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On a Stokes System Arising in a Free Surface Viscous Flow of a Horizontally Periodic Fluid with Fractional Boundary Operators

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Abstract. In this note we investigate the initial-boundary value problem for a Stokes system arising in a free surface viscous flow of a horizontally periodic fluid with fractional boundary operators. We derive an integral representation of solutions by making use of the multiple Fourier series. Moreover, we demonstrate a unique solvability in the framework of the Sobolev space of L^2-type.

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1. Introduction

In this note we investigate a Stokes system on the horizontally periodic domain of the mean depth $\frac{\pi}{2}$:

$$\Omega := \left\{ x = (x_1, x') \in \mathbb{R}^3 : 0 < x_1 < \frac{\pi}{2},\ x' := (x_2, x_3) \in \mathbb{T}^2 \right\},$$

where x_1 is the vertical coordinate and $x' = (x_2, x_3)$ is the horizontal coordinates in the 2-dimensional torus $\mathbb{T}^2 = (-\pi, \pi]^2 \cong \mathbb{R}^2/(2\pi\mathbb{Z})^2$. The upper and lower boundaries are denoted by

$$S_F = \left\{ (\frac{\pi}{2}, x') : x' \in \mathbb{T}^2 \right\} \equiv \left\{ x_1 = \frac{\pi}{2} \right\},$$
$$S_B = \{(0, x') : x' \in \mathbb{T}^2\} \equiv \{x_1 = 0\}.$$

Our target system is the governing equations:

$$\frac{\partial \eta}{\partial t} - u_1 = 0 \qquad \text{on } S_F \times (0, \infty), \tag{1.1}$$

$$\frac{\partial u}{\partial t} - \nu \Delta u + \nabla p = 0 \qquad \text{in } \Omega \times (0, \infty), \tag{1.2}$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega \times (0, \infty) \tag{1.3}$$

subject to the boundary conditions:

$$p - 2\nu \frac{\partial u_1}{\partial x_1} - g\left(\eta - \frac{\pi}{2}\right) - \sigma(-\Delta')^{\alpha}\eta = 0 \qquad \text{on } S_F \times (0, \infty), \tag{1.4}$$

$$\frac{\partial u_i}{\partial x_1} + \frac{\partial u_1}{\partial x_i} = 0 \quad (i = 2, 3) \quad \text{on } S_F \times (0, \infty), \tag{1.5}$$

$$u = 0 \qquad \text{on } S_B \times (0, \infty) \tag{1.6}$$

with $\Delta' := \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ and the initial conditions: for $x = (x_1, x') \in \Omega$,

$$(\eta, u)|_{t=0} = (\eta_0, a) = (\eta_0(x'), a_1(x), a_2(x), a_3(x)) \tag{1.7}$$

satisfying certain compatibility conditions. Here $\eta = \eta(x', t)$ is an unknown graph of the free surface on S_F and $u = (u_i(x, t))_{1 \le i \le 3}$ is an unknown velocity field of the fluid with a pressure function $p = p(x, t)$. Furthermore, $\nu > 0$ is a constant coefficient of the viscosity of the fluid, $\sigma > 0$ is the surface tension coefficient, and $g \ge 0$ is a constant describing the effect of gravitational acceleration on the fluid. For simplicity, we assume that only gravity acts on the fluid as an external force.

The motion of a viscous incompressible fluid bounded above by an atmospheric pressure on an upper free surface and below by a rigid bottom is modelled by the *Navier–Stokes equations* with appropriate boundary conditions (cf. [10]), where the effect of the surface tension is taken into account by using the mean curvature of the free interface $x_1 = \eta(x', t)$:

$$\sigma\mathcal{H}[\eta] = \sigma\nabla' \cdot \left(\frac{\nabla'\eta}{\sqrt{1 + |\nabla'\eta|^2}}\right), \qquad \nabla' := (\partial_2, \partial_3). \tag{1.8}$$

In [2], J. T. Beale studied the initial-boundary value problem of an infinite layer of a viscous fluid having a non-compact upper free surface $\{x_1 = \eta(x', t)\}$ and a rigid bottom surface $\{x_1 = -b(x)\}$ under the effect of surface tension at free surface, where he showed by using the contraction mapping principle that there exists a unique global solution to the free surface Navier-Stokes problem for a sufficiently small initial data with certain compatibility conditions. For this task, he transformed the Navier-Stokes system to a linearized system on the infinite layer $(-b(x), 0) \times \mathbb{R}^2$ by stretching/compressing on vertical line segments instead of using the Lagrangian formulation (cf. [2, (2.1)–(2.6) in Sect. 2]).

In [6], Nishida–Teramoto–Yoshihara studied the motion of horizontally periodic Navier–Stokes fluid with surface tension. Then they showed such a global-in-time existence for small initial data and the exponential-in-time decay of energy via the linearized problem with $\alpha = 1$ at the equilibrium domain $(-b, 0) \times \mathbb{T}^2$ (cf. [6, (2.4)–(2.9) in Sect. 2]).

In [9], Tice and Zbarsky introduced the generalized boundary condition of fractional Laplacian type (1.4) in the Stokes system and then studied the decay rate of energy when $0 < \alpha \le 1$ in both cases of the infinite layer $(0, b) \times \mathbb{R}^{d-1}$ and the horizontally periodic domain $(0, b) \times \mathbb{T}^{d-1}$ for arbitrary dimension $d \ge 3$.

In this note, we are concerned with construction of solution operator for the initial-boundary value problem (1.1)–(1.7) constituted of the horizontally periodic Stokes system with fractional boundary operators. Indeed, we show that the problem possesses an integral representation of solutions by making use of the multiple Fourier series (cf. Theorem 3.1). From the alternative formulation, we demonstrate the unique solvability, provided that $0 < \alpha < 3/2$ (cf. Theorem 4.1).

This note is organized as follows. In Sect. 2, as a preliminary, we review the almost-everywhere convergence results of the multiple Fourier series in the Sobolev spaces and also introduce the Sobolev–Slobodeckij spaces of L^2-type. In Sect. 3, we derive a Fourier representation of solutions to the IBVP (1.1)–(1.7). In Sect. 4, we establish the unique solvablity of the IBVP (1.1)–(1.7).

2. Preliminaries

Let $\partial_i = \partial/\partial x_i$ for $i = 1, \dots, d$. We use the standard multi-index notation for the spatial differential operator: $\partial^\beta = \partial_1^{\beta_1} \dots \partial_d^{\beta_d}$ with the order $|\beta| = \beta_1 + \cdots + \beta_d$ for $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$, and $\partial^0 = Id$.

