{\Omega}_0)}).$$

Then (5.9) yields

$$X_{n+1} \le t^a \phi(\gamma D)\gamma D + D. \tag{5.11}$$

Let t be so small that

$$t^a \phi(\gamma D)\gamma \le \gamma - 1.$$

Then (5.11) yields

$$X_{n+1} \le \gamma D. \tag{5.12}$$

Assuming that $X_0 = 0$ estimate (5.12) implies

$$X_n \le \gamma D \quad \text{for any } n \in \mathbb{N}. \tag{5.13}$$

Introducing differences

$$X_n = \overset{2}{\bar{v}}_n - \overset{2}{\bar{v}}_{n-1} + \bar{q}_{n,\xi} - \bar{q}_{n-1,\xi}$$

we can show convergence of the constructed sequence. This concludes the proof of Lemma 5.2. □

In D introduced in assumptions of Lemma 5.2 we have the term $\overset{2}{v}(0)$ which is a solution to problem (5.6). Therefore, we need

Lemma 5.3. *Assume that* $\overset{1}{v}(0) \in H^{1/2+\alpha}(S_0)$. *Then there exists a solution to* (5.6) *such that* $\overset{2}{v}(0) \in H^{1+\alpha}(\overset{2}{\Omega}_0)$ *and the estimate holds*

$$\|\overset{2}{v}(0)\|_{H^{1+\alpha}(\overset{2}{\Omega}_0)} \le c\|\overset{1}{v}(0)\|_{H^{1/2+\alpha}(S_0)} \le c\|\overset{1}{\bar{v}}\|_{H^{2+\alpha,1+\alpha/2}(\overset{1}{\Omega}{}_0^t)}. \tag{5.14}$$

Corollary 5.4. *Using* (5.8) *and* (5.14) *in* (5.1) *yields*

$$\begin{aligned}\sum_{i=1}^2 \|\overset{i}{\bar{H}}_{n+1}\|_{V_2^{2+\alpha}(\overset{i}{\Omega}{}_0^t)} &\le t^a(\phi(\delta_{\overset{1}{\bar{v}}}(t))\cdot \\ &\cdot \sum_{i=1}^2 (1 + \|\overset{1}{\bar{v}}_n\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)})\|\overset{i}{\bar{H}}_n\|_{V_2^{2+\alpha}(\overset{i}{\Omega}{}_0^t)}) \\ &+ c\sum_{i=1}^2 (\|H_{*i}\|_{H^{\frac{3}{2}+\alpha,\frac{3}{4}+\alpha/2}(B^t)} + \|\overset{i}{H}(0)\|_{H^{1+\alpha}(\overset{i}{\Omega}_0)}).\end{aligned} \tag{5.15}$$

6. Existence of Local Solutions to Problem (1.18), (1.20)

We prove existence of local solutions to problem (1.18), (1.20) by the method of successive approximations defined in Sect. 3.

Lemma 6.1. (uniform boundedness of the sequence introduced in Sect. 3) *Assume that the quantity*

$$\begin{aligned}D_0 &= \|f\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} + \|v(0)\|_{H^{1+\alpha}(\Omega_0)} \\ &+ \sum_{i=1}^2 (\|H_{*i}\|_{H^{3/2+\alpha,3/4+\alpha/2}(B^t)} + \|\overset{i}{H}(0)\|_{H^{1+\alpha}(\overset{i}{\Omega}_0)})\end{aligned} \tag{6.1}$$

is finite, where $\alpha > 5/8$.

Let

$$\begin{aligned} X_n(t) = \|\bar{v}_n\|_{V_2^{2+\alpha}(\overset{1}{\Omega}{}_0^t)} + \|\bar{p}_n\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} + \|\nabla\bar{p}_n\|_{H^{\alpha,\alpha/2}(\overset{1}{\Omega}{}_0^t)} \\ + \|\bar{p}_n\|_{H^{1/2+\alpha,1/4+\alpha/2}(S_0^t)} + \sum_{i=1}^2 \|\overset{i}{\bar{H}}_n\|_{V_2^{2+\alpha}(\overset{i}{\Omega}{}_0^t)}. \end{aligned} \tag{6.2}$$

Then for t sufficiently small the following bound holds

$$X_n(t) \le c_0\gamma D_0 \ \forall n \in \mathbb{N}, \tag{6.3}$$

where $\gamma > 1$ and c_0 is the constant which estimates constants in (4.59) and (5.15).

Proof. Inequalities (4.59) and (5.15) imply

$$X_{n+1}(t) \le t^a \phi(t^a X_n(t)) X_n(t) + c_0 D_0. \tag{6.4}$$

Let $\gamma > 1$. Assume that

$$X_n(t) \le \gamma c_0 D_0. \tag{6.5}$$

Let t be so small that

$$t^a \phi(t^a \gamma c_0 D_0)\gamma \le \gamma - 1 \tag{6.6}$$

Then (6.4) implies

$$X_{n+1}(t) \le \gamma c_0 D_o. \tag{6.7}$$

Assume that $X_0 = 0$. Then (6.5) and (6.7) imply (6.3). This ends the proof. □

To show convergence of the above sequence we need

Lemma 6.2. *Let the sequence $\{\bar{v}_n, \bar{p}_n, \overset{1}{\bar{H}}_n, \overset{2}{\bar{H}}_n\}$ be examined in Lemma 6.1. Let*

$$\begin{aligned} Y_n(t) = \|\bar{V}_n\|_{V_2^2(\overset{1}{\Omega}{}_0^t)} + \|\bar{P}_{n,\xi}\|_{L_2(\overset{1}{\Omega}{}_0^t)} + \|\bar{P}_n\|_{L_2(\overset{1}{\Omega}{}_0^t)} \\ + \|\bar{P}_n\|_{H^{1/2,1/4}(S_0^t)} + \sum_{i=1}^2 \|\overset{i}{\bar{K}}_n\|_{V_2^2(\overset{i}{\Omega}{}_0^t)} \end{aligned} \tag{6.8}$$

where

$$\bar{V}_n = \bar{v}_n - v_{n-1}, \quad \bar{P}_n = \bar{p}_n - \bar{p}_{n-1}, \quad \overset{i}{\bar{K}}_n = \overset{i}{\bar{H}}_n - \overset{i}{\bar{H}}_{n-1}, \quad j = 1, 2.$$

Then

$$Y_{n+1}(t) \le t^a \phi(c_0 D_0) Y_n, \tag{6.9}$$

where $a > 0$.

Proof. Taking the difference of (3.8) for n and for $n-1$ we obtain

$$\begin{aligned}
&\bar V_{n+1,t}-\operatorname{div}_\xi\mathbb{T}_\xi(\bar V_{n+1},\bar P_{n+1})\\
&\quad=-[\operatorname{div}_\xi\mathbb{T}_\xi(\bar V_n,\bar P_n)-\operatorname{div}_{\bar v_n}\mathbb{T}_{\bar v_n}(\bar V_n,\bar P_n)]\\
&\qquad+[\operatorname{div}_{\bar v_n}\mathbb{T}_{\bar v_n}(\bar v_{n-1},\bar p_{n-1})-\operatorname{div}_{\bar v_{n-1}}\mathbb{T}_{\bar v_{n-1}}(\bar v_{n-1},\bar p_{n-1})]\\
&\qquad+\mu_1[\operatorname{div}_{\bar v_n}\mathbb{T}(\overset{1}{\bar H}_n)-\operatorname{div}_{\bar v_n}\mathbb{T}(\overset{1}{\bar H}_{n-1})]\\
&\qquad+\mu_1[\operatorname{div}_{\bar v_n}\mathbb{T}(\overset{1}{\bar H}_{n-1})-\operatorname{div}_{\bar v_{n-1}}\mathbb{T}(\overset{1}{\bar H}_{n-1})]\equiv J_n\equiv\sum_{i=1}^4 I_i,\\
&\operatorname{div}_\xi\bar V_{n+1}=\operatorname{div}_\xi\bar V_n-[\operatorname{div}_{\bar v_n}\bar v_n-\operatorname{div}_{\bar v_n}\bar v_{n-1}]\\
&\qquad-[\operatorname{div}_{\bar v_n}\bar v_{n-1}-\operatorname{div}_{\bar v_{n-1}}\bar v_{n-1}]\equiv G_n\\
&\quad\equiv\operatorname{div}_\xi\bar V_n+I_5+I_6,\\
&\bar v_\xi\cdot\mathbb{T}_\xi(\bar V_{n+1},\bar P_{n+1})=\bar n_\xi\cdot\mathbb{T}_\xi(\bar V_n,\bar P_n)\\
&\qquad-(\bar n_{\bar v_n}\mathbb{T}_{\bar v_n}(\bar v_n,\bar p_n)-\bar n_{\bar v_{n-1}}\mathbb{T}_{\bar v_{n-1}}(\bar v_{n-1},\bar p_{n-1}))\\
&\qquad-\mu_1(\bar n_{\bar v_n}\cdot\mathbb{T}(\overset{1}{\bar H}_n)-\bar n_{\bar v_{n-1}}\cdot\mathbb{T}(\overset{1}{\bar H}_{n-1}))\equiv L_n\\
&\bar V_{n+1}|_{t=0}=0
\end{aligned}\tag{6.10}$$

