Ill-Posedness of the Novikov Equation in the Critical Besov Space $B^1_{\infty,1}(\mathbb{R})$

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Abstract. It is shown that both the Camassa–Holm and Novikov equations are ill-posed in $B^{1+1/p}_{p,r}(\mathbb{R})$ with $(p,r) \in [1,\infty] \times (1,\infty]$ in Guo et al. (J Differ Equ 266:1698–1707, 2019) and well-posed in $B^{1+1/p}_{p,1}(\mathbb{R})$ with $p \in [1,\infty)$ in Ye et al. (J Differ Equ 367: 729–748, 2023). Recently, the ill-posedness for the Camassa–Holm equation in $B^1_{\infty,1}(\mathbb{R})$ has been proved in Guo et al. (J Differ Equ 327: 127–144, 2022). In this paper, we shall solve the only left an endpoint case $r = 1$ for the Novikov equation. More precisely, we prove the ill-posedness for the Novikov equation in $B^1_{\infty,1}(\mathbb{R})$ by exhibiting the norm inflation phenomena.

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1. Introduction

Vladimir Novikov [30] investigated the question of integrability for Camassa–Holm type equations of the form

$$(1 - \partial_x^2)u_t = P(u, u_x, u_{xx}, u_{xxx}, \cdots),$$

where P is a polynomial of u and its x-derivatives. Using as test for integrability the existence of an infinite hierarchy of quasi-local higher symmetries, he produced about 20 integrable equations with quadratic nonlinearities that include the Camassa–Holm (CH) equation

$$(1 - \partial_x^2)u_t = uu_{xxx} + 2u_x u_{xx} - 3uu_x \tag{1.1}$$

and the Degasperis–Procesi (DP) equation

$$(1 - \partial_x^2)u_t = uu_{xxx} + 3u_x u_{xx} - 4uu_x. \tag{1.2}$$

Moreover, he produced about 10 integrable equations with cubic nonlinearities that include the Novikov equation (NE)

$$(1 - \partial_x^2)u_t = u^2 u_{xxx} + 3uu_x u_{xx} - 4u^2 u_x. \tag{1.3}$$

The Camassa–Holm equation was originally derived as a bi-Hamiltonian system by Fokas and Fuchssteiner [12] in the context of the KdV model and gained prominence after Camassa–Holm [2] independently re-derived it from the Euler equations of hydrodynamics using asymptotic expansions. (1.1) is completely integrable [2,5] with a bi-Hamiltonian structure [4,12] and infinitely many conservation laws [2,12]. Also, it admits exact peaked soliton solutions (peakons) of the form $ce^{-|x-ct|}$ with $c > 0$, which are orbitally stable [8] and models wave breaking (i.e., the solution remains bounded, while its slope becomes unbounded in finite time [3,6,7]). The Degasperis–Procesi equation with a bi-Hamiltonian structure is integrable [10] and has traveling wave solutions [22]. Although DP is similar to CH in several aspects, these two equations are truly different. One of the novel features of DP different from CH is that it has

not only peakon solutions [10] and periodic peakon solutions [37], but also shock peakons [23] and the periodic shock waves [11].

For the Novikov equation, Hone-Wang [21] derived the Lax pair. Also, NE has infinitely many conserved quantities. Like CH, the most important quantity conserved by a solution u to NE is its H^1-norm $\|u\|_{H^1}^2 = \int_{\mathbb{R}}(u^2 + u_x^2)\mathrm{d}x$. NE possesses peakon traveling wave solutions [13,19,20], which on the real line are given by the formula $u(x,t) = \pm\sqrt{c}e^{-|x-ct|}$ where $c > 0$ is the wave speed. In fact, NE admits multi-peakon traveling wave solutions on both the line and the circle. More precisely, on the line the n-peakon

$$u(x,t) = \sum_{j=1}^{n} p_j(t)e^{-|x-q_j(t)|}$$

is a solution to NE if and only if the positions $(q_1,\ldots,q_n)$ and the momenta $(p_1,\ldots,p_n)$ satisfy the following system of $2n$ differential equations

$$\begin{cases} \dfrac{\mathrm{d}q_j}{\mathrm{d}t} = u^2(q_j), \\ \dfrac{\mathrm{d}p_j}{\mathrm{d}t} = -u(q_j)u_x(q_j)p_j. \end{cases}$$

We would like to mention that, Himonas–Holliman–Kenig [18] constructed a 2-peakon solution with an asymmetric antipeakon-peakon initial data and showed the Cauchy problem for NE on both the line and the circle is ill-posed in Sobolev spaces H^s with $s < 3/2$.