We denote by $\mathbb{T}^d$ the d-dimensional torus $(-\pi, \pi]^d \cong \mathbb{R}^d/(2\pi\mathbb{Z})^d$, and specially, $\mathbb{T} := \mathbb{T}^1$.

Let $\{\mathcal{F}[f](n)\}_{n\in\mathbb{Z}^d}$ denote the Fourier coefficients of an absolutely integral function $f(x) = f(x_1, \dots, x_d)$ on $\mathbb{T}^d$:

$$\mathcal{F}[f](n) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-\sqrt{-1}n\cdot x} f(x)dx, \tag{2.1}$$

where $n \cdot x = n_1x_1 + \cdots + n_dx_d$ and $dx = dx_1 \dots dx_d$. The multiple Fourier series is formally defined by the series

$$\mathcal{F}^*[\mathcal{F}[f]](x) := \sum_{n\in\mathbb{Z}^d} \mathcal{F}[f](n)e^{\sqrt{-1}n\cdot x}. \tag{2.2}$$

Here we recall the celebrated almost-everywhere convergence result of the multiple Fourier series on $\mathbb{T}^d$ (cf. [3,4,7,8]). If $f = f(x)$ lies in the Lebesgue space $L^p(\mathbb{T}^d)$ with $p > 1$, the rectangular partial sum

$$S_N^{\square}[f](x) = \sum_{|n_1|,\dots,|n_d|\le N} \mathcal{F}[f](n)e^{\sqrt{-1}n\cdot x}$$

converges to $f(x)$ for almost every $x \in \mathbb{T}^d$ as $N \to \infty$. Recently, for the *Sobolev* (or *Bessel potential*) *space* $H^{r,p}(\mathbb{T}^d)$ with the norm

$$\|f\|_{H^{r,p}(\mathbb{T}^d)} = \left\| \sum_{n\in\mathbb{Z}^d} (1+|n|^2)^{\frac{r}{2}} \mathcal{F}[f](n)e^{\sqrt{-1}n\cdot x} \right\|_{L^p(\mathbb{T}^d)},$$

Ashurov [1] proved that the spherical partial sum

$$S_N[f](x) = \sum_{|n|^2=n_1^2+\cdots+n_d^2\le N} \mathcal{F}[f](n)e^{\sqrt{-1}n\cdot x}$$

converges to $f(x)$ in $H^{r,p}(\mathbb{T}^d)$ for almost every $x \in \mathbb{T}^d$ as $N \to \infty$, provided that $1 < p \le 2$ and $r > (d-1)(\frac{1}{p} - \frac{1}{2})$. Therefore, when $p = 2$, if $f \in H^{r,2}(\mathbb{T}^d)$ for $r \ge 0$, then we obtain the inversion formula $f(x) = \mathcal{F}^*[\mathcal{F}[f]](x)$ for a.e. $x \in \mathbb{T}^d$, where the Fourier series $\mathcal{F}^*[\,\cdot\,](x)$ corresponds to the pointwise limit of the spherical (resp. rectangular) partial sum when $r > 0$ (resp. $r = 0$). Hence, from the Parseval formula

$$\|f\|^2_{H^{r,2}(\mathbb{T}^d)} = \frac{1}{(2\pi)^d} \sum_{n\in\mathbb{Z}^d} (1+|n|^2)^r \left|\mathcal{F}[f](n)\right|^2,$$

we get the characterization of $H^{r,2}(\mathbb{T}^d) = \{f \in L^2(\mathbb{T}^d) : \|f\|_{H^{r,2}(\mathbb{T}^d)} < \infty\}$. From now on, we shall use the shorthand notation $H^r(\mathbb{T}^d) := H^{r,2}(\mathbb{T}^d)$.

For $m \in \mathbb{N}_0$, we set the *Slobodeckij space* of L^2-type on $D = \mathbb{T}^d$ or Ω:

$$W^m(D) := \left\{ f \in L^2(D) : \|f\|_{W^m(D)} := \sum_{|\beta|=0}^{m} \|\partial^\beta f\|_{L^2(D)} < \infty \right\}.$$

In particular, $W^m(\mathbb{T}^d)$ (resp. $W^0(D)$) coincides with $H^m(\mathbb{T}^d)$ (resp. $L^2(D)$). We also define the space of the $\frac{1}{2}$-fractional type on $\mathbb{T}^d$:

$$W^{m+\frac{1}{2}}(\mathbb{T}^d) := \left\{ f \in W^m(\mathbb{T}^d) : \frac{|f(x)-f(y)|}{|x-y|^{\frac{d+1}{2}}} \in L^2(\mathbb{T}^d \times \mathbb{T}^d) \right\}$$

with the norm

$$\|f\|_{W^{m+\frac{1}{2}}(\mathbb{T}^d)} := \|f\|_{W^m(\mathbb{T}^d)} + \left(\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+1}} dxdy \right)^{\frac{1}{2}}.$$

We should notice that there exists a trace operator $\gamma : W^m(\Omega) \to W^{m-\frac{1}{2}}(S_B \cup S_F)$ such that $\gamma f = f|_{S_B\cup S_F}$ for $m \in \mathbb{N}$ (cf. [5]).

Let us introduce the subspace of $W^m(\mathbb{T}^3)$ (resp. $H^r(\mathbb{T}^3)$) consisting of functions with the sine-like symmetry in $x_1 \in \mathbb{T}$:

$$W^m_{s_1}(\mathbb{T}^3) := \left\{ f \in W^m(\mathbb{T}^3) : f(-x_1, x') = -f(x),\ f(\pi - x_1, x') = f(x) \text{ for } x \in \mathbb{T}^3 \right\}$$
$$\left(\text{resp. } H^r_{s_1}(\mathbb{T}^3) := \left\{ f \in H^r(\mathbb{T}^3) : f(-x_1, x') = -f(x),\ f(\pi - x_1, x') = f(x) \text{ for } x \in \mathbb{T}^3 \right\}\right).$$

The space restricted to Ω is denoted by

$$W^m_{s_1}(\Omega) := \left\{ f = f^\star|_{\{0<x_1<\frac{\pi}{2}\}} : f^\star \in W^m_{s_1}(\mathbb{T}^3) \right\} \tag{2.3}$$

with the norm

$$\|f\|_{W^m_{s_1}(\Omega)} = \|f^\star\|_{W^m(\mathbb{T}^d)}. \tag{2.4}$$

In particular, when $m = 2$ (resp. $m = 3$), $f \in W^m_{s_1}(\mathbb{T}^3)$ satisfies $f = 0$ on S_B (resp. $f = 0$ on S_B and $\partial_1 f = 0$ on S_F), since $W^2(\mathbb{T}^3) \subset C(\mathbb{T}^3)$ (resp. $W^3(\mathbb{T}^3) \subset C^1(\mathbb{T}^3)$) by the embedding theorem.