Taking the difference of (3.9) for n and for $n-1$ yields

$$\begin{aligned}
\mu_1\overset{1}{\bar K}_{n+1,t}+\frac{1}{\sigma_1}\operatorname{rot}^2_\xi\overset{1}{\bar K}_{n+1}&=\frac{1}{\sigma_1}\operatorname{rot}^2_\xi\overset{1}{\bar K}_n-\frac{1}{\sigma_1}(\operatorname{rot}^2_{\overset{1}{\bar v}_n}\overset{1}{\bar H}_n-\operatorname{rot}^2_{\overset{1}{\bar v}_{n-1}}\overset{1}{\bar H}_{n-1})\\
&\quad+\mu_1(\operatorname{rot}_{\overset{1}{\bar v}_n}(\overset{1}{\bar v}_n\times\overset{1}{\bar H}_n)-\operatorname{rot}_{\overset{1}{\bar v}_{n-1}}(\overset{1}{\bar v}_{n-1}\times\overset{1}{\bar H}_{n-1})\\
&\quad+\mu_1(\overset{1}{\bar v}_n\cdot\nabla_{\overset{1}{\bar v}_n}\overset{1}{\bar H}_n-\overset{1}{\bar v}_{n-1}\cdot\nabla_{\overset{1}{\bar v}_{n-1}}\overset{1}{\bar H}_{n-1})\equiv\overset{1}{M}_n,\\
\operatorname{div}_\xi\overset{1}{\bar K}_{n+1}&=\operatorname{div}_\xi\overset{1}{\bar K}_n-(\operatorname{div}_{\overset{1}{\bar v}_n}\overset{1}{\bar H}_n-\operatorname{div}_{\overset{1}{\bar v}_{n-1}}\overset{1}{\bar H}_{n-1})\equiv\overset{1}{N}_n,\\
\mu_2\overset{2}{\bar K}_{n+1,t}+\frac{1}{\sigma_2}\operatorname{rot}^2_\xi\overset{2}{\bar K}_{n+1}&=\frac{1}{\sigma_2}\operatorname{rot}^2_\xi\overset{2}{\bar K}\frac{1}{\sigma_2}(\operatorname{rot}^2_{\overset{2}{\bar v}_n}\overset{2}{\bar H}_n-\operatorname{rot}_{\overset{2}{\bar v}_{n-1}}\overset{2}{\bar H}_{n-1})\\
&\quad+\mu_2(\overset{2}{\bar v}_n\cdot\nabla_{\overset{2}{\bar v}_n}\overset{2}{\bar H}_n-\overset{2}{\bar v}_{n-1}\cdot\nabla_{\overset{2}{\bar v}_{n-1}}\overset{2}{\bar H}_{n-1})\equiv\overset{2}{M}_n,\\
\operatorname{div}_\xi\overset{2}{\bar K}_{n+1}&=\operatorname{div}_\xi\overset{2}{\bar K}_n-(\operatorname{div}_{\overset{2}{\bar v}_n}\overset{2}{\bar H}_n-\operatorname{div}_{\overset{2}{\bar v}_{n-1}}\overset{2}{\bar H}_{n-1})\equiv\overset{2}{N}_n,\\
&\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi\overset{1}{\bar K}_{n+1}-\frac{1}{\sigma_2}\operatorname{rot}_\xi\overset{2}{\bar K}_{n+1}\right)\cdot\bar\tau_\alpha\\
&=\left(\frac{1}{\sigma_1}\operatorname{rot}_\xi\overset{1}{\bar K}_n-\frac{1}{\sigma_2}\operatorname{rot}_\xi\overset{2}{\bar K}_n\right)\cdot\bar\tau_\alpha\\
&\quad-\left(\frac{1}{\sigma_1}\operatorname{rot}_{\bar v_n}\overset{1}{\bar H}_n-\frac{1}{\sigma_1}\operatorname{rot}_{\bar v_{n-1}}\overset{1}{\bar H}_{n-1}\right)\cdot\bar\tau_\alpha\\
&\quad-\frac{1}{\sigma_2}(\operatorname{rot}_{\bar v_n}\overset{2}{\bar H}_n-\operatorname{rot}_{\bar v_{n-1}}\overset{2}{\bar H}_{n-1})\cdot\bar\tau_\alpha\\
&\quad+\frac{1}{\sigma_1}\operatorname{rot}_{\bar v_n}\overset{1}{\bar H}_n\cdot(\bar\tau_\alpha-\bar\tau_{\bar v_n\alpha})-\frac{1}{\sigma_1}\operatorname{rot}_{\bar v_{n-1}}\overset{1}{\bar H}_{n-1}\cdot(\tau_\alpha-\bar\tau_{\bar v_{n-1\alpha}})
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\sigma_2}\mathrm{rot}_{\bar v_n}\overset{2}{\bar H}_n\cdot(\bar\tau_\alpha-\bar\tau_{\bar v_n\alpha})+\frac{1}{\sigma_2}\mathrm{rot}_{\bar v_{n-1}}\overset{2}{\bar H}_{n-1}\cdot(\bar\tau_\alpha-\bar\tau_{\bar v_{n-1}\alpha})\\
&+\mu_1\bar v_n\times\overset{1}{\bar H}_n\cdot\bar\tau_{\bar v_n\alpha}-\mu_1\bar v_{n-1}\times\overset{1}{\bar H}_{n-1}\cdot\bar\tau_{\bar v_{n-1}\alpha}\equiv\overset{1}{P}_n,\\
(\overset{1}{\bar K}_{n+1}-\overset{2}{\bar K}_{n+1})\cdot\bar n\times\bar\tau_\alpha&=(\overset{1}{\bar K}_n-\overset{2}{\bar K}_n)\cdot\bar n\times\bar\tau_\alpha-(\overset{1}{\bar K}_n-\overset{2}{\bar K}_n)\cdot\bar n_{\bar v_n}\times\bar\tau_{\bar v_n\alpha}\\
&-(\overset{1}{\bar H}_{n-1}-\overset{2}{\bar H}_{n-1})\cdot(\bar n_{\bar v_n}\times\bar\tau_{\bar v_n\alpha}-\bar n_{\bar v_{n-1}}\times\bar\tau_{\bar v_{n-1}\alpha})\equiv\overset{2}{P}_n,\\
&\overset{2}{\bar K}_{n+1}\cdot\bar\tau_\alpha|_B=0,\quad\alpha=1,2,\\
\mathrm{div}_{\overset{2}{\bar v}_n}\overset{2}{\bar K}_{n+1}&=-(\mathrm{div}_{\overset{2}{\bar v}_n}\overset{2}{\bar H}_n-\mathrm{div}_{\overset{2}{\bar v}_{n-1}}\overset{2}{\bar H}_n)\equiv Q_n,\\
&\overset{i}{\bar K}_{n+1}|_{t=0}=0.
\end{aligned}\tag{6.11}
$$

First we examine problem (6.10).