The well-posedness of the Camassa–Holm type equations has been widely investigated during the past 20 years. The local well-posedness for the Cauchy problem of CH [9,24,25,31] and NE [16,17,29,32–35,38] in Sobolev and Besov spaces $B_{p,r}^s(\mathbb{R})$ with $s > \max\{1 + 1/p, 3/2\}$ and $(p,r) \in [1,\infty] \times [1,\infty)$ has been established. In our recent papers [27,28], we established the ill-posedness for CH in $B_{p,\infty}^s(\mathbb{R})$ with $p \in [1,\infty]$ by proving the solution map starting from u_0 is discontinuous at $t = 0$ in the metric of $B_{p,\infty}^s(\mathbb{R})$. Guo–Liu–Molinet–Yin [14] established the ill-posedness for the Camassa–Holm type equations in $B_{p,r}^{1+1/p}(\mathbb{R})$ with $(p,r) \in [1,\infty] \times (1,\infty]$ by proving the norm inflation, which implies that $B_{p,1}^{1+1/p}$ is the critical Besov space for both CH and NE. Ye–Yin–Guo [36] obtained the local well-posedness for the Camassa–Holm type equation in critical Besov spaces $B_{p,1}^{1+1/p}(\mathbb{R})$ with $p \in [1,\infty)$. We should mention that the well-posedness for DP in $B_{\infty,1}^1(\mathbb{R})$ has been established in our recent paper [26]. Recently, Guo–Ye–Yin [15] obtained the ill-posedness for CH in $B_{\infty,1}^1(\mathbb{R})$ by constructing a special initial data which leads to the norm inflation. However, their initial data seems to be invalid when proving the ill-posedness for NE in $B_{\infty,1}^1(\mathbb{R})$. To the best of our knowledge, whether NE is well-posed or not in $B_{\infty,1}^1(\mathbb{R})$ is still an open problem. We shall present the negative result in this paper.

Setting $\Lambda^{-2} = (1 - \partial_x^2)^{-1}$, then $\Lambda^{-2}f = G * f$ where $G(x) = \frac{1}{2}e^{-|x|}$ is the kernel of the operator Λ^{-2}. We can transform the Novikov equation into the following transport type equation

$$\begin{cases} u_t + u^2 u_x = \mathbf{P}_1(u) + \mathbf{P}_2(u), \\ u(x, t = 0) = u_0(x), \end{cases} \tag{1.4}$$

where

$$\mathbf{P}_1(u) = -\frac{1}{2}\Lambda^{-2}u_x^3 \quad \text{and} \quad \mathbf{P}_2(u) = -\partial_x\Lambda^{-2}\left(\frac{3}{2}uu_x^2 + u^3\right). \tag{1.5}$$

We can now state our main result as follows.

Theorem 1.1. *For any $n \in \mathbb{Z}^+$ large enough, there exist u_0 and $T > 0$ such that the Novikov equation* (1.4) *has a solution $u \in \mathcal{C}([0,T); H^\infty)$ satisfying*

$$\|u_0\|_{B_{\infty,1}^1} \leq \frac{1}{\log\log n} \quad but \quad \|u(t_0)\|_{B_{\infty,1}^1} \geq \log\log n \quad with \quad t_0 \in \left(0, \frac{1}{\log n}\right].$$

Since the norm inflation implies discontinuous of the data-to-solution map at the trivial function $u_0 \equiv 0$, Theorem 1.1 demonstrates that

Corollary 1.1. *The Cauchy problem for the Novikov equation is ill-posed in $B^1_{\infty,1}(\mathbb{R})$ in the sense of Hadamard.*

This paper is structured as follows. In Sect. 2, we list some notations and recall some Lemmas which will be used in the sequel. In Sect. 3 we present the proof of Theorem 1.1 by dividing it into several parts: (1) Construction of initial data; (2) Key Estimation for Discontinuity; (3) The Equation Along the Flow; (4) Norm inflation.

2. Preliminaries

Notation C stands for some positive constant independent of n, which may vary from line to line. The symbol $A \approx B$ means that $C^{-1}B \le A \le CB$. Given a Banach space X, we denote its norm by $\|\cdot\|_X$. We shall use the simplified notation $\|f, \cdots, g\|_X = \|f\|_X + \cdots + \|g\|_X$ if there is no ambiguity. We will also define the Lipschitz space $C^{0,1}$ using the norm $\|f\|_{C^{0,1}} = \|f\|_{L^\infty} + \|\partial_x f\|_{L^\infty}$. For $I \subset \mathbb{R}$, we denote by $\mathcal{C}(I; X)$ the set of continuous functions on I with values in X. Let us recall that for all $f \in \mathcal{S}'$, the Fourier transform $\widehat{f}$, is defined by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\mathrm{d}x \quad \text{for any } \xi \in \mathbb{R}.$$

Next, we will recall some facts about the Littlewood-Paley decomposition and the nonhomogeneous Besov spaces (see [1] for more details). Choose a radial, non-negative, smooth function $\vartheta : \mathbb{R} \mapsto [0,1]$ such that $\operatorname{supp} \vartheta \subset B(0, 4/3)$ and $\vartheta(\xi) \equiv 1$ for $|\xi| \le 3/4$. Setting $\varphi(\xi) := \vartheta(\xi/2) - \vartheta(\xi)$, then we deduce that φ has the following properties

- $\operatorname{supp} \varphi \subset \{\xi \in \mathbb{R} : 3/4 \le |\xi| \le 8/3\}$;
- $\varphi(\xi) \equiv 1$ for $4/3 \le |\xi| \le 3/2$;
- $\vartheta(\xi) + \sum_{j \ge 0} \varphi(2^{-j}\xi) = 1$ for any $\xi \in \mathbb{R}$.

For every $u \in \mathcal{S}'(\mathbb{R})$, the inhomogeneous dyadic blocks Δ_j are defined as follows

$$\Delta_j u = 0, \text{ if } j \le -2; \quad \Delta_{-1} u = \vartheta(D)u; \quad \Delta_j u = \varphi(2^{-j}D)u, \text{ if } j \ge 0,$$

where the pseudo-differential operator $\sigma(D) : u \to \mathcal{F}^{-1}(\sigma \mathcal{F}u)$.