For a function $f(x) = f(x_1, x_2, x_3) = f(x_1, x') : \mathbb{T}^3 \to \mathbb{R}$, we denote the tangential Fourier coefficients by

$$\mathcal{F}'[f](x_1, n') \equiv \hat{f}(x_1, n') := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{-\sqrt{-1}n'\cdot x'} f(x) dx' \tag{2.5}$$

for $(x_1, n') \in \mathbb{T} \times \mathbb{Z}^2$, where $n' = (n_2, n_3)$, $n' \cdot x' = n_2x_2 + n_3x_3$ and $dx' = dx_2dx_3$. The Fourier series of a given function $\hat{f}(x_1, n') : \mathbb{T} \times \mathbb{Z}^2 \to \mathbb{C}$ is denoted by

$$\mathcal{F}'^*[\hat{f}](x) := \sum_{n'\in\mathbb{Z}^2} \hat{f}(x_1, n') e^{\sqrt{-1}n'\cdot x'}. \tag{2.6}$$

We define the x_1-tangential fractional Laplacian acting on $f(x) : \Omega \to \mathbb{R}$ by

$$\mathcal{F}'[(-\Delta')^\alpha f](x_1, n') := |n'|^{2\alpha} \hat{f}(x_1, n') \qquad (\alpha > 0) \tag{2.7}$$

with $|n'|^2 = n_2^2 + n_3^2$ and $\hat{f}(x_1, n') = \mathcal{F}'[f](x_1, n')$.

For a function $g(x_1) : (0, \frac{\pi}{2}) \to \mathbb{R}$, we set the sine-type (resp. cosine-type) Fourier coefficients

$$\begin{aligned} \mathcal{S}_1[g](n_1) :=& \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin((2n_1 - 1)x_1) g(x_1) dx_1 \\ \Bigg(\mathcal{C}_1[g](n_1) :=& \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \cos((2n_1 - 1)x_1) g(x_1) dx_1\Bigg) \end{aligned} \tag{2.8}$$

for $n_1 \in \mathbb{N}$ and the associated Fourier series

$$\begin{aligned} \mathcal{S}_1^*[\mathcal{S}_1[g]](x_1) &:= \sum_{n\in\mathbb{N}} \mathcal{S}_1[g](n_1) \sin((2n_1 - 1)x_1) \\ \Bigg(\mathcal{C}_1^*[\mathcal{C}_1[g]](x_1) &:= \sum_{n\in\mathbb{N}} \mathcal{C}_1[g](n_1) \cos((2n_1 - 1)x_1)\Bigg) \end{aligned} \tag{2.9}$$

for $x_1 \in \mathbb{T}$.

For a function $f(x) : \Omega \to \mathbb{R}$, we define the coefficients of the hybrid x_1-sine-type Fourier series by

$$\mathcal{F}_{s_1}[f](n) := \mathcal{S}_1[\mathcal{F}'[f]](n) = \frac{1}{\pi^3} \int_\Omega \sin((2n_1 - 1)x_1) e^{-\sqrt{-1}n'\cdot x'} f(x) dx \tag{2.10}$$

for $n = (n_1, n') \in \mathbb{N} \times \mathbb{Z}^2$ and the associated series of $\mathcal{F}_{s_1}[f] : \mathbb{N} \times \mathbb{Z}^2 \to \mathbb{C}$ by

$$\mathcal{F}^*_{s_1}[\mathcal{F}_{s_1}[f]](x) := \sum_{n\in\mathbb{N}\times\mathbb{Z}^2} \mathcal{F}_{s_1}[f](n) \sin((2n_1 - 1)x_1) e^{\sqrt{-1}n'\cdot x'}. \tag{2.11}$$

Here we note that for $f(x) : \mathbb{T}^3 \to \mathbb{R}$ and $l_1 \in \mathbb{Z}$,

$$\begin{aligned}\mathcal{F}[f](l_1, n') &= \frac{1}{2\pi}\int_{\mathbb{T}} e^{-\sqrt{-1}l_1 x_1}\mathcal{F}'[f](x_1, n')dx_1 \\ &= \frac{1}{\pi}\int_0^{\pi} \sin(l_1 x_1)\mathcal{F}'[f](x_1, n')dx_1 \\ &= \begin{cases} 0 & (\text{for } |l_1| = 2n_1), \\ \frac{1}{2}\mathcal{F}_{s_1}[f](n_1, n') & (\text{for } |l_1| = 2n_1 - 1)\end{cases}\end{aligned}$$

for $n_1 \in \mathbb{N}_0$, and thus, $\mathcal{F}^*[\mathcal{F}[f]] = \mathcal{F}^*_{s_1}[\mathcal{F}_{s_1}[f]]$. On the other hand, thanks to the convergence results as mentioned above, the inversion formula $\mathcal{F}^*_{s_1}[\mathcal{F}_{s_1}[f]] = f$ holds on $W^m_{s_1}(\Omega) = H^m_{s_1}(\mathbb{T}^3)$ for $m \in \mathbb{N}_0$ under the suitable limit operation. Therefore, we may redefine

$$\begin{aligned}\|f\|_{W^m_{s_1}(\Omega)} &:= \left(\sum_{n\in\mathbb{N}\times\mathbb{Z}^2} ((2n_1-1)^2 + |n'|^2)^{\frac{m}{2}}\, |\mathcal{F}_{s_1}[f](n)|^2\right)^{\frac{1}{2}} \\ &= \left\|((2n_1-1)^2 + |n'|^2)^{\frac{m}{2}}\mathcal{F}_{s_1}[f](n)\right\|_{l^2(\mathbb{N}\times\mathbb{Z}^2)}.\end{aligned} \tag{2.12}$$

For $\rho, r > 0$, we define the fractional differential operator $|\partial_1|^\rho|\nabla'|^r$ acting on $f \in W^0_{s_1}(\Omega) \cong L^2(\Omega)$ by

$$\mathcal{F}_{s_1}[|\partial_1|^\rho|\nabla'|^r f](n) := (2n_1-1)^\rho|n'|^r\mathcal{F}_{s_1}[f](n) \tag{2.13}$$

for $n = (n_1, n') \in \mathbb{N}\times\mathbb{Z}^2$.