Let Φ_n be a solution to the problem

$$
\begin{aligned}
&\Delta\Phi_n=G_n &&\text{in }\ \overset{1}{\Omega}_0,\\
&\Phi_n=0 &&\text{on }\ S_0.
\end{aligned}\tag{6.12}
$$

There exists the Green function $G(x,y)$ to problem (6.12) such that $G(x,y)|_{S_0}=0$ and

$$
\Phi_n(x,t)=\int\limits_{\overset{1}{\Omega}_0}G(x,y)G_n(y,t)dy.\tag{6.13}
$$

Applying Lemma 2.3.2 with estimate (2.3.9) to problem (6.10) yields

$$
\begin{aligned}
\|\bar V_{n+1}\|_{V_2^2(\overset{1}{\Omega}{}_0^t)}&\le c(\|\nabla\Phi_n\|_{H^{2,1}(\overset{1}{\Omega}{}_0^t)}\\
&+\|J_n\|_{L_2(\overset{1}{\Omega}{}_0^t)}+\|\nabla G_n\|_{L_2(\overset{1}{\Omega}{}_0^t)}+\|L_n\|_{H^{1/2,1/4}(S_0^t)}).
\end{aligned}\tag{6.14}
$$

We estimate a few terms from the r.h.s. of (6.14)

$$
\begin{aligned}
&\|\mathrm{div}_\xi\mathbb{T}_\xi(\bar V_n,\bar P_n)-\mathrm{div}_{\bar v_n}\mathbb{T}_{\bar v_n}(\bar V_n,\bar P_n)\|_{L_2(\overset{1}{\Omega}{}_0^t)}\\
&\le\phi(\delta_{\bar v_n}(t))\bigg(\bigg\|\int_0^t\bar v_{n,\xi}(\xi,\tau)d\tau\bar V_{n,\xi\xi}\bigg\|_{L_2(\overset{1}{\Omega}{}_0^t)}\\
&\quad+\bigg\|\int_0^t\bar v_{n,\xi\xi}(\xi,\tau)d\tau\bar V_{n,\xi}\bigg\|_{L_2(\overset{1}{\Omega}{}_0^t)}+\bigg\|\int_0^t\bar v_{n,\xi}(\xi,\tau)d\tau P_{n,\xi}\bigg\|_{L_2(\overset{1}{\Omega}{}_0^t)}\\
&\le t^a\phi(\delta_{\bar v_n}(t))\bigg(\int_0^t\|\bar v_n\|^2_{H^{2+\alpha}(\overset{1}{\Omega}_0)}d\tau\bigg)^{1/2}(\|\bar V_{n,\xi}\|_{L_2(0,t;H^1(\overset{1}{\Omega}_0))}+\|\bar P_{n,\xi}\|_{L_2(\overset{1}{\Omega}{}_0^t)}).
\end{aligned}
$$

Continuing the considerations we derive

$$
\begin{aligned}
&\|\bar V_{n+1}\|_{V_2^2(\overset{1}{\Omega}{}_0^t)}+\|\bar P_{n+1,\xi}\|_{L_2(\overset{1}{\Omega}{}_0^t)}+\|\bar P_{n+1}\|_{L_2(\overset{1}{\Omega}{}_0^t)}+\|\bar P_{n+1}\|_{H^{1,1/4}(S_0^t)}\\
&\quad\le t^a\phi(c_0D_0)(\|\bar V_n\|_{V_2^2(\overset{1}{\Omega}{}_0^t)}+\|\bar P_{n,\xi}\|_{L_2(\overset{1}{\Omega}{}_0^t)}+\|\bar P_n\|_{L_2(\overset{1}{\Omega}{}_0^t)}\\
&\qquad+\|\bar P_n\|_{H^{1/2,1/4}(S_0^t)}+\|\overset{1}{\bar K}_n\|_{V_2^2(\overset{1}{\Omega}{}_0^t)}).
\end{aligned}\tag{6.15}
$$

Applying Lemma 2.5.1 in the case $\alpha = 0$ to problem (6.11) yields

$$\begin{aligned}\sum_{i=1}^{2}\|\stackrel{i}{\bar K}_{n+1}\|_{V_2^2(\stackrel{i}{\Omega}{}_0^t)} &\le c\sum_{i=1}^{2}(\|\stackrel{i}{M}_n\|_{L_2(\stackrel{i}{\Omega}{}_0^t)}\\ &\quad+\|\nabla\stackrel{i}{N}_n\|_{L_2(\stackrel{i}{\Omega}{}_0^t)})+c(\|\stackrel{1}{P}_n\|_{H^{\frac12,\frac14}(S_0^t)}+\|\stackrel{2}{P}_n\|_{H^{\frac32,\frac34}(S_0^t)}\\ &\quad+\|Q_n\|_{H^{1/2,1/4}(B^t)}).\end{aligned} \tag{6.16}$$

We shall estimate some terms from the r.h.s. of (6.16). Hence, we examine the term

$$\begin{aligned}I &\equiv \|\mathrm{rot}\,{}^2_{\xi}\stackrel{1}{\bar K}_n-(\mathrm{rot}\,{}^2_{\stackrel{1}{\bar v}_n}\stackrel{1}{\bar H}_n-\mathrm{rot}\,{}^2_{\stackrel{1}{\bar v}_{n-1}}\stackrel{1}{\bar H}_{n-1})\|_{L_2(\stackrel{1}{\Omega}{}_0^t)}\\ &=\|\mathrm{rot}\,{}^2_{\xi}\stackrel{1}{\bar K}_n-\mathrm{rot}\,{}^2_{\stackrel{1}{\bar v}_n}\stackrel{1}{\bar K}_n-(\mathrm{rot}\,{}^2_{\stackrel{1}{\bar v}_n}\stackrel{1}{\bar H}_{n-1}-\mathrm{rot}\,{}^2_{\stackrel{1}{\bar v}_{n-1}}\stackrel{1}{\bar H}_{n-1})\|_{L_2(\stackrel{1}{\Omega}{}_0^t)}\\ &\le\|\mathrm{rot}\,{}^2_{\xi}\stackrel{1}{\bar K}_n-\mathrm{rot}\,{}^2_{\stackrel{1}{\bar v}_n}\stackrel{1}{\bar K}_n\|_{L_2(\stackrel{1}{\Omega}{}_0^t)}+\|\mathrm{rot}\,{}^2_{\stackrel{1}{\bar v}_n}\stackrel{1}{\bar H}_{n-1}-\mathrm{rot}\,{}^2_{\stackrel{1}{\bar v}_{n-1}}\stackrel{1}{\bar H}_{n-1}\|_{L_2(\stackrel{1}{\Omega}{}_0^t)}\\ &\le\phi(\delta_{\stackrel{1}{\bar v}_n}(t))\bigg\|\int_0^t\stackrel{1}{\bar v}_{n,\xi}(\xi,\tau)d\tau\stackrel{1}{\bar K}_{n,\xi\xi}\bigg\|_{L_2(\stackrel{1}{\Omega}{}_0^t)}\\ &\quad+\phi(\delta_{\stackrel{1}{\bar v}_n}(t))\bigg\|\int_0^t\stackrel{1}{\bar v}_{n,\xi\xi}(\xi,\tau)d\tau\stackrel{1}{\bar K}_{n,\xi}\bigg\|_{L_2(\stackrel{1}{\Omega}{}_0^t)}\\ &\quad+\phi(\delta_{\stackrel{1}{\bar v}_n}(t),\delta_{\stackrel{1}{\bar v}_{n-1}}(t))\bigg\|\int_0^t\stackrel{1}{\bar V}_{n,\xi}(\xi,\tau)d\tau\stackrel{1}{\bar H}_{n-1,\xi\xi}\bigg\|_{L_2(\stackrel{1}{\Omega}{}_0^t)}\\ &\quad+\phi(\delta_{\stackrel{1}{\bar v}_n}(t),\delta_{\stackrel{1}{\bar v}_{n-1}}(t))\bigg\|\int_0^t\stackrel{1}{\bar V}_{n,\xi\xi}(\xi,\tau)d\tau\stackrel{1}{\bar H}_{n-1,\xi}\bigg\|_{L_2(\stackrel{1}{\Omega}{}_0^t)}\\ &\equiv\sum_{i=1}^{4}I_i,\end{aligned}$$

where

$$I_1\le t^{1/2}\phi(\delta_{\stackrel{1}{\bar v}_n}(t))\bigg(\int_0^t\|\stackrel{1}{\bar v}_{n,\xi}(\cdot,\tau)\|^2_{L_\infty(\stackrel{1}{\Omega}_0)}d\tau\bigg)^{1/2}\|\stackrel{1}{\bar K}_n\|_{L_2(0,t;H^2(\stackrel{1}{\Omega}_0))}$$

and applying the Hölder inequality with respect to ξ yields

$$\begin{aligned}I_2&\le t^{1/2}\phi(\delta_{\stackrel{1}{\bar v}_n}(t))\bigg(\int_0^t\|\stackrel{1}{\bar v}_{n,\xi\xi}(\cdot,\tau)\|^2_{L_{2p}(\stackrel{1}{\Omega}_0}d\tau\bigg)^{1/2}\cdot\\ &\quad\cdot\bigg(\int_0^t\|\stackrel{1}{\bar K}_{n,\xi}\|^2_{L_{2p'}(\stackrel{1}{\Omega}_0)}dt'\bigg)^{1/2}\equiv I_2^1,\end{aligned}$$

where $1/p+1/p'=1$. Using the imbeddings

$$\begin{aligned}&\|u\|_{L_{2p}(\stackrel{1}{\Omega}_0)}\le c\|u\|_{H^\alpha(\stackrel{1}{\Omega}_0)}, &&\frac32-\frac3{2p}\le\alpha,\\ &\|u\|_{L_{2p'}(\stackrel{1}{\Omega}_0)}\le c\|u\|_{H^1(\stackrel{1}{\Omega}_0)}, &&\frac32-\frac3{2p'}\le1.\end{aligned} \tag{6.17}$$