Let $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]^2$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R})$ is defined by

$$B^s_{p,r}(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{B^s_{p,r}(\mathbb{R})} := \left\| 2^{js} \|\Delta_j u\|_{L^p} \right\|_{\ell^r(j \ge -1)} < \infty \right\}.$$

The following Bernstein's inequalities will be used in the sequel.

Lemma 2.1 ([1], Lemma 2.1). *Let $\mathcal{B}$ be a ball and $\mathcal{C}$ be an annulus. There exists a constant $C > 0$ such that for all $k \in \mathbb{N} \cup \{0\}$, any $\lambda \in \mathbb{R}^+$ and any function $f \in L^p$ with $1 \le p \le q \le \infty$, we have*

$$\operatorname{supp} \widehat{f} \subset \lambda\mathcal{B} \;\Rightarrow\; \|\partial_x^k f\|_{L^q} \le C^{k+1} \lambda^{k+(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p},$$

$$\operatorname{supp} \widehat{f} \subset \lambda\mathcal{C} \;\Rightarrow\; C^{-k-1} \lambda^k \|f\|_{L^p} \le \|\partial_x^k f\|_{L^p} \le C^{k+1} \lambda^k \|f\|_{L^p}.$$

Let us complete this section by recalling the useful commutator estimate.

Lemma 2.2 ([1], Lemma 2.100). *Let $1 \le r \le \infty$, $1 \le p \le p_1 \le \infty$ and $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1}$. There exists a constant C depending continuously on p and p_1 such that*

$$\left\| 2^j \|[\Delta_j, v]\partial_x f\|_{L^p} \right\|_{\ell^r(j \ge -1)} \le C \left(\|\partial_x v\|_{L^\infty} \|f\|_{B^1_{p,r}} + \|\partial_x f\|_{L^{p_2}} \|\partial_x v\|_{B^0_{p_1,r}} \right),$$

where we denote the standard commutator $[\Delta_j, v]\partial_x f = \Delta_j(v\partial_x f) - v\Delta_j \partial_x f$.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

3.1. Construction of Initial Data

Define a radial, smooth cut-off function χ with values in $[0,1]$ which satisfies

$$\chi(x) = \begin{cases} 1, & \text{if } |x| \leq \frac{1}{4}, \\ 0, & \text{if } |x| \geq \frac{1}{2}. \end{cases}$$

From now on, we set $\gamma := \frac{17}{24}$ just for the sake of simplicity. Letting

$$n \in 16\mathbb{N} = \{16, 32, 48, \cdots\} \quad \text{and} \quad \mathbb{N}(n) = \left\{ k \in 8\mathbb{N} : \frac{n}{4} \leq k \leq \frac{n}{2} \right\}.$$

We introduce the following new notation which will be used often throughout this paper

$$\|f\|_{B^k_{\infty,1}(\mathbb{N}(n))} = \sum_{j \in \mathbb{N}(n)} 2^{kj} \|\Delta_j f\|_{L^\infty}, \quad k \in \{0, 1\}.$$

Now, we can define the initial data u_0 by

$$u_0 = n^{-\frac{1}{3}} \left(u_0^{\mathrm{H}} + u_0^{\mathrm{L}} \right),$$

where

$$u_0^{\mathrm{H}} := 2^{-n} \log n \sum_{\ell \in \mathbb{N}(n)} \cos\left(2^n \gamma (x + 2^{\ell+1}\gamma)\right) \cdot \cos\left(2^\ell \gamma (x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}(x + 2^{\ell+1}\gamma), \quad (3.6)$$

$$u_0^{\mathrm{L}} := \sum_{\ell \in \mathbb{N}(n)} \check{\chi}(x + 2^{\ell+1}\gamma). \quad (3.7)$$

Some Observations

1. Obviously,

$$\operatorname{supp} \widehat{u_0^{\mathrm{L}}} \subset \left\{ \xi \in \mathbb{R} : \ |\xi| \leq \frac{1}{2} \right\}. \quad (3.8)$$

2. It is not difficult to check that

$$\operatorname{supp} \mathcal{F}\left(\cos\left(2^\ell \gamma (x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}(x + 2^{\ell+1}\gamma)\right)$$
$$\subset \left\{ \xi \in \mathbb{R} : \ 2^\ell \gamma - \frac{1}{2} \leq |\xi| \leq 2^\ell \gamma + \frac{1}{2} \right\}, \quad (3.9)$$

which in turn gives

$$\operatorname{supp} \mathcal{F}\left(\cos\left(2^n \gamma (x + 2^{\ell+1}\gamma)\right) \cdot \cos\left(2^\ell \gamma (x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}(x + 2^{\ell+1}\gamma)\right)$$
$$\subset \left\{ \xi \in \mathbb{R} : \ 2^n \gamma - 2^\ell \gamma - \frac{1}{2} \leq |\xi| \leq 2^n \gamma + 2^\ell \gamma + \frac{1}{2} \right\}. \quad (3.10)$$

Thus

$$\operatorname{supp} \widehat{u_0^{\mathrm{H}}} \subset \left\{ \xi \in \mathbb{R} : \ \frac{4}{3} 2^{n-1} \leq |\xi| \leq \frac{3}{2} 2^{n-1} \right\}. \quad (3.11)$$

3. Since $\check{\chi}$ is a Schwartz function, we have

$$|\check{\chi}(x)| + |\partial_x \check{\chi}(x)| \leq C(1 + |x|)^{-M}, \qquad M \gg 1. \quad (3.12)$$