We finish this section to provide basic properties regarding the heat semigroups.

Proposition 2.1. *For $f : \mathbb{T}^d \to \mathbb{R}$, let*

$$e^{\nu t\Delta}f := \mathcal{F}^*[e^{-\nu|n|^2 t}\mathcal{F}[f]] \qquad (t \geq 0). \tag{2.14}$$

Then $\{e^{\nu t\Delta}\}_{t\geq 0}$ is a strongly continuous semigroup on $H^r(\mathbb{T}^d)$.

Moreover, for every $m \in \mathbb{N}$, there exists a constant $C = C(m) > 0$ such that

$$\sup_{t>0} t^{\frac{m}{2}} \sup_{|\beta|=m} \|\partial^\beta e^{\nu t\Delta}f\|_{L^2(\mathbb{T}^d)} \leq C\|f\|_{L^2(\mathbb{T}^d)},$$

and $v := e^{\nu t\Delta}f$ is a smooth solution to the heat equation $\partial_t v = \nu\Delta v$ on $\mathbb{T}^d\times(0,\infty)$, where $\Delta = \partial_1^2 + \cdots + \partial_d^2$.

In addition, $\|e^{\nu t\Delta}f\|_{H^r(\mathbb{T}^2)} \leq \|f\|_{H^r(\mathbb{T}^2)}$ for $t \geq 0$.

Proof. We can verify the main statement by using the inversion formula and the Parseval formula. We omit the detail of the proof. □

Proposition 2.2. *Let $m \in \mathbb{N}_0$ be arbitrary. For $f : \Omega \to \mathbb{R}$, let*

$$e^{\nu t\Delta_{s_1}}f := \mathcal{F}^*_{s_1}[e^{-\nu((2n_1-1)^2+|n'|^2)t}\mathcal{F}_{s_1}[f]] \qquad (t \geq 0). \tag{2.15}$$

Then $\{e^{\nu t\Delta_{s_1}}\}_{t\geq 0}$ is a strongly continuous semigroup on $W^m_{s_1}(\Omega)$ such that $v := e^{\nu t\Delta_{s_1}}f$ is a smooth solution to the heat equation $\partial_t v = \nu\Delta v$ in $\Omega\times(0,\infty)$. Moreover, the solution $v = v(t)$ satisfies $\partial_1^{2k}v = 0$ on S_B and $\partial_1^{2k+1}v = 0$ on S_F for all $t > 0$ and every $k \in \mathbb{N}_0$.

Proof. For $f \in W^m_{s_1}(\Omega)$, let $f^\star \in W^m_{s_1}(\mathbb{T}^d)$ with $f^\star|_{\{0<x_1<\frac{\pi}{2}\}} = f$. Let $v := e^{\nu t\Delta}f^\star$. Since $v = e^{\nu t\Delta_{s_1}}f$, the main assertion follows from Proposition 2.1. Furthermore, it is easily verified that $\partial_1^{2k}v|_{x_1=0} = \partial_1^{2k+1}v|_{x_1=\frac{\pi}{2}} = 0$. □

3. Fourier Representation of Solutions

In this section, we present a Fourier representation of solutions to the IBVP (1.1)–(1.7).

For a function $z = z(x_2, x_3) = z(x') : \mathbb{T}^2 \to \mathbb{R}$, we set the heat semigroups:

$$e^{\nu t\Delta'} z := \mathcal{F}'^*[e^{-\nu|n'|^2 t}\mathcal{F}'[z]], \qquad |n'|^2 = n_2^2 + n_3^2, \tag{3.1}$$

and

$$e^{\nu t\Delta_{s_1}}(x_1 z) := \mathcal{F}^*_{s_1}\left[\frac{4}{\pi}\frac{(-1)^{n_1-1}}{(2n_1-1)^2}e^{-\nu\kappa(n)t}\mathcal{F}'[z]\right], \tag{3.2}$$

which is consistent with the semigroup (2.15) when $f = x_1 z(x_2, x_3)$.

For a set of functions $\{\eta(x',t), \eta_0(x'), u(x,t), a(x), p(x,t)\}$ in (1.1)–(1.7), we set

$$\begin{cases} \hat{\eta} = \hat{\eta}(n',t) := \mathcal{F}'[\eta], \\ \hat{\eta}_0 = \hat{\eta}_0(n',t) := \mathcal{F}'[\eta_0], \\ \hat{u} = (\hat{u}_i(x_1,n',t))_{1\le i\le 3} := \mathcal{F}'[u] = (\mathcal{F}'[u_i])_{1\le i\le 3}, \\ \hat{a} = (\hat{a}_i(x_1,n',t))_{1\le i\le 3} := \mathcal{F}'[a] = (\mathcal{F}'[a_i])_{1\le i\le 3}, \\ \hat{p} = \hat{p}(x_1,n',t) := \mathcal{F}'[p]. \end{cases} \tag{3.3}$$

Theorem 3.1. *Consider the initial-boundary value problem (1.1)–(1.7) with the additional boundary condition:*

$$\partial_1^2 u = 0 \quad \text{on } S_B, \qquad \partial_1^3 u = 0 \quad \text{on } S_F. \tag{3.4}$$

Let

$$\kappa(n) := (2n_1-1)^2 + |n'|^2, \qquad \lambda^\alpha(n') := g + \sigma|n'|^{2\alpha}, \tag{3.5}$$

and let

$$h(x',t) := \eta(x',t) - \frac{\pi}{2}, \qquad \hat{h}(n',t) := \mathcal{F}'[h](n') = \hat{\eta}(n',t) - \frac{\pi}{2}. \tag{3.6}$$