The above imbeddings can hold together because they imply the restriction

$$\frac{3}{2} \le 1 + \alpha$$

which holds for $\alpha > 1/2$. Then

$$I_2^1 \le t^{1/2}\phi(\delta_{\overset{1}{\bar v}_n}(t))\bigg(\int_0^t \|\overset{1}{\bar v}_n(\cdot,\tau)\|^2_{H^{2+\alpha}(\overset{1}{\Omega}_0)}d\tau\bigg)\cdot\|\overset{1}{\bar K}_n\|_{V_2^2(\overset{1}{\Omega}{}_0^t)}.$$

Next, we estimate

$$\begin{aligned} I_3 \le{}& t^{1/2}\phi(\delta_{\overset{1}{\bar v}_n}(t),\delta_{\overset{1}{\bar v}_{n-1}}(t))\bigg(\int_0^t \|\overset{1}{\bar V}_{n,\xi}(\cdot,\tau)\|_{L_{2p'}(\overset{1}{\Omega}_0)}d\tau\bigg)^{1/2}\cdot \\ &\cdot\bigg(\int_0^t \|\overset{1}{\bar H}_{n-1,\xi\xi}(\cdot,\tau')\|^2_{L_{2p}(\overset{1}{\Omega}_0)}dt'\bigg)^{1/2} \equiv I_3^1. \end{aligned}$$

Applying imbeddings (6.17) yields

$$\begin{aligned} I_3^1 \le{}& t^{1/2}\phi(\delta_{\overset{1}{\bar v}_n}(t),\delta_{\overset{1}{\bar v}_{n-1}}(t))\bigg(\int_0^t \|\overset{1}{\bar V}_n(\cdot,\tau)\|^2_{H^2(\overset{1}{\Omega}_0)}d\tau\bigg)^{1/2}\cdot \\ &\cdot\bigg(\int_0^t \|\overset{1}{\bar H}_{n-1}(\cdot,t')\|^2_{H^{2+\alpha}(\overset{1}{\Omega}_0)}dt'\bigg)^{1/2}. \end{aligned}$$

Finally,

$$\begin{aligned} I_4 \le{}& t^{1/2}\phi(\delta_{\overset{1}{\bar v}_n}(t),\delta_{\overset{1}{\bar v}_{n-1}}(t))\bigg(\int_0^t \|\overset{1}{\bar V}_{n,\xi\xi}(\cdot,\tau)\|^2_{L_2(\overset{1}{\Omega}_0)}d\tau\bigg)^{1/2}\cdot \\ &\cdot\bigg(\int_0^t \|\overset{1}{\bar H}_{n-1,\xi}(\cdot,t')\|^2_{L_\infty(\overset{1}{\Omega}_0)}dt'\bigg)^{1/2}. \end{aligned}$$

Summarizing and using Lemma 6.1 we obtain

$$I \le t^{1/2}\phi(c_0D_0)(\|\overset{1}{\bar K}_n\|_{L_2(0,t;H^2(\overset{1}{\Omega}_0))} + \|\overset{1}{\bar V}_n\|_{L_2(0,t;H^2(\overset{1}{\Omega}_0))}).$$

Similar considerations can be applied to other terms from the r.h.s. of (6.16).

Estimating the terms from the r.h.s. of (6.16) we obtain

$$\begin{aligned} \sum_{i=1}^2 \|\overset{i}{\bar K}_{n+1}\|_{V_2^2(\overset{i}{\Omega}{}_0^t)} \le{}& t^a\phi(c_0D_0)\bigg(\sum_{i=1}^2 \|\overset{i}{\bar K}_n\|_{V_2^2(\overset{i}{\Omega}{}_0^t)} \\ &+ \|\bar V_n\|_{V_2^2(\overset{i}{\Omega}{}_0^t)}\bigg), \end{aligned} \tag{6.18}$$

where $a > 0$. Estimates (6.15) and (6.18) imply (6.9). This ends the proof. □

7. Problem (2.5.2) in a Neighborhood of S_0

In this Section we consider problem (2.5.2) localized to a neighborhood of any pooint of S_0. The problem is complicated because it describes interaction of magnetic fields through the free surface S_0. The localization means that problem (2.5.2) is multiplied by a function ζ from the partition of unity with a support in a neighborhood of a point of S_0. Then we obtain a problem for functions $\overset{i}{\bar{H}}\zeta$, $i=1,2$. Next, we apply a transformation which makes S_0 locally flat. Moreover, we consider a more general transmission condition. Therefore, we derive the problem

$$\begin{aligned} &\mu_1\overset{1}{u}_{,t}-\frac{1}{\sigma_1}\Delta_z\overset{1}{u}=\overset{1}{\bar{f}} && z_3>0,\\ &\operatorname{div}\overset{1}{u}=0 && z_3>0,\\ &\mu_2\overset{2}{u}_{,t}-\frac{1}{\sigma_2}\Delta_z\overset{2}{u}=\overset{2}{\bar{f}} && z_3<0,\\ &\operatorname{div}\overset{2}{u}=0 && z_3<0, \end{aligned} \tag{7.1}$$

with transmission conditions

$$\begin{aligned} &(a_1\operatorname{rot}_z\overset{1}{u}-a_2\operatorname{rot}_z\overset{2}{u})\cdot\bar{\tau}_\alpha=\bar{k}_\alpha, \quad \alpha=1,2, \qquad z_3=0,\\ &b_1\overset{1}{u}_\alpha-b_2\overset{2}{u}_\alpha=\bar{l}_\alpha, \quad \alpha=1,2, \quad z_3=0, \end{aligned} \tag{7.2}$$

where a_i, b_i, $i=1,2$, are some constants, $\bar{\tau}_1=(1,0,0)$, $\bar{\tau}_2=(0,1,0)$ and with the initial conditions

$$\overset{i}{u}|_{t=0}=\overset{i}{u}(0), \quad i=1,2, \tag{7.3}$$

where $\overset{i}{u}(0)$, $i=1,2$, are divergence free.

Since problem (7.1)–(7.3) is derived by the localization all data functions $(\overset{i}{\bar{f}},\bar{k}_i,\bar{l}_i,\overset{i}{u}(0),i=1,2)$ have compact supports.

In this Section we find estimates for solutions to problem (7.1)–(7.3) assuming that $\overset{i}{\bar{f}}\in H^{\alpha,\alpha/2}(\mathbb{R}^3_i\times(0,T))$, $\mathbb{R}^3_1=\{z\colon z_3>0\}$, $\mathbb{R}^3_2=\{z\colon z_3<0\}$, $\bar{k}_i\in H^{1+\alpha-1/2,1/2+\alpha/2-1/4}(\mathbb{R}^2)$, $\bar{l}_i\in H^{2+\alpha-1/2,1+\alpha/2-1/4}(\mathbb{R}^2)$, $\mathbb{R}^2=\{z\colon z_3=0\}$, $\overset{i}{u}(0)\in H^{1+\alpha}(\mathbb{R}^3_i)$, $i=1,2$, $\alpha\in(0,1)$.