4. By $\check{\chi}(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \chi(x)\mathrm{d}x \geq \frac{1}{4\pi}$, we have

$$\left\| \cos\left(2^{\ell+1}\gamma(x+2^{\ell+1}\gamma)\right)\check{\chi}^3(x+2^{\ell+1}\gamma) \right\|_{L^\infty} \geq \check{\chi}^3(0) \geq \frac{1}{64\pi^3}. \tag{3.13}$$

Lemma 3.1. *There exists a positive constant C independent of n such that*

$$2^n\|u_0^{\mathrm{H}}\|_{L^\infty} + \|\partial_x u_0^{\mathrm{H}}\|_{L^\infty} \leq C\log n,$$
$$\|u_0^{\mathrm{L}}\|_{C^{0,1}} \leq C,$$
$$\|u_0\|_{B^1_{\infty,1}} \leq Cn^{-\frac{1}{3}}\log n.$$

Proof. Due to (3.11)–(3.12), by Bernstein's inequality, we have

$$2^n\|u_0^{\mathrm{H}}\|_{L^\infty} + \|\partial_x u_0^{\mathrm{H}}\|_{L^\infty} + \log n\|u_0^{\mathrm{L}}\|_{C^{0,1}}$$

$$\leq C\log n \left\| \sum_{\ell\in\mathbb{N}(n)} \frac{1}{(1+|x+2^{\ell+1}\gamma|)^M} \right\|_{L^\infty}$$

$$\leq C\log n$$

and

$$\|u_0\|_{B^1_{\infty,1}} \leq Cn^{-\frac{1}{3}}\left(\|u_0^{\mathrm{H}}\|_{B^1_{\infty,1}} + \|u_0^{\mathrm{L}}\|_{B^1_{\infty,1}} \right)$$
$$\leq Cn^{-\frac{1}{3}}\left(2^n\|u_0^{\mathrm{H}}\|_{L^\infty} + \|u_0^{\mathrm{L}}\|_{L^\infty} \right)$$
$$\leq Cn^{-\frac{1}{3}}\log n.$$

This completes the proof of Lemma 3.1. $\square$

3.2. Key Estimation for Discontinuity

The following Lemma is crucial for the proof of Theorem 1.1.

Lemma 3.2. *There exists a positive constant c independent of n such that*

$$\left\| u_0(\partial_x u_0)^2 \right\|_{B^0_{\infty,1}(\mathbb{N}(n))} \geq c(\log n)^2, \qquad n \gg 1.$$

Proof. Obviously,

$$n \cdot u_0(\partial_x u_0)^2 = \underbrace{u_0^{\mathrm{L}}(\partial_x u_0^{\mathrm{H}})^2}_{=:\,\mathbf{I}_1} + \underbrace{u_0^{\mathrm{H}}(\partial_x u_0^{\mathrm{H}} + \partial_x u_0^{\mathrm{L}})^2}_{=:\,\mathbf{I}_2} + \underbrace{u_0^{\mathrm{L}}\left((\partial_x u_0^{\mathrm{L}})^2 + 2\partial_x u_0^{\mathrm{H}}\partial_x u_0^{\mathrm{L}}\right)}_{=:\,\mathbf{I}_3}.$$

Next, we need to estimate the above three terms.

Estimation of $\mathbf{I}_2$. Using Lemma 3.1 yields

$$\|\mathbf{I}_2\|_{B^0_{\infty,1}(\mathbb{N}(n))} \leq Cn\|\mathbf{I}_2\|_{L^\infty} \leq Cn\|u_0^{\mathrm{H}}\|_{L^\infty}\|\partial_x u_0^{\mathrm{H}}, \partial_x u_0^{\mathrm{L}}\|_{L^\infty}^2 \leq Cn2^{-n}(\log n)^2.$$

Estimation of $\mathbf{I}_3$. Notice that the support conditions (3.8) and (3.11), one has

$$\Delta_j\mathbf{I}_3 = 0 \quad \text{for } j \in \mathbb{N}(n) \quad \Rightarrow \quad \|\mathbf{I}_3\|_{B^0_{\infty,1}(\mathbb{N}(n))} = 0.$$

Estimation of $\mathbf{I}_1$. Now we focus on the estimation of $\mathbf{I}_1$. Obviously,

$$\partial_x u_0^{\mathrm{H}} = -\gamma\log n \sum_{\ell\in\mathbb{N}(n)} \sin\left(2^n\gamma(x+2^{\ell+1}\gamma)\right) \cdot \cos\left(2^\ell\gamma(x+2^{\ell+1}\gamma)\right) \cdot \check{\chi}(x+2^{\ell+1}\gamma)$$

$$+ 2^{-n}\log n \sum_{\ell\in\mathbb{N}(n)} \cos\left(2^n\gamma(x+2^{\ell+1}\gamma)\right) \cdot \partial_x\Big(\cos\left(2^\ell\gamma(x+2^{\ell+1}\gamma)\right) \cdot \check{\chi}(x+2^{\ell+1}\gamma)\Big).$$