We set the auxiliary functions:

$$z_0(x',t) = e^{\nu t\Delta'}a_1(\tfrac{\pi}{2},x') - \int_0^t e^{\nu(t-s)\Delta'}\mathcal{F}'^*\left[|n'|\tanh\frac{\pi}{2}|n'|\big(2\nu\hat{z}_1(n',s) + \lambda^\alpha(n')\hat{h}(n',s)\big)\right]ds, \tag{3.7}$$

$$z_1(x',t) = e^{3\nu t\Delta'}\partial_1 a_1(\tfrac{\pi}{2},x') + \big(g + (-\Delta')^\alpha\big)\Delta'\int_0^t e^{3\nu(t-s)\Delta'}h(s)ds, \tag{3.8}$$

and

$$Q_1(x,t) := \mathcal{F}^*_{s_1}\left[\frac{4}{\pi}\frac{(-1)^{n_1-1}|n'|^4}{(2n_1-1)^2\kappa(n)}\int_0^t e^{-\nu\kappa(n)(t-s)}\big(2\nu\hat{z}_1(n',s) + \lambda^\alpha(n')\hat{h}(n',s)\big)ds\right], \tag{3.9}$$

$$Q_i(x,t) := \mathcal{F}^*_{s_1}\Bigg[\frac{4}{\pi}\sqrt{-1}n_i|n'|\frac{(-1)^{n_1-1}(2(2n_1-1)^2+|n'|^2)}{(2n_1-1)\kappa(n)}\tanh\frac{\pi}{2}|n'| \times\int_0^t e^{-\nu\kappa(n)(t-s)}\big(2\nu\hat{z}_1(n',s)+\lambda^\alpha(n')\hat{h}(n',s)\big)ds\Bigg] \tag{3.10}$$

for $i = 2, 3$. Then the solution (η, u) satisfies the equations

$$\eta(x',t) = \eta_0(x') + \int_0^t z_0(x',s)ds \tag{3.11}$$

and

$$u_1(x,t) = x_1 z_1(t) - e^{\nu t\Delta_{s_1}}\left(x_1\partial_1 a_1(\tfrac{\pi}{2},x')\right) + e^{\nu t\Delta_{s_1}}a_1 + Q_1(x,t), \tag{3.12}$$

$$u_i(x,t) = -x_1\partial_i z_0(t) + e^{\nu t\Delta_{s_1}}\left(x_1\partial_i a_1(\tfrac{\pi}{2},x')\right) + e^{\nu t\Delta_{s_1}}a_i + Q_i(x,t) \tag{3.13}$$

for $i = 2, 3$.

Proof. We have that the initial datum $a = u|_{t=0}$ satisfies the conditions:

$$\begin{cases} \nabla \cdot a = 0, \quad a|_{x_1=0} = 0, \quad \partial_1^2 a|_{x_1=0} = 0, \quad \partial_1^3 a|_{x_1=\frac{\pi}{2}} = 0, \\ \partial_1 a_i|_{x_1=\frac{\pi}{2}} = -\partial_i a_1|_{x_1=\frac{\pi}{2}} \quad (i = 2, 3). \end{cases} \tag{3.14}$$

Set

$$\begin{cases} v = (v_i(x,t))_{1\le i\le 3} := (\partial_1 u_i)_{1\le i\le 3}, \\ \hat{v} = (\hat{v}_i(x_1, n', t))_{1\le i\le 3} := \mathcal{F}'[v] = (\partial_1 \hat{u}_i)_{1\le i\le 3}, \\ z_1 = z_1(x', t) := v_1|_{x_1=\frac{\pi}{2}} = \partial_1 u_1|_{x_1=\frac{\pi}{2}}, \\ \hat{z}_1 = \hat{z}_1(n', t) := \mathcal{F}'[z_1] = \hat{v}_1|_{x_1=\frac{\pi}{2}} = \partial_1 \hat{u}_1|_{x_1=\frac{\pi}{2}}, \\ z_0 = z_0(x', t) := u_1|_{x_1=\frac{\pi}{2}}, \\ \hat{z}_0 = \hat{z}_0(n', t) := \mathcal{F}'[z_0] = \hat{u}_1|_{x_1=\frac{\pi}{2}}. \end{cases} \tag{3.15}$$

The boundary condition (1.4) can be rewritten as

$$p|_{x_1=\frac{\pi}{2}} = 2\nu z_1 + gh + \sigma(-\Delta')^\alpha h, \tag{3.16}$$

which gives

$$\hat{p}(\tfrac{\pi}{2}, n', t) = 2\nu \hat{z}_1(n', t) + \lambda^\alpha(n')\hat{h}(n', t).$$

Applying the divergence operator $\nabla\cdot$ to the second equation (1.2), we obtain that $\Delta p = 0$, i.e., $(\partial_1^2 - |n'|^2)\hat{p} = 0$. Therefore, $\hat{p} = C_1(n', t)e^{x_1|n'|} + C_2(n', t)e^{-x_1|n'|}$. Here we impose the boundary condition on S_B: $\partial_1 p|_{x_1=0} = 0$, that is, $\partial_1 \hat{p}|_{x_1=0} = 0$. We thus deduce from the two boundary data that

$$\hat{p}(x_1, n', t) = \frac{\cosh x_1|n'|}{\cosh \frac{\pi}{2}|n'|}\big(2\nu \hat{z}_1(n', t) + \lambda^\alpha(n')\hat{h}(n', t)\big). \tag{3.17}$$

Since the second equation (1.2) yields

$$\partial_t v_i - \nu\Delta v_i + \partial_i \partial_1 p = 0 \qquad (i = 1, 2, 3), \tag{3.18}$$

we deduce from the additional boundary condition $\partial_1^3 u|_{x_1=\frac{\pi}{2}} = 0$ in (3.4) that the function $z_1(x', t) = v_1(\frac{\pi}{2}, x', t)$ satisfies

$$\partial_t z_1 - \nu\Delta' z_1 + \partial_1^2 p|_{x_1=\frac{\pi}{2}} = 0, \qquad \Delta' = \partial_2^2 + \partial_3^2. \tag{3.19}$$

On the other hand, we can see from (3.18) with the pressure (3.17) that $\hat{v}_1(x_1, n', t)$ is a solution to the IBVP of the 1-D heat equation:

$$\partial_t \hat{v}_1 - \nu\partial_1^2 \hat{v}_1 + \nu|n'|^2 \hat{v}_1 + |n'|^2 \hat{p} = 0 \tag{3.20}$$

for $(x_1, n', t) \in [0, \ \frac{\pi}{2}] \times \mathbb{Z}^2 \times [0, \infty)$ subject to the boundary conditions

$$\hat{v}_1|_{x_1=\frac{\pi}{2}} = \hat{z}_1, \qquad \partial_1 \hat{v}_1|_{x_1=0} = 0, \qquad \partial_1^2 \hat{v}_1|_{x_1=\frac{\pi}{2}} = 0 \tag{3.21}$$

and the initial condition $\hat{v}_1|_{t=0} = \partial_1 \hat{a}_1$. In particular, it follows from (3.19) with the pressure (3.17) that

$$\frac{d\hat{z}_1}{dt} + \nu|n'|^2 \hat{z}_1 + |n'|^2(2\nu\hat{z}_1 + \lambda^\alpha \hat{h}) = 0, \qquad \hat{z}_1|_{t=0} = \partial_1 \hat{a}_1|_{x_1=\frac{\pi}{2}} \tag{3.22}$$

for $(n', t) \in \mathbb{Z}^2 \times [0, \infty)$, which yields

$$\hat{z}_1(n', t) = e^{-3\nu|n'|^2 t}\partial_1 \hat{a}_1|_{x_1=\frac{\pi}{2}} - \lambda^\alpha(n')|n'|^2 \int_0^t e^{-3\nu|n'|^2(t-s)}\hat{h}(n', s)ds. \tag{3.23}$$

Therefore, we obtain (3.8).