Remark 7.1. Functions $\overset{i}{\bar{f}}$, $i=1,2$, are divergence free. This follows from the following construction of problem (7.1). Multiply equations $(2.5.1)_1$–$(2.5.1)_4$ by cut-off functions $\overset{1}{\zeta}$ and $\overset{2}{\zeta}$, respectively. Introduce the notation $\overset{i}{\bar{v}}=\overset{i}{H}\overset{i}{\zeta}$, $i=1,2,$. Then $(2.5.1)_2$ and $(2.5.1)_4$ take the form

$$\operatorname{div}\overset{i}{\bar{v}}=\overset{i}{H}\cdot\nabla\overset{i}{\zeta}, \quad i=1,2.$$

Introducing function $\overset{i}{\varphi}$ as a solution to the problem

$$\begin{aligned} &\Delta\overset{i}{\varphi}=\overset{i}{H}\cdot\nabla\overset{i}{\zeta}, \quad i=1,2,\\ &\varphi^i=0 \quad \text{on} \quad \partial\operatorname{supp}\overset{i}{\zeta}, \quad i=1,2, \end{aligned}$$

we construct the function

$$\overset{i}{u}=\overset{i}{\bar{v}}-\nabla\overset{i}{\varphi}, \quad i=1,2,$$

which is divergence free and satisfies the equation

$$\begin{aligned} \mu_i \overset{i}{u}_{,t} - \frac{1}{\sigma_i}\Delta \overset{i}{u} &= -\mu_i \nabla \overset{i}{\varphi}_{,t} + \frac{1}{\sigma_i}\Delta\nabla\overset{i}{\varphi} \\ &\quad - \frac{1}{\sigma_i}(2\nabla \overset{i}{H}\cdot\nabla\overset{i}{\zeta} + \overset{i}{H}\Delta\overset{i}{\zeta}) + \delta_{1i}\mathrm{rot}\,(\overset{1}{v}\times\overset{1}{H})\overset{1}{\zeta}\nabla\overset{i}{\psi} \equiv \overset{i}{f}, \end{aligned} \tag{7.4}$$

where

$$\begin{aligned} \Delta\overset{i}{\psi} &= \mu_i\Delta\overset{i}{\varphi}_{,t} - \frac{1}{\sigma_i}\Delta^2\overset{i}{\varphi} + \frac{1}{\sigma_i}(2\nabla_j\overset{i}{H}_k\nabla_k\nabla_j\overset{i}{\zeta} + \overset{i}{H}_k\nabla_k\Delta\overset{i}{\zeta}) \\ &\quad - \delta_{i1}\mathrm{rot}\,(v\times\overset{1}{H})\cdot\nabla\overset{i}{\zeta}, \\ \overset{i}{\psi} &= 0 \quad \text{on} \quad \partial\mathrm{supp}\,\overset{i}{\zeta}, \end{aligned}$$

$i=1,2$, δ_{1i} is the Kronecker delta and $\overset{i}{f}$, $i=1,2$, are divergence free. Passing to such coordinates that S_0 becomes flat we derive system (7.1). Functions $\overset{i}{\psi}$ depend on $\nabla\overset{i}{H}$, $\overset{i}{H}$, $i=1,2$, so the dependence is not important in the proof of existence of solutions by the technique of regularizer. This ends Remark 7.1.

To make the initial data homogeneous we construct divergence free extensions $\overset{i}{\tilde{u}}$, $i=1,2$, of initial data $\overset{i}{u}(0)$, $i=1,2$, such that

$$\overset{i}{\tilde{u}}|_{t=0} = \overset{i}{u}(0), \quad i=1,2. \tag{7.5}$$

Set

$$\overset{i}{v} = \overset{i}{u} - \overset{i}{\tilde{u}}, \quad i=1,2. \tag{7.6}$$

Then problem (7.1)–(7.3) takes the form

$$\begin{aligned} &\sigma_i\mu_i\overset{i}{v}_{,t} - \Delta\overset{i}{v} = \overset{i}{\sigma}\overset{i}{\bar{f}} - (\sigma_i\mu_i\overset{i}{\tilde{u}}_t - \Delta\overset{i}{\tilde{u}}) \equiv \overset{i}{f}, \quad i=1,2, \\ &\mathrm{div}\,\overset{i}{v} = 0, \\ &(a_1\mathrm{rot}\,\overset{1}{v} - a_2\mathrm{rot}\,\overset{2}{v})\cdot\bar{\tau}_\alpha \\ &= \bar{k}_\alpha - (a_1\mathrm{rot}\,\overset{1}{\tilde{u}} - a_2\mathrm{rot}\,\overset{2}{\tilde{u}})\cdot\bar{\tau}_\alpha \equiv k'_\alpha, \quad \alpha=1,2, \quad z_3=0, \\ &b_1\overset{1}{v}_\alpha - b_2\overset{2}{v}_\alpha = \bar{l}_\alpha - (b_1\overset{1}{\tilde{u}}_\alpha - b_2\overset{2}{\tilde{u}}_\alpha) \equiv l'_\alpha, \quad \alpha=1,2, \quad z_3=0. \end{aligned} \tag{7.7}$$

Expressing the transimission condition explicitly and extending (7.7) for $t<0$ we have

$$\begin{aligned} &\sigma_1\mu_1\overset{1}{v}_{,t} - \Delta\overset{1}{v} = \overset{1}{f}, \quad \mathrm{div}\,\overset{1}{v} = 0, && z_3>0, \\ &\sigma_2\mu_2\overset{2}{v}_{,t} - \Delta\overset{2}{v} = \overset{2}{f}, \quad \mathrm{div}\,\overset{2}{v} = 0, && z_3<0, \\ &a_1(\overset{1}{v}_{2,z_3} - \overset{1}{v}_{3,z_2}) - a_2(\overset{2}{v}_{2,z_3} - \overset{2}{v}_{3,z_2}) = k'_1, && z_3=0, \\ &a_1(\overset{1}{v}_{3,z_1} - \overset{1}{v}_{1,z_3}) - a_2(\overset{2}{v}_{3,z_1} - \overset{2}{v}_{1,z_3}) = k'_2, && z_3=0, \\ &b_1\overset{1}{v}_i - b_2\overset{2}{v}_i = l'_i, \quad i=1,2, && z_3=0. \end{aligned} \tag{7.8}$$

Since $\overset{i}{f} \in H^{\alpha,\alpha/2}(\mathbb{R}^3_i \times \mathbb{R}_+)$, $\alpha \in (0,1)$, we can extend them by zero on $\mathbb{R}^3$, respectively. We denote the extnsions by $\overset{i}{f}{}'$, $i = 1,2$. Then we are looking for solutions to the problems

$$
\begin{aligned}
&\sigma_1\mu_1 \overset{1}{v}{}'_{,t} - \Delta \overset{1}{v}{}' = \overset{1}{f}{}', \quad \operatorname{div} \overset{1}{v}{}' = 0 \qquad \text{in } \ \mathbb{R}^3,\\
&\sigma_2\mu_2 \overset{2}{v}{}'_{,t} - \Delta \overset{2}{v}{}' = \overset{2}{f}{}', \quad \operatorname{div} \overset{2}{v}{}' = 0 \qquad \text{in } \ \mathbb{R}^3.
\end{aligned} \tag{7.9}
$$

Then the functions

$$
\overset{i}{w} = \overset{i}{v} - \overset{i}{v}{}', \quad i = 1,2, \tag{7.10}
$$

are solutions to the problem with vanishing initial data

$$
\begin{aligned}
&\sigma_1\mu_1 \overset{1}{w}_{,t} - \Delta \overset{1}{w} = 0, \quad \operatorname{div} \overset{1}{w} = 0, && z_3 > 0,\\
&\sigma_2\mu_2 \overset{2}{w}_{,t} - \Delta \overset{2}{w} = 0, \quad \operatorname{div} \overset{2}{w} = 0, && z_3 < 0,\\
&a_1(\overset{1}{w}_{2,z_3} - \overset{1}{w}_{3,z_2}) - a_2(\overset{2}{w}_{2,z_3} - \overset{2}{w}_{3,z_2}) &&\\
&\quad = k_1' - a_1(\overset{1}{v}{}'_{2,z_3} - \overset{1}{v}{}'_{3,z_2}) + a_2(\overset{2}{v}{}'_{2,z_3} - \overset{3}{v}{}'_{1,z_2}) \equiv k_1, && z_3 = 0,\\
&a_1(\overset{1}{w}_{3,z_1} - \overset{1}{w}_{1,z_3}) - a_2(\overset{2}{w}_{3,z_1} - \overset{2}{w}_{1,z_3}) &&\\
&\quad = k_2' - a_1(\overset{1}{v}{}'_{3,z_1} - \overset{1}{v}{}'_{1,z_3}) + a_2(\overset{2}{v}{}'_{3,z_1} - \overset{2}{v}{}'_{1,z_3}) \equiv k_2, && z_3 = 0,\\
&b_1\overset{1}{w}_i - b_2\overset{2}{w}_i = l_i' - (b_1\overset{1}{v}{}'_i - b_2\overset{2}{v}{}'_i \equiv l_i, \quad i = 1,2, && z_3 = 0.
\end{aligned} \tag{7.11}
$$

Lemma 7.2. *Assume that* $(\overset{1}{w}, \overset{2}{w})$ *is a solution to* (7.11). *Let* $\alpha \in (0,1)$. *Assume that* $k_i \in H^{1/2+\alpha,1/4+\alpha/2}$ $(\mathbb{R}^2 \times \mathbb{R}_+)$, $i = 1,2$, $l_i \in H^{3/2+\alpha,3/4+\alpha/2}(\mathbb{R}^2 \times \mathbb{R}_+)$, $i = 1,2$.