We decompose $\mathbf{I}_1$ into three terms

$$\mathbf{I}_1 = (\log n)^2 \left(\mathbf{I}_{11} + \mathbf{I}_{12} - \mathbf{I}_{13}\right),$$

where

$$\mathbf{I}_{11} = \gamma^2 u_0^{\mathrm{L}} \left(\sum_{\ell \in \mathbb{N}(n)} \sin\left(2^n \gamma(x + 2^{\ell+1}\gamma)\right) \cdot \cos\left(2^\ell \gamma(x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}(x + 2^{\ell+1}\gamma) \right)^2,$$

$$\mathbf{I}_{12} = 2^{-2n} u_0^{\mathrm{L}} \left(\sum_{\ell \in \mathbb{N}(n)} \cos\left(2^n \gamma(x + 2^{\ell+1}\gamma)\right) \cdot \partial_x\left(\cos\left(2^\ell \gamma(x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}(x + 2^{\ell+1}\gamma) \right) \right)^2,$$

$$\mathbf{I}_{13} = 2\gamma 2^{-n} u_0^{\mathrm{L}} \sum_{\ell \in \mathbb{N}(n)} \sin\left(2^n \gamma(x + 2^{\ell+1}\gamma)\right) \cdot \cos\left(2^\ell \gamma(x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}(x + 2^{\ell+1}\gamma)$$

$$\times \sum_{\ell \in \mathbb{N}(n)} \cos\left(2^n \gamma(x + 2^{\ell+1}\gamma)\right) \cdot \partial_x\left(\cos\left(2^\ell \gamma(x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}(x + 2^{\ell+1}\gamma) \right).$$

Easy computations give that

$$\|\mathbf{I}_{12}\|_{L^\infty} \le C 2^{-2n} \|u_0^{\mathrm{L}}\|_{L^\infty} \left\| \sum_{\ell \in \mathbb{N}(n)} \partial_x\left(\cos\left(2^\ell \gamma(x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}(x + 2^{\ell+1}\gamma) \right) \right\|_{L^\infty}^2$$

$$\le C 2^{-2n} \left\| \sum_{\ell \in \mathbb{N}(n)} \frac{2^\ell}{(1 + |x + 2^{\ell+1}\gamma|)^M} \right\|_{L^\infty}^2$$

$$\le C 2^{-n}.$$

Similarly, we have $\|\mathbf{I}_{13}\|_{L^\infty} \le C 2^{-\frac{n}{2}}$. Thus

$$\|\mathbf{I}_{12}, \mathbf{I}_{13}\|_{B^0_{\infty,1}(\mathbb{N}(n))} \le C n 2^{-\frac{n}{2}}.$$

By the simple equality $\sin^2(a)\cos^2(b) = \frac{1}{4}(1 - \cos(2a))(1 + \cos(2b))$, we break $\mathbf{I}_{11}$ down into some easy-to-handle terms

$$\mathbf{I}_{11} = \frac{\gamma^2}{4} \sum_{i=1}^5 \mathbf{I}_{11i}, \quad \text{where}$$

$$\mathbf{I}_{111} = u_0^{\mathrm{L}} \sum_{\ell \in \mathbb{N}(n)} \cos\left(2^{\ell+1}\gamma(x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}^2(x + 2^{\ell+1}\gamma),$$

$$\mathbf{I}_{112} = u_0^{\mathrm{L}} \sum_{\ell \in \mathbb{N}(n)} \check{\chi}^2(x + 2^{\ell+1}\gamma),$$

$$\mathbf{I}_{113} = -u_0^{\mathrm{L}} \sum_{\ell \in \mathbb{N}(n)} \cos\left(2^{n+1}\gamma(x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}^2(x + 2^{\ell+1}\gamma),$$

$$\mathbf{I}_{114} = -u_0^{\mathrm{L}} \sum_{\ell \in \mathbb{N}(n)} \cos\left(2^{n+1}\gamma(x + 2^{\ell+1}\gamma)\right) \cdot \cos\left(2^{\ell+1}\gamma(x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}^2(x + 2^{\ell+1}\gamma),$$

$$\mathbf{I}_{115} = 4 u_0^{\mathrm{L}} \sum_{\substack{\ell, j \in \mathbb{N}(n) \\ \ell \ne j}} \Big(\sin\left(2^n \gamma(x + 2^{\ell+1}\gamma)\right) \cdot \cos\left(2^\ell \gamma(x + 2^{\ell+1}\gamma)\right) \cdot \check{\chi}(x + 2^{\ell+1}\gamma)$$

$$\times \sin\left(2^n \gamma(x + 2^{j+1}\gamma)\right) \cdot \cos\left(2^j \gamma(x + 2^{j+1}\gamma)\right) \cdot \check{\chi}(x + 2^{j+1}\gamma) \Big).$$

Notice that the support conditions (3.9) and (3.10), one has

$$\Delta_j \mathbf{I}_{112} = \Delta_j \mathbf{I}_{113} = \Delta_j \mathbf{I}_{114} = 0 \quad \text{for } j \in \mathbb{N}(n),$$

which implies directly that

$$\|\mathbf{I}_{112}, \mathbf{I}_{113}, \mathbf{I}_{114}\|_{B^0_{\infty,1}(\mathbb{N}(n))} = 0.$$

Using Lemma 3.1 and (3.12), we obtain that for $n \gg 1$

$$\|\mathbf{I}_{115}\|_{B^0_{\infty,1}(\mathbb{N}(n))} \leq Cn\|\mathbf{I}_{115}\|_{L^\infty} \leq Cn\|u_0^{\mathrm{L}}\|_{L^\infty} \sum_{\substack{\ell,j \in \mathbb{N}(n) \\ \ell \neq j}} \left\|\check{\chi}(x+2^{\ell+1}\gamma) \cdot \check{\chi}(x+2^{j+1}\gamma)\right\|_{L^\infty}$$

$$\leq Cn \sum_{\substack{\ell,j \in \mathbb{N}(n) \\ j > \ell}} \left\|(1+|x+2^{j+1}\gamma|^2)^{-M}(1+|x+2^{\ell+1}\gamma|^2)^{-M}\right\|_{L^\infty}$$

$$\leq Cn \sum_{\substack{\ell,j \in \mathbb{N}(n) \\ j > \ell}} \left\|(1+|x|^2)^{-M}(1+|x-(2^{j+1}-2^{\ell+1})\gamma|^2))^{-M}\right\|_{L^\infty}$$

$$\leq Cn \sum_{\substack{\ell,j \in \mathbb{N}(n) \\ j > \ell}} \left(\gamma(2^j - 2^\ell)\right)^{-2M}$$

$$\leq C2^{-\frac{M}{4}n},$$

where we have separated $\mathbb{R}$ into two different regions $\{x : |x| \leq \gamma(2^j - 2^\ell)\}$ and $\{x : |x| > \gamma(2^j - 2^\ell)\}$.