Set $\hat{w}_1 := \hat{v}_1 - \hat{z}_1$. We deduce from (3.20)–(3.22) with (3.17) that

$$\partial_t \hat{w}_1 - \nu\partial_1^2 \hat{w}_1 + \nu|n'|^2 \hat{w}_1 + |n'|^2 \left(\frac{\cosh x_1|n'|}{\cosh \frac{\pi}{2}|n'|} - 1\right)(2\nu\hat{z}_1 + \lambda^\alpha \hat{h}) = 0 \tag{3.24}$$

subject to the boundary conditions

$$\hat{w}_1|_{x_1=\frac{\pi}{2}}=0, \qquad \partial_1\hat{w}_1|_{x_1=0}=0, \qquad \partial_1^2\hat{w}_1|_{x_1=\frac{\pi}{2}}=0 \tag{3.25}$$

and the initial condition $\hat{w}_1|_{t=0}=\partial_1\hat{a}_1-\partial_1\hat{a}_1|_{x_1=\frac{\pi}{2}}$. By using the Fourier method and the Duhamel formula, we get

$$\begin{aligned}
&\hat{w}_1(x_1,n',t)\\
&\quad=\mathcal{C}_1^*\big[e^{-\nu\kappa(n)t}\mathcal{C}_1[\partial_1\hat{a}_1-\partial_1\hat{a}_1|_{x_1=\frac{\pi}{2}}]\big]\\
&\qquad-|n'|^2\mathcal{C}_1^*\left[\mathcal{C}_1\left[\frac{\cosh x_1|n'|}{\cosh\frac{\pi}{2}|n'|}-1\right]\int_0^t e^{-\nu\kappa(n)(t-s)}(2\nu\hat{z}_1(s)+\lambda^\alpha\hat{h}(s))ds\right]\\
&\quad=\mathcal{C}_1^*\big[(2n_1-1)e^{-\nu\kappa(n)t}\mathcal{F}_{s_1}[a_1]\big]-\mathcal{C}_1^*\left[\frac{4}{\pi}\frac{(-1)^{n_1-1}}{2n_1-1}e^{-\nu\kappa(n)t}\partial_1\hat{a}_1|_{x_1=\frac{\pi}{2}}\right]\\
&\qquad-\mathcal{C}_1^*\left[\frac{4}{\pi}(-1)^{n_1-1}|n'|^2\left(\frac{2n_1-1}{(2n_1-1)^2+|n'|^2}-\frac{1}{2n_1-1}\right)\int_0^t e^{-\nu\kappa(n)(t-s)}(2\nu\hat{z}_1(s)+\lambda^\alpha\hat{h}(s))ds\right]\\
&\quad=\mathcal{C}_1^*\big[(2n_1-1)e^{-\nu\kappa(n)t}\mathcal{F}_{s_1}[a_1]\big]-\frac{4}{\pi}\mathcal{C}_1^*\left[\frac{(-1)^{n_1-1}}{2n_1-1}e^{-\nu\kappa(n)t}\partial_1\hat{a}_1|_{x_1=\frac{\pi}{2}}\right]\\
&\qquad+\mathcal{C}_1^*\left[\frac{4}{\pi}\frac{(-1)^{n_1-1}|n'|^4}{(2n_1-1)\kappa(n)}\int_0^t e^{-\nu\kappa(n)(t-s)}(2\nu\hat{z}_1(s)+\lambda^\alpha\hat{h}(s))ds\right],
\end{aligned}$$

since

$$\mathcal{C}_1[\partial_1\hat{a}_1]=(2n_1-1)\mathcal{F}_{s_1}[a_1], \qquad \mathcal{C}_1[1]=\frac{4}{\pi}\frac{(-1)^{n_1-1}}{2n_1-1},$$

and

$$\mathcal{C}_1\big[\cosh x_1|n'|\big]=\frac{4}{\pi}\frac{(-1)^{n_1-1}(2n_1-1)}{(2n_1-1)^2+|n'|^2}\cosh\frac{\pi}{2}|n'|.$$

On the other hand, we have

$$\hat{u}_1(t)=\int_0^{x_1}\hat{v}_1(y_1,n',t)dy_1=x_1\hat{z}_1(t)+\int_0^{x_1}\hat{w}_1(y_1,n',t)dy_1, \tag{3.26}$$

since $\partial_1\hat{u}_1=\hat{v}_1=\hat{z}_1+\hat{w}_1$ and $\hat{u}_1|_{x_1=0}=0$. We compute

$$\begin{aligned}
\int_0^{x_1}\hat{w}_1(y_1,n',t)dy_1&=\mathcal{S}_1^*\big[e^{-\nu\kappa(n)t}\mathcal{F}_{s_1}[a_1]\big]-\mathcal{S}_1^*\left[\frac{4}{\pi}\frac{(-1)^{n_1-1}}{(2n_1-1)^2}e^{-\nu\kappa(n)t}\partial_1\hat{a}_1|_{x_1=\frac{\pi}{2}}\right]\\
&\quad+\mathcal{S}_1^*\left[\frac{4}{\pi}\frac{(-1)^{n_1-1}|n'|^4}{(2n_1-1)^2\kappa(n)}\int_0^t e^{-\nu\kappa(n)(t-s)}\big(2\nu\hat{z}_1(s)+\lambda^\alpha\hat{h}(s)\big)ds\right].
\end{aligned}$$

We apply $\mathcal{F}'^*[\,\cdot\,]$ to the both sides in (3.26) to obtain (3.12).