Then there exists a solution to problem (7.11) *such that* $\overset{i}{w} \in H^{2+\alpha,1+\alpha/2}(\mathbb{R}^3_i \times \mathbb{R}_+)$, $\overset{i}{w}|_{t=0} = 0$, $i = 1,2$, *and*

$$
\begin{aligned}
&\sum_{i=1}^2 \|\overset{i}{w}\|_{H^{2+\alpha,1+\alpha/2}(\mathbb{R}^3_i \times \mathbb{R}_+)}\\
&\quad \le c\sum_{i=1}^2 \|k_i\|_{H^{1/2+\alpha,1/4+\alpha/2}(\mathbb{R}^2 \times \mathbb{R}_+)}\\
&\qquad + c\sum_{i=1}^2 \|l_i\|_{H^{3/2+\alpha,3/4+\alpha/2}(\mathbb{R}^2 \times \mathbb{R}_+)}.
\end{aligned} \tag{7.12}
$$

Proof. Apply the Fourier-Laplace transform

$$
(Ff)(\xi, z_3, s) = \tilde{f}(\xi, z_3, s) = \int_0^\infty e^{-st} \int_{\mathbb{R}^2} f(z', z_3, t) e^{-i\xi\cdot z'} dz' dt, \tag{7.13}
$$

$\operatorname{Re} s > 0$, $s = i\xi_0 + \gamma$, $\xi = (\xi_1, \xi_2)$, $z' = (z_1, z_2)$, $z' \cdot \xi = z_1\xi_1 + z_2\xi_2$, $z = (z_1, z_2, z_3)$, to problem (7.11). Then we have

$$\begin{aligned} &\tau_1^2 \overset{1}{\tilde{w}} - \overset{1}{\tilde{w}}_{z_3,z_3} = 0, \quad i\xi_\alpha \overset{1}{\tilde{w}}_\alpha + \overset{1}{\tilde{w}}_{3,z_3} = 0, && z_3 > 0,\\ &\tau_2^2 \overset{2}{\tilde{w}} - \overset{2}{\tilde{w}}_{z_3z_3} = 0, \quad i\xi_\alpha \overset{2}{\tilde{w}}_\alpha + \overset{2}{\tilde{w}}_{3,z_3} = 0, && z_3 < 0,\\ &a_1(\overset{1}{\tilde{w}}_{2,z_3} - i\xi_2 \overset{1}{\tilde{w}}_3) - a_2(\overset{2}{\tilde{w}}_{2,z_3} - i\xi_2\overset{2}{\tilde{w}}_3) = k_1, && z_3 = 0,\\ &a_1(i\xi_1 \overset{1}{\tilde{w}}_3 - \overset{1}{\tilde{w}}_{1,z_3}) - a_2(i\xi_1\overset{2}{\tilde{w}}_3 - \overset{2}{\tilde{w}}_{1,z_3}) = k_2, && z_3 = 0,\\ &b_1\overset{1}{\tilde{w}}_i - b_2\overset{2}{\tilde{w}}_i = l_i, \quad i = 1,2, && z_3 = 0, \end{aligned} \tag{7.14}$$

where $\tau_1^2 = \mu_1\sigma_1 s + \xi^2$, $\tau_2^2 = \mu_2\sigma_2 s + \xi^2$.

Solving $(7.14)_{1,2}$ and using the Shapiro-Lopatinskii condition we obtain

$$\begin{aligned} &\overset{1}{\tilde{w}} = \overset{1}{A} e^{-\tau_1 z_3}, && \overset{2}{\tilde{w}} = \overset{2}{A} e^{\tau_2 z_3},\\ &-\tau_1 \overset{1}{A}_3 + i\xi_\alpha \overset{1}{A}_\alpha = 0, && \tau_2\overset{2}{A}_3 + i\xi_\alpha\overset{2}{A}_\alpha = 0. \end{aligned} \tag{7.15}$$

Inserting $(7.15)_1$ in the transmission conditions $(7.14)_{3,4,5}$ yields

$$\begin{aligned} &a_1(-\tau_1\overset{1}{A}_2 - i\xi_2\overset{1}{A}_3) - a_2(\tau_2\overset{2}{A}_2 - i\xi_2\overset{2}{A}_3) = \tilde{k}_1,\\ &a_1(i\xi_1\overset{1}{A}_3 + \tau_1\overset{1}{A}_1) - a_2(i\xi_1\overset{2}{A}_3 - \tau_2\overset{2}{A}_1) = \tilde{k}_2,\\ &b_1\overset{1}{A}_j - b_2\overset{2}{A}_j = \tilde{l}_j, \quad j = 1,2. \end{aligned} \tag{7.16}$$

Using (7.15), we get

$$\begin{aligned} &\frac{a_1}{\tau_1}[(\xi_2^2 - \tau_1^2)\overset{1}{A}_2 + \xi_1\xi_2\overset{1}{A}_1] - \frac{a_2}{\tau_2}[(\tau_2^2 - \xi_2^2)\overset{2}{A}_2 - \xi_1\xi_2\overset{2}{A}_1] = \tilde{k}_1,\\ &\frac{a_1}{\tau_1}[(\tau_1^2 - \xi_1^2)\overset{1}{A}_1 - \xi_1\xi_2\overset{1}{A}_2] - \frac{a_2}{\tau_2}[(\xi_1^2 - \tau_2^2)\overset{2}{A}_1 + \xi_1\xi_2\overset{2}{A}_2] = \tilde{k}_2,\\ &b_1\overset{1}{A}_j - b_2\overset{2}{A}_j = \tilde{l}_j, \quad j = 1,2. \end{aligned} \tag{7.17}$$

Using $(7.17)_3$ in $(7.17)_{1,2}$ and setting

$$d_1 = \frac{a_1b_2}{\tau_1} + \frac{a_2b_1}{\tau_2}, \quad d_2 = a_1b_2\tau_1 + a_2b_1\tau_2 \tag{7.18}$$

we obtain

$$\begin{aligned} &-(d_2 - d_1\xi_2^2)\overset{2}{A}_2 + d_1\xi_1\xi_2\overset{2}{A}_1 = \tilde{h}_1,\\ &-d_1\xi_1\xi_2\overset{2}{A}_2 + (d_2 - d_1\xi_1^2)\overset{2}{A}_1 = \tilde{h}_2, \end{aligned} \tag{7.19}$$

where

$$\begin{aligned} &\tilde{h}_1 = b_1\tilde{k}_1 - \frac{a_1}{\tau_1}(\xi_2^2 - \tau_1^2)\tilde{l}_2 - \frac{a_1}{\tau_1}\xi_1\xi_2\tilde{l}_1,\\ &\tilde{h}_2 = b_1\tilde{k}_2 - \frac{a_1}{\tau_1}(\tau_1^2 - \xi_1^2)\tilde{l}_1 + \frac{a_1}{\tau_1}\xi_1\xi_2\tilde{l}_2. \end{aligned} \tag{7.20}$$

Solving (7.19) yields

$$\begin{aligned} &\overset{2}{A}_1 = \frac{\tilde{h}_1d_1\xi_1\xi_2 - \tilde{h}_2(d_2 - d_1\xi_2^2)}{-d_2(d_2 - d_1\xi^2)}\\ &\overset{2}{A}_2 = \frac{\tilde{h}_1(d_2 - d_1\xi_1^2) - \tilde{h}_2d_1\xi_1\xi_2}{-d_2(d_2 - d_1\xi^2)}. \end{aligned} \tag{7.21}$$

We have the qualitative relations

$$\begin{aligned} &|d_1| \sim \frac{c}{|\tau|}, \quad |d_2| \sim c|\tau|, \quad |\tilde{h}| \sim |\tilde{k}| + |\tau||\tilde{l}|, \\ &|\overset{2}{A}_1| \sim \frac{|\tilde{h}|}{|\tau|}, \quad |\overset{2}{A}_2| \sim \frac{|\tilde{h}|}{|\tau|}, \end{aligned} \tag{7.22}$$

where $\tilde{h}$, τ replace $(\tilde{h}_1, \tilde{h}_2)$, (τ_1, τ_2), respectively.