Finally, we can break $\mathbf{I}_{111}$ down into two parts, where the first part contributes the main part.

$$\mathbf{I}_{111} = \sum_{\ell \in \mathbb{N}(n)} \cos\left(2^{\ell+1}\gamma(x+2^{\ell+1}\gamma)\right) \cdot \check{\chi}^3(x+2^{\ell+1}\gamma)$$

$$+ \sum_{\substack{\ell,j \in \mathbb{N}(n) \\ \ell \neq j}} \cos\left(2^{\ell+1}\gamma(x+2^{\ell+1}\gamma)\right) \check{\chi}^2(x+2^{\ell+1}\gamma) \cdot \check{\chi}(x+2^{j+1}\gamma)$$

$$:= \mathbf{I}_{1111} + \mathbf{I}_{1112}.$$

Due to (3.9) and the support condition of φ and for all $k \in \mathbb{Z}$

$$\varphi(2^{-k}\xi) \equiv 1 \quad \text{for} \quad \xi \in \mathcal{C}_k = \left\{\xi \in \mathbb{R} : \frac{4}{3}2^k \leq |\xi| \leq \frac{3}{2}2^k\right\},$$

we have

$$\dot{\Delta}_j \mathbf{I}_{1111} = \mathcal{F}^{-1}\left(\varphi(2^{-j}\cdot)\mathcal{F}\mathbf{I}_{1111}\right)$$

$$= \begin{cases} \cos\left(2^{j+1}\gamma(x+2^{j+1}\gamma)\right) \cdot \check{\chi}^3(x+2^{j+1}\gamma), & \text{if } \ell = j, \\ 0, & \text{if } \ell \neq j, \end{cases}$$

which combining (3.13) implies that

$$\|\mathbf{I}_{1111}\|_{B^0_{\infty,1}(\mathbb{N}(n))} = \sum_{j \in \mathbb{N}(n)} \left\|\cos\left(2^{j+1}\gamma(x+2^{j+1}\gamma)\right) \cdot \check{\chi}^3(x+2^{j+1}\gamma)\right\|_{L^\infty} \geq cn.$$

Following the same procedure as $\mathbf{I}_{115}$, we get for $n \gg 1$

$$\|\mathbf{I}_{1112}\|_{B^0_{\infty,1}(\mathbb{N}(n))} \leq C2^{-\frac{M}{4}n}.$$

Gathering all the above estimates, we obtain that for large enough n

$$n\|u_0(\partial_x u_0)^2\|_{B^0_{\infty,1}(\mathbb{N}(n))} \geq \|\mathbf{I}_1\|_{B^0_{\infty,1}(\mathbb{N}(n))} - \|\mathbf{I}_2\|_{B^0_{\infty,1}(\mathbb{N}(n))}$$

$$\geq C(\log n)^2 \left(\|\mathbf{I}_{1111}\|_{B^0_{\infty,1}(\mathbb{N}(n))} - \|\mathbf{I}_{12}, \mathbf{I}_{13}, \mathbf{I}_{115}, \mathbf{I}_{1112}\|_{B^0_{\infty,1}(\mathbb{N}(n))} - n2^{-n}\right)$$

$$\geq cn(\log n)^2.$$

This completes the proof of Lemma 3.2. $\qquad\qquad\square$

Remark 3.1. Setting $u_0 = n^{-\frac{1}{2}} u_0^{\mathrm{H}}$ where u_0^{H} is given by (3.6) and following the same argument as the proof of Lemma 3.2, we can establish

$$\|u_0\|_{B_{\infty,1}^1(\mathbb{R})} \le C_1 n^{-\frac{1}{2}} \log n \quad \text{and} \quad \left\|(\partial_x u_0)^2\right\|_{B_{\infty,1}^0(\mathbb{N}(n))} \ge c_2 (\log n)^2, \qquad n \gg 1.$$

3.3. The Equation Along the Flow

Given a Lipschitz velocity field u, we may solve the following ODE to find the flow induced by u^2:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\phi(t,x) = u^2(t,\phi(t,x)), \\ \phi(0,x) = x, \end{cases} \tag{3.14}$$

which is equivalent to the integral form

$$\phi(t,x) = x + \int_0^t u^2(\tau, \phi(\tau,x))\mathrm{d}\tau. \tag{3.15}$$

Considering

$$\begin{cases} \partial_t v + u^2 \partial_x v = P, \\ v(0,x) = v_0(x), \end{cases} \tag{3.16}$$

then, we get from (3.16) that

$$\partial_t(\Delta_j v) + u^2 \partial_x \Delta_j v = R_j + \Delta_j P \quad \text{with} \quad R_j = -[\Delta_j, u^2]\partial_x v.$$