We deduce from the first component in equation (1.2) with the pressure (3.17) that

$$\partial_t\hat{u}_1-\nu\partial_1^2\hat{u}_1+\nu|n'|^2\hat{u}_1+|n'|\frac{\sinh x_1|n'|}{\cosh\frac{\pi}{2}|n'|}(2\nu\hat{z}_1+\lambda^\alpha\hat{h})=0.$$

Thus $\hat{z}_0(n',t)=\hat{u}_1(\frac{\pi}{2},n',t)$ is governed by

$$\frac{d\hat{z}_0}{dt}+\nu|n'|^2\hat{z}_0+|n'|\tanh\frac{\pi}{2}|n'|(2\nu\hat{z}_1+\lambda^\alpha\hat{h})=0, \qquad \hat{z}_0|_{t=0}=\hat{a}_1|_{x_1=\frac{\pi}{2}}. \tag{3.27}$$

since $\partial_1^2\hat{u}_1|_{x_1=\frac{\pi}{2}}=0$ (cf. (3.4)). That is,

$$\hat{z}_0(t)=e^{-\nu|n'|^2t}\hat{a}_1|_{x_1=\frac{\pi}{2}}-|n'|\tanh\frac{\pi}{2}|n'|\int_0^t e^{-\nu|n'|^2(t-s)}\big(2\nu\hat{z}_1(s)+\lambda^\alpha\hat{h}(s)\big)ds, \tag{3.28}$$

and we get Eq. (3.7). Furthermore, Eq. (3.11) follows from the first equation (1.1).

Let us derive integral representations of $\hat{u}_i = \hat{u}_i(t)$ for $i = 2, 3$ similarly as above. For $\hat{v}_i$ for each $i = 2, 3$, we deduce from (3.18) with the pressure (3.17) the governing equation

$$\partial_t \hat{v}_i - \nu \partial_1^2 \hat{v}_i + \nu |n'|^2 \hat{v}_i + \sqrt{-1} n_i |n'| \frac{\sinh x_1 |n'|}{\cosh \frac{\pi}{2} |n'|} (2\nu \hat{z}_1 + \lambda^\alpha \hat{h}) = 0 \tag{3.29}$$

for all $(x_1, n', t) \in [0, \frac{\pi}{2}] \times \mathbb{Z}^2 \times [0, \infty)$ and from (1.5) and (3.4) the boundary conditions

$$\hat{v}_i|_{x_1=\frac{\pi}{2}} = -\sqrt{-1} n_i \hat{u}_1|_{x_1=\frac{\pi}{2}}, \qquad \partial_1 \hat{v}_i|_{x_1=0} = 0, \qquad \partial_1^2 \hat{v}_i|_{x_1=\frac{\pi}{2}} = 0 \tag{3.30}$$

and from (1.7) the initial condition

$$\hat{v}_i|_{t=0} = \partial_1 \hat{a}_i. \tag{3.31}$$

Set

$$\hat{w}_i := \hat{v}_i + \sqrt{-1} n_i \hat{z}_0 \qquad (i = 2, 3).$$

From (3.27)–(3.31), we have the IBVP of the 1–D heat equation on $[0, \frac{\pi}{2}]$: for $i = 2, 3$,

$$\partial_t \hat{w}_i - \nu \partial_1^2 \hat{w}_i + \nu |n'|^2 \hat{w}_i + \sqrt{-1} n_i |n'| \left(\frac{\sinh x_1 |n'|}{\cosh \frac{\pi}{2} |n'|} + \tanh \frac{\pi}{2} |n'| \right) (2\nu \hat{z}_1 + \lambda^\alpha \hat{h}) = 0$$

subject to the boundary conditions

$$\hat{w}_i|_{x_1=\frac{\pi}{2}} = 0, \qquad \partial_1 \hat{w}_i|_{x_1=0} = 0, \qquad \partial_1^2 \hat{w}_i|_{x_1=\frac{\pi}{2}} = 0$$

and the initial condition

$$\hat{w}_i|_{t=0} = \partial_1 \hat{a}_i + \sqrt{-1} n_i \hat{a}_1|_{x_1=\frac{\pi}{2}}.$$

Again, by using the Fourier method and the Duhamel formula, we get

$$\begin{aligned}
&\hat{w}_i(x_1, n', t) \\
&= \mathcal{C}_1^* \left[e^{-\nu\kappa(n)t} \mathcal{C}_1 [\partial_1 \hat{a}_i + \sqrt{-1} n_i \hat{a}_1|_{x_1=\frac{\pi}{2}}] \right] \\
&\quad - \mathcal{C}_1^* \left[\mathcal{C}_1 \left[\frac{\sinh x_1 |n'|}{\cosh \frac{\pi}{2} |n'|} + \tanh \frac{\pi}{2} |n'| \right] \sqrt{-1} n_i |n'| \int_0^t e^{-\nu\kappa(n)(t-s)} (2\nu \hat{z}_1(s) + \lambda^\alpha \hat{h}(s)) ds \right] \\
&= \mathcal{C}_1^* \left[(2n_1 - 1) e^{-\nu\kappa(n)t} \mathcal{F}_{s_1}[a_i] \right] + \mathcal{C}_1^* \left[\frac{4}{\pi} \frac{(-1)^{n_1-1}}{2n_1 - 1} e^{-\nu\kappa(n)t} \mathcal{F}' [\partial_i a_1] (\tfrac{\pi}{2}, n') \right] \\
&\quad - \mathcal{C}_1^* \left[\frac{4}{\pi} \sqrt{-1} n_i |n'| \left(\frac{(-1)^{n_1-1}(2n_1 - 1)}{(2n_1 - 1)^2 + |n'|^2} + \frac{(-1)^{n_1-1}}{2n_1 - 1} \right) \tanh \frac{\pi}{2} |n'| \right. \\
&\quad \left. \times \int_0^t e^{-\nu\kappa(n)(t-s)} (2\nu \hat{z}_1(s) + \lambda^\alpha \hat{h}(s)) ds \right],
\end{aligned}$$

since

$$\mathcal{C}_1 \left[\sinh x_1 |n'| \right] = \frac{4}{\pi} \frac{(-1)^{n_1-1}(2n_1 - 1)}{(2n_1 - 1)^2 + |n'|^2} \sinh \frac{\pi}{2} |n'|.$$