From (7.15) we have

$$\begin{aligned} &\overset{1}{\tilde{w}}_\alpha = \overset{1}{A}_\alpha e^{-\tau_1 z_3} = \left(\frac{b_2}{b_1}\overset{2}{A}_\alpha + \frac{1}{b_1}\tilde{l}_\alpha\right)e^{-\tau_1 z_3}, \quad \alpha = 1, 2, \\ &\overset{2}{\tilde{w}}_\alpha = \overset{2}{A}_\alpha e^{\tau_2 z_3}, \quad \alpha = 1, 2. \end{aligned} \tag{7.23}$$

Continuing,

$$\begin{aligned} &\overset{1}{\tilde{w}}_3 = \frac{i\xi_\alpha}{\tau_1}\overset{1}{A}_\alpha e^{-\tau_1 z_3} = \frac{i\xi_\alpha}{\tau_1}\left(\frac{b_2}{b_1}\overset{2}{A}_\alpha + \frac{1}{b_1}\tilde{l}_\alpha\right)e^{-\tau_1 z_3}, \\ &\overset{2}{\tilde{w}}_3 = -\frac{i\xi_\alpha}{\tau_2}\overset{2}{A}_\alpha e^{\tau_2 z_3}, \end{aligned} \tag{7.24}$$

where the summation over $\alpha \in \{1, 2\}$ is assumed. Using Lemmas 2.6.1 and 2.6.2 we conclude the proof of Lemma 7.2. □

8. Initial-Boundary Value Problem Near *B*

In the proof of Lemma 2.5.1 we distinguish a local problem near B (see 1_4). To examine the problem we localize (2.5.2) to a neighborhood of some point of B using an appropriate function from the partition of unity. Introducing a new system of coordinates with the origin at the point of B and flattening locally B we obtain

$$\begin{aligned} &v_{,t} - \Delta v + \nabla \mathrm{div}\, v = f \quad t > 0, \quad x_3 > 0, \\ &\mathrm{div}\, v|_{x_3=0} = h, \quad v_i|_{x_3} = b_i, \quad i = 1, 2. \end{aligned} \tag{8.1}$$

First we construct a function $\overset{1}{v}$ such that $\mathrm{div}\,\overset{1}{v}|_{x_3} = h$ so $\overset{1}{v}_{1,x_1} = h_1$, $\overset{1}{v}_{2,x_2} = h_2$, $\overset{1}{v}_{3,x_3} = h_3$, $h = h_1 + h_2 + h_3$ on $x_3 = 0$.

Next, we construct a function $\overset{2}{v}$ as a solution to the problem

$$\begin{aligned} &\overset{2}{v}_{,t} - \Delta\overset{2}{v} = f - \nabla \mathrm{div}\,\overset{1}{v} \equiv g, \quad t > 0, \quad x_3 > 0, \\ &\overset{2}{v}_1|_{x_3=0} = \overset{2}{v}_2|_{x_3=0} = 0, \quad \mathrm{div}\,\overset{2}{v}|_{x_3=0} = 0. \end{aligned} \tag{8.2}$$

Introducing the function

$$w = v - \overset{1}{v} - \overset{2}{v} \tag{8.3}$$

we see that it is a solution to the following initial-boundary value problem

$$\begin{aligned} &w_{,t} - \Delta w + \nabla \mathrm{div}\, w = 0, && z_3 > 0 \\ &w|_{t=0} = 0, && z_3 > 0, \\ &w_j = a_j, \quad j = 1, 2, \quad \mathrm{div}\, w = 0 && \text{on} \quad z_3 = 0. \end{aligned} \tag{8.4}$$

Lemma 8.1. *Let $\mathbb{R}^2 = \{x \in \mathbb{R}^3 \colon x_3 = 0\}$. Assume that $h \in H^{\frac{1}{2}+\alpha, \frac{1}{4}+\frac{\alpha}{2}}(\mathbb{R}^2 \times \mathbb{R}_+)$. Then $\overset{1}{v} \in H^{2+\alpha, 1+\alpha/2}$ $(\mathbb{R}^3_+ \times \mathbb{R}_+)$ and*

$$\|\overset{1}{v}\|_{H^{2+\alpha,1+\alpha/2}(\mathbb{R}^3_+ \times \mathbb{R}_+)} \le c\|h\|_{H^{\frac{1}{2}+\alpha, \frac{1}{4}+\frac{\alpha}{2}}(\mathbb{R}^2 \times \mathbb{R}_+)} \tag{8.5}$$

Assume that $g \in H^{\alpha,\alpha/2}(\mathbb{R}^3_+ \times \mathbb{R}_+)$. *Then there exists a solution to* (8.2) *such that* $\overset{2}{v} \in H^{2+\alpha,1+\alpha/2}(\mathbb{R}^3_+ \times \mathbb{R}_+)$ *and*

$$\|\overset{2}{v}\|_{H^{2+\alpha,1+\alpha/2}(\mathbb{R}^3_+\times\mathbb{R}_+)} \le c\|g\|_{H^{\alpha,\alpha/2}(\mathbb{R}^3_+\times\mathbb{R}_+)}. \tag{8.6}$$

Lemma 8.2. *Assume that* $a_j \in H^{3/2+\alpha,3/4+\alpha/2}(\mathbb{R}^2 \times \mathbb{R}_+)$, $j = 1, 2$, $\alpha \in (0, 1)$. *Then there exists a solution to problem* (8.4) *such that* $w \in H^{2+\alpha,1+\alpha/2}(\mathbb{R}^3_+ \times \mathbb{R}_+)$, w *is divergence free and*

$$\|w\|_{H^{2+\alpha,1+\alpha/2}(\mathbb{R}^3_+\times\mathbb{R}_+)} \le c\sum_{j=1}^{2}\|a_j\|_{H^{3/2+\alpha,3/4+\alpha/2}(\mathbb{R}^2\times\mathbb{R}_+)}. \tag{8.7}$$

Proof. Applying the Fourier–Laplace transform (7.13) to (8.4) yields

$$\begin{aligned}
&s\tilde{w}_\beta - \tilde{w}_{\beta,z_3z_3} + \xi^2\tilde{w}_\beta + i\xi_\beta(i\xi_\gamma\tilde{w}_\gamma + \tilde{w}_{3,z_3}) = 0,\\
&s\tilde{w}_3 - \tilde{w}_{3,z_3z_3} + \xi^2\tilde{w}_3 + \partial_{z_3}(i\xi_\gamma\tilde{w}_\gamma + \tilde{w}_{3,z_3}) = 0,\\
&\tilde{w}|_{z_3=0} = (\tilde{a}_1, \tilde{a}_2, \tilde{b}),
\end{aligned} \tag{8.8}$$

where the summation convention over repeated Greek indices from 1 to 2 is assumed.