Due to (3.14), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((\Delta_j v) \circ \phi\right) = R_j \circ \phi + \Delta_j P \circ \phi,$$

which means that

$$\Delta_j v \circ \phi = \Delta_j v_0 + \int_0^t R_j \circ \phi \mathrm{d}\tau + \int_0^t \Delta_j P \circ \phi \mathrm{d}\tau. \tag{3.17}$$

3.4. Norm Inflation

For $n \gg 1$, we have for $t \in [0,1]$

$$\|u\|_{C^{0,1}} \le C\|u_0\|_{C^{0,1}} \le C n^{-\frac{1}{3}} \log n.$$

To prove Theorem 1.1, it suffices to show that there exists $t_0 \in \left(0, \frac{1}{\log n}\right]$ such that

$$\|u(t_0,\cdot)\|_{B_{\infty,1}^1} \ge \log \log n. \tag{3.18}$$

We prove (3.18) by contradiction. If (3.18) were not true, then

$$\sup_{t \in \left(0, \frac{1}{\log n}\right]} \|u(t,\cdot)\|_{B_{\infty,1}^1} < \log \log n. \tag{3.19}$$

We divide the proof into two steps.

Step 1: Lower bound for $(\Delta_j u) \circ \phi$.

Now we consider the equation along the Lagrangian flow-map associated to u^2. Utilizing (3.17) to (1.4) yields

$$(\Delta_j u) \circ \phi = \Delta_j u_0 + \int_0^t R_j^1 \circ \phi \mathrm{d}\tau + \int_0^t \Delta_j F \circ \phi \mathrm{d}\tau$$

$$+ \int_0^t \left(\Delta_j E \circ \phi - \Delta_j E_0\right) \mathrm{d}\tau + t\Delta_j E_0,$$

where

$$R_j^1 = -[\Delta_j, u^2]\partial_x u, \qquad F = -\Lambda^{-2}\left(\frac{1}{2}(\partial_x u)^3 + \partial_x(u^3)\right),$$

$$E = -\frac{3}{2}\partial_x\Lambda^{-2}\left(u(\partial_x u)^2\right) \quad \text{with} \quad E_0 = -\frac{3}{2}\partial_x\Lambda^{-2}\left(u_0(\partial_x u_0)^2\right).$$

Due to Lemma 3.2, we deduce

$$\sum_{j\in\mathbb{N}(n)} 2^j \|\Delta_j E_0\|_{L^\infty} \approx \sum_{j\in\mathbb{N}(n)} \|\Delta_j \partial_x E_0\|_{L^\infty} \geq C \sum_{j\in\mathbb{N}(n)} \left\|\Delta_j[u_0(\partial_x u_0)^2]\right\|_{L^\infty} \geq c(\log n)^2. \qquad (3.20)$$

Notice that the L^∞-norm of any function f is preserved under the flow ϕ, i.e.

$$\|f(t,\phi(t,x))\|_{L^\infty(\mathbb{R})} = \|f(t,x)\|_{L^\infty(\mathbb{R})},$$

then, using the commutator estimate from Lemma 2.2, we have

$$\sum_{j\in\mathbb{N}(n)} 2^j \|R_j^1 \circ \phi\|_{L^\infty} \leq \left\|2^j\|R_j^1\|_{L^\infty}\right\|_{\ell^1(j\geq-1)} \leq C\|\partial_x(u^2)\|_{B^0_{\infty,1}}\|u\|_{B^1_{\infty,1}}$$

$$\leq C\|u\|_{C^{0,1}}\|u\|_{B^1_{\infty,1}}^2 \leq Cn^{-\frac{1}{3}}(\log n)^3. \qquad (3.21)$$

Also, we have

$$\sum_{j\in\mathbb{N}(n)} 2^j \|\Delta_j F \circ \phi\|_{L^\infty} = \sum_{j\in\mathbb{N}(n)} 2^j \|\Delta_j F\|_{L^\infty} \leq C\|(\partial_x u)^3 + \partial_x(u^3)\|_{L^\infty}$$

$$\leq C\|u\|_{C^{0,1}}^3 \leq Cn^{-1}(\log n)^3. \qquad (3.22)$$

Combining (3.20)–(3.22) and using Lemma 3.1 yields

$$\sum_{j\in\mathbb{N}(n)} 2^j \|(\Delta_j u) \circ \phi\|_{L^\infty} \geq t \sum_{j\in\mathbb{N}(n)} 2^j \|\Delta_j E_0\|_{L^\infty} - \sum_{j\in\mathbb{N}(n)} 2^j \|\Delta_j E \circ \phi - \Delta_j E_0\|_{L^\infty}$$

$$- Cn^{-\frac{1}{3}}(\log n)^3 - C\|u_0\|_{B^1_{\infty,1}}$$

$$\geq ct(\log n)^2 - \sum_{j\in\mathbb{N}(n)} 2^j \|\Delta_j E \circ \phi - \Delta_j E_0\|_{L^\infty} - Cn^{-\frac{1}{3}}(\log n)^3. \qquad (3.23)$$

Step 2: Upper bound for $\Delta_j E \circ \phi - \Delta_j E_0$.

By easy computations

$$\partial_x\Lambda^{-2}[u^2 u_x^3 + 2u\partial_x(u^2 u_x)u_x] = \partial_x\Lambda^{-2}[2u^2 u_x^3 + \partial_x(u^3 u_x^2)]$$

$$= 2\partial_x\Lambda^{-2}(u^2 u_x^3) + \Lambda^{-2}(u^3 u_x^2) - u^3 u_x^2,$$

then we find that

$$\partial_t E + u^2\partial_x E = -\frac{3}{2}\left(\partial_x\Lambda^{-2}\partial_t(uu_x^2) + u^2\partial_x^2\Lambda^{-2}(uu_x^2)\right)$$

$$= \mathrm{J} + \frac{3}{2}\left(\partial_x\Lambda^{-2}[u^2 u_x^3 + 2u\partial_x(u^2 u_x)u_x] + u^3 u_x^2 - u^2\Lambda^{-2}(uu_x^2)\right)$$

$$= \mathrm{J} + \mathrm{K}, \qquad (3.24)$$

where

$$\mathrm{J} = -\frac{3}{2}\partial_x\Lambda^{-2}\left(\mathbf{P}_1(u)u_x^2 + \mathbf{P}_2(u)u_x^2 + 2\partial_x\mathbf{P}_1(u)uu_x + 2\partial_x\mathbf{P}_2(u)uu_x\right),$$

$$\mathrm{K} = \frac{3}{2}\left(2\partial_x\Lambda^{-2}(u^2 u_x^3) + \Lambda^{-2}(u^3 u_x^2) - u^2\Lambda^{-2}(uu_x^2)\right).$$

Utilizing (3.17) to (3.24) yields

$$\Delta_j E \circ \phi - \Delta_j E_0 = \int_0^t [u^2, \Delta_j] \partial_x E \circ \phi \mathrm{d}\tau + \int_0^t \Delta_j (\mathrm{J} + \mathrm{K}) \circ \phi \mathrm{d}\tau.$$

Using the commutator estimate from Lemma 2.2, one has

$$2^j \|[u^2, \Delta_j] \partial_x E\|_{L^\infty} \le C \left(\|\partial_x (u^2)\|_{L^\infty} \|E\|_{B^1_{\infty,\infty}} + \|\partial_x E\|_{L^\infty} \|u^2\|_{B^1_{\infty,\infty}} \right)$$
$$\le C\|u\|_{C^{0,1}}^5 \le Cn^{-\frac{5}{3}} (\log n)^5. \tag{3.25}$$

Due to the facts

$$\|\Lambda^{-2} f\|_{L^\infty} \le \|f\|_{L^\infty} \quad \text{and} \quad \|\partial_x \Lambda^{-2} f\|_{L^\infty} \le \|f\|_{L^\infty} \quad \Rightarrow \quad \|\partial_x^2 \Lambda^{-2} f\|_{L^\infty} \le 2\|f\|_{L^\infty},$$

then we have

$$2^j \|\Delta_j \mathrm{J}\|_{L^\infty} \approx \|\partial_x \mathrm{J}\|_{L^\infty} \le C\|u\|_{C^{0,1}}^5 \le Cn^{-\frac{5}{3}} (\log n)^5. \tag{3.26}$$

Similarly,

$$2^j \|\Delta_j \mathrm{K}\|_{L^\infty} \le C\|u\|_{C^{0,1}}^5 \le Cn^{-\frac{5}{3}} (\log n)^5. \tag{3.27}$$

Then, we deduce from (3.25)–(3.27) that

$$2^j \|\Delta_j E \circ \phi - \Delta_j E_0\|_{L^\infty} \le Cn^{-\frac{5}{3}} (\log n)^5,$$

which leads to

$$\sum_{j \in \mathbb{N}(n)} 2^j \|\Delta_j E \circ \phi - \Delta_j E_0\|_{L^\infty} \le Cn^{-\frac{2}{3}} (\log n)^5. \tag{3.28}$$

Combining (3.23) and (3.28), then for $t = \frac{1}{\log n}$, we obtain for $n \gg 1$

$$\|u(t)\|_{B^1_{\infty,1}} \ge \|u(t)\|_{B^1_{\infty,1}(\mathbb{N}(n))} = \sum_{j \in \mathbb{N}(n)} 2^j \|(\Delta_j u) \circ \phi\|_{L^\infty}$$
$$\ge ct(\log n)^2 - Cn^{-\frac{2}{3}} (\log n)^5 - Cn^{-\frac{1}{3}} (\log n)^3$$
$$\ge \log \log n,$$

which contradicts the hypothesis (3.19). Thus, Theorem 1.1 is proved.   □

4. Discussion

By the Lagrangian coordinate transformation used cleverly in [15] and constructing a new initial data, we prove that the Novikov equation is ill-posed in critical Besov spaces $B^1_{\infty,1}(\mathbb{R})$. Thus our results (Theorem 1.1 and Corollary 1.1) indicate that the local well-posedness and ill-posedness for the Novikov equation in all critical Besov spaces $B^{1+1/p}_{p,r}(\mathbb{R})$ have been solved completely. Since the Novikov equation has cubic nonlinear term, we expect that norm inflation is stemmed from the worst term $u(\partial_x u)^2$, which is different from the quadratic term $(\partial_x u)^2$ for the Camassa–Holm equation. Our new idea is to construct a initial data which includes two parts, one of whose Fourier transform is supported at high frequencies and the other is supported at low frequencies. Then the cubic nonlinear term $u(\partial_x u)^2$ will generate the low-high-high frequency interaction, which contributes a large quantity lead to the norm inflation. Lastly, we should mention that, by dropping the low frequency term, the initial data $u_0 = n^{-\frac{1}{2}} u_0^{\mathrm{H}}$ can be as an example which leads to the norm inflation for the Camassa–Holm equation (see Remark 3.1).

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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