Simplifying (8.8) yields

$$\begin{aligned}
&(s+\xi^2)\tilde{w}_\beta - \tilde{w}_{\beta,z_3z_3} - \xi_\beta\xi_\gamma\tilde{w}_\gamma + i\xi_\beta\tilde{w}_{3,z_3} = 0,\\
&(s+\xi^2)\tilde{w}_3 + i\xi_\gamma\tilde{w}_{\gamma,z_3} = 0,\\
&\tilde{w}|_{z_3=0} = (\tilde{a}_1, \tilde{a}_2, \tilde{b}).
\end{aligned} \tag{8.9}$$

Multiplying $(8.9)_1$ by ξ_β and summing with respect to β gives

$$(s+\xi^2)\xi_\beta\tilde{w}_\xi - \xi_\beta\tilde{w}_{\beta,z_3z_3} - \xi^2\xi_\beta\tilde{w}_\beta + i\xi^2\tilde{w}_{3,z_3} = 0. \tag{8.10}$$

Introducing the quantity

$$\tilde{G} = \xi_\beta\tilde{w}_\beta \tag{8.11}$$

we obtain from (8.10), $(8.9)_{1,2}$ the following problem:

$$\begin{aligned}
&(s+\xi^2)\tilde{G} - \tilde{G}_{,z_3z_3} - \xi^2\tilde{G} + i\xi^2\tilde{w}_{3,z_3} = 0,\\
&(s+\xi^2)\tilde{w}_3 + i\tilde{G}_{,z_3} = 0,\\
&\tilde{G}|_{z_3=0} = \xi_\beta\tilde{a}_\beta.
\end{aligned} \tag{8.12}$$

From $(8.12)_2$ we have

$$\tilde{w}_3 = -\frac{i}{s+\xi^2}\tilde{G}_{,z_3}. \tag{8.13}$$

Inserting this in $(8.12)_1$ yields

$$s(s+\xi^2)\tilde{G} - s\tilde{G}_{,z_3z_3} = 0. \tag{8.14}$$

Since Re $s > 0$ we get

$$(s+\xi^2)\tilde{G} - \tilde{G}_{,z_3z_3} = 0 \quad \tilde{G}|_{z_3=0} = \xi_\beta\tilde{a}_\beta. \tag{8.15}$$

Solving (8.15) gives

$$\tilde{G} = c_1\exp(\sqrt{s+\xi^2}z_3) + c_2\exp(-\sqrt{s+\xi^2}z_3), \quad z_3 > 0. \tag{8.16}$$

Since $\mathrm{Re}\sqrt{s+\xi^2} > 0$, we have to assume that $c_1 = 0$. From boundary condition $(8.15)_2$ we obtain $c_2 = \xi_\beta\tilde{a}_\beta$. Hence

$$\tilde{G} = \xi_\beta\tilde{a}_\beta\exp(-\sqrt{s+\xi^2}z_3). \tag{8.17}$$

In view of (8.11) we have

$$\xi_\beta \tilde{w}_\beta = \xi_\beta \tilde{a}_\beta \exp(-\sqrt{s+\xi^2} z_3),$$

so

$$\xi_\beta (\tilde{w}_\beta - \tilde{a}_\beta \exp(-\sqrt{s+\xi^2} z_3)) = 0.$$

Hence

$$\tilde{w}_\beta = \tilde{a}_\beta \exp(-\sqrt{s+\xi^2} z_3), \quad \beta = 1, 2. \tag{8.18}$$

In view of (8.13) and (8.17) we have

$$\tilde{w}_3 = \frac{i\xi_\beta \tilde{a}_\beta}{\sqrt{s+\xi^2}} \exp(-\sqrt{s+\xi^2} z_3). \tag{8.19}$$

Differentiating (8.19) with respect to z_3 and projecting on the plane $z_3 = 0$ we obtain

$$\tilde{w}_{3,z_3} - i\xi_\beta \tilde{a}_\beta = 0 \quad \text{on} \quad z_3 = 0. \tag{8.20}$$

Applying the inverse Laplace–Fourier transform we obtain condition $(8.4)_3$.

Finally, applying Lemmas 2.6.1 and 2.6.2 we derive (8.7). This ends the proof. □

Data Availability Statement The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

[1] Kacprzyk, P., Zajączkowski, W.M.: On the Faedo–Galerkin method for a free boundary problem for incompressible viscous magnetohydrodynamics. Topol. Methods Nonlinear Anal. **52**, 69–98 (2018)

[2] Shibata, Y., Zajączkowski, W.M.: On local solutions to a free boundary problem for incompressible viscous magnetohydrodynamics in the L_p-approach. Diss. Math. **566**, 1–102 (2021)

[3] Cole, G.H.A.: Fluid Dynamics. London (1962)

[4] Landau, L.D., Lifshitz, E.M., Pitaevskii, L.P.: Electrodynamics of Continuous Media. Landau and Lifshitz Course of Theoretical Physics, vol. 8, 2nd edn. Elsevier, Amsterdam (2013)

[5] Ladyzhenskaya, O.A., Solonnikov, V.A.: Solvability of some nonstationary problems of magnetohydrodynamics for viscous incompressible fluids. Trudy Mat. Inst. Steklova **59**, 115–173 (1960). ((**in Russian**))

[6] Padula, M., Solonnikov, V.A.: On free boundary problem of mhd. Zap. Nauchn Sem. POMI **385**, 135–186 (2010). ((**in Russian); English transl. J. Math. Sc. (N.Y.) 178, 313–344 (2011)**))

[7] Frolova, E.V.: Free boundary problem of magnetohydrodynamics. Zap. Nauchn. Sem. POMI **425**, 149–178 (2014). ((**in Russian**))

[8] Frolova, E.V., Solonnikov, V.A.: Solvability of a free boundary problem of magnetohydrodynamics in an infinite time interval. Zap. Nauchn. Sem. POMI **410**, 131–167 (2013). ((**in Russian), English transl. J. Math. Sci. (N.Y.)195 (2013), 76–97**))

[9] Kacprzyk, P.: Local free boundary problem for incompressible magnetohydrodynamics. Diss. Math. **509**, 1–52 (2015)

[10] Kacprzyk, P.: Local existence of solutions of the free boundary problem for the equations of a magnetohydrodynamic incompressible fluid. Appl. Math. **30**(4), 461–488 (2003)
[11] Zajączkowski, W.M.: On nonstationary motion of a compressible barotropic viscous fluid bounded by a free surface. Diss. Math. **324**, 101 (1993)
[12] Frolova, E.V., Shibata, Y.: On the maximal $L_p - L_q$ regularity theorem for the linearized electromagnetic field equations with interface condition. Zap. Nauchn. Sem. POMI **489**, 130–172 (2020). (**English transl.: J. Math. Sci. 260 (2022), 87–117**)
[13] Frolova, E.V., Shibata, Y.: Local well posedness for the magnetohydrodynamics in the different two liquids case. Mathematics **10**, 4751 (2022)
[14] Frolova, E.V.: Linearization of a free boundary problem of magnetohydrodynamics. Probl. Math. Anal. **95**, 69–78 (2018). (**English transl.: J. Math. Sci. 235 (2018), 322–333**)
[15] Kacprzyk, P.: Global free boundary problem for incompressible magnetohydrodynamics. Diss. Math. **510**, 1–44 (2015)
[16] Kacprzyk, P.: Almost global solutions of the free boundary problem for the equations of a magnetohydrodynamic incompressible fluid. Appl. Math. **31**(1), 69–77 (2004)
[17] Kacprzyk, P.: Free boundary problem for the equations of magnetohydrodynamic incompressible viscous fluid. Appl. Math. (Warsaw) **34**, 75–95 (2007)
[18] Besov, O.V., Il'in, V.P., Nikolskii, S.M.: Integral Representations of Functions and Imbedding Theorems. Nauka, Moscow (1975). ((**in Russian); English transl.: Scripta Series in Mathematics, Winston and Halsted Press, 1979**)
[19] Golovkin, K.K.: On equivalent norms for fractional spaces. Trudy Mat. Inst. Stekolva **66**, 364–383 (1962). ((**in Russian**))
[20] Bugrov, Y.S.: Function spaces with mixed norm. Izv. AN SSSR Ser. Mat. **35**, 1137–1158 (1971). (**English transl.: Math. USSR-Izv., 5, 1145–1167 (1971)**)
[21] Rencławowicz, J., Zajączkowski, W.M.: On the Stokes system in cylindrical domains. J. Math. Fluid Mech. **24**, 64 (2022). https://doi.org/10.1007/s00021-022-00698-z
[22] Triebel, H.: Theory of Function Spaces. Birkhäuser, Basel (1983)
[23] Solonnikov, V.A.: Estimates of solutions to nonstationary linearized Navier–Stokes system. Trudy Mat. Inst. Steklova **70**, 213–317 (1964). ((**in Russian**))
[24] Solonnikov, V.A.: On an unsteady motion of an isolated volume of a visous incompressible fluid. Izv. Akad. Nauk SSSR Ser. Mat. **51**, 1065–1087 (1987). ((**in Russian**))
[25] Solonnikov, V.A.: An initial boundary value problem for Stokes system that arises in the study of a problem with a free boundary. Trudy Mat. Inst. Steklova **188**, 150–188 (1990). ((**Russian); English transl.: Proc. Steklov Inst. Math. 3 (1991), 191–239**)
[26] Ladyzhenskaya, O.A., Solonnikov, V.A., Uraltseva, N.N.: Linear and Quasilinear Equations of Parabolic Type. Nauka, Moscow (1967). ((**in Russian**))

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