On the other hand, we have

$$\hat{u}_i(t) = \int_0^{x_1} \hat{v}_i(y_1, n', t) dy_1 = -x_1 \sqrt{-1} n_i \hat{z}_0(t) + \int_0^{x_1} \hat{w}_i(y_1, n', t) dy_1, \tag{3.32}$$

since $\partial_1 \hat{u}_i = \hat{v}_i = -\sqrt{-1}n_i\hat{z}_0 + \hat{w}_i$ and $\hat{u}_i|_{x_1=0} = 0$. We compute

$$\begin{aligned}\int_0^{x_1} &\hat{w}_1(y_1, n', t)dy_1 \\ &= \mathcal{S}_1^*\big[e^{-\nu\kappa(n)t}\mathcal{F}_{s_1}[a_1]\big] + \mathcal{S}_1^*\left[\frac{4}{\pi}\frac{(-1)^{n_1-1}}{(2n_1-1)^2}e^{-\nu\kappa(n)t}\mathcal{F}'[\partial_i a_1](\tfrac{\pi}{2}, n')\right] \\ &\quad - \mathcal{S}_1^*\left[\frac{4}{\pi}\sqrt{-1}n_i|n'|\frac{(-1)^{n_1-1}(2(2n_1-1)^2+|n'|^2)}{(2n_1-1)\kappa(n)}\tanh\frac{\pi}{2}|n'|\right. \\ &\quad\quad \left.\times\int_0^t e^{-\nu\kappa(n)(t-s)}\big(2\nu\hat{z}_1(s) + \lambda^\alpha\hat{h}(s)\big)ds\right].\end{aligned}$$

Therefore, (3.13) is obtained by applying $\mathcal{F}'^*$ to the both sides in (3.32). □

4. Unique Solvability

In this section, we establish the unique solvability of the IBVP (1.1)–(1.7) under the condition $0 < \alpha < 3/2$.

Theorem 4.1. *Assume $0 < \alpha < 3/2$. Let an integer $m \geq 2$ and $T > 0$ be arbitrary. Suppose that $\eta_0 \in H^{m+1}(\mathbb{T}^2)$ and that $a = (a_i)_{1\leq i\leq 3}$, $a_i \in W^m_{s_1}(\Omega)$ satisfies the compatibility conditions:*

$$\nabla\cdot a = 0 \quad \text{in } \Omega, \qquad \partial_1 a_i + \partial_i a_1 = 0 \quad (i = 2,3) \quad \text{on } S_F. \tag{4.1}$$

Then there exist unique functions h, z_0, z_1 and $Q = (Q_i)_{1\leq i\leq 3}$ in (3.6)–(3.10) such that

$$h \in L^\infty([0,T), H^{m+1}(\mathbb{T}^2)), \quad h|_{t=0} = \eta_0 - \frac{\pi}{2}, \tag{4.2}$$

$$z_0 \in C([0,T), H^{m-1}(\mathbb{T}^2)), \quad z_0|_{t=0} = a_1|_{S_F}, \tag{4.3}$$

$$z_1 \in C([0,T), H^{m-2}(\mathbb{T}^2)), \quad z_1|_{t=0} = \partial_1 a_1|_{S_F}, \tag{4.4}$$

$$|\partial_1|^\rho|\nabla'|^{m-2}Q_1 \in L^\infty([0,T), L^2(\mathbb{T}^3)) \quad (\forall\rho < 3/2), \quad Q_1|_{t=0} = 0, \tag{4.5}$$

$$|\partial_1|^{\rho'}|\nabla'|^{m-2}Q_i \in L^\infty([0,T), L^2(\mathbb{T}^3)) \quad (\forall\rho' < 1/2), \quad Q_i|_{t=0} = 0 \tag{4.6}$$

for $i = 2,3$, and the velocity field $u = (u_i(x,t))_{1\leq i\leq 3}$ given by

$$u_1(x,t) = x_1 z_1 + e^{\nu t\Delta_{s_1}}a_1 + Q_1(x,t), \tag{4.7}$$

$$u_i(x,t) = -x_1\partial_i z_0 + e^{\nu t\Delta_{s_1}}\big(x_1\partial_i a_1|_{S_F}\big) + e^{\nu t\Delta_{s_1}}a_i + Q_i(x,t) \quad (i = 2,3) \tag{4.8}$$

is a solution with $\eta(x',t) = h(x',t) + \frac{\pi}{2}$ to the problem (1.1)–(1.7).

Proof. Let us redefine the H^r-norm of a function $z(x') = z(x_2,x_3) : \mathbb{T}^2 \to \mathbb{R}$ for $r \geq 0$ by

$$\begin{aligned}\|z\|_{H^r(\mathbb{T}^2)} &:= |\mathcal{F}'[z](0)| + \left(\sum_{n'\in\mathbb{Z}^2\setminus\{0\}}|n'|^r\,|\mathcal{F}'[z](n')|^2\right)^{\frac{1}{2}} \\ &\equiv |\mathcal{F}'[z](0)| + \big\||n'|^r\mathcal{F}'[z](n')\big\|_{l^2(\mathbb{Z}^2)}.\end{aligned} \tag{4.9}$$

We have from Eq. (1.1) that $\{\hat{h}(n',t)\}_{n'\in\mathbb{Z}^2,t>0}$ is governed by

$$\hat{h}(n',t) = \hat{h}_0(n') + \int_0^t \hat{z}_0(n',t), \qquad \hat{h}_0(n') := \hat{\eta}_0(n') - \frac{\pi}{2}, \tag{4.10}$$

where

$$\hat{z}_0(n',t) = e^{-\nu|n'|^2 t}\hat{a}_1(\tfrac{\pi}{2}, n') - |n'|\tanh\frac{\pi}{2}|n'|\int_0^t e^{-\nu|n'|^2(t-s)}\big(2\nu\hat{z}_1(n',s) + \lambda^\alpha(n')\hat{h}(n',s)\big)ds \tag{4.11}$$

associated with

$$\hat{z}_1(n',t) = e^{-\nu|n'|^2 t}\partial_1\hat{a}_1(\tfrac{\pi}{2},n') - \lambda^\alpha(n')|n'|^2\int_0^t e^{-3\nu|n'|^2(t-s)}\hat{h}(n',s)ds, \tag{4.12}$$

corresponding to equations (3.28) and (3.23) respectively.

For $n'=0$, we deduce from (4.10) with (4.11) that

$$\hat{h}(0,t) = \hat{h}_0(0) + t\hat{a}_1(\tfrac{\pi}{2},0) = \hat{\eta}_0(0) - \frac{\pi}{2} + t\hat{a}_1(\tfrac{\pi}{2},0). \tag{4.13}$$

For every $n'\neq 0$, by substituting (4.11) into (4.10) and integrating by parts, we obtain that

$$\begin{aligned}\hat{h}(n',t) &= \hat{h}_0(n') + \frac{1-e^{-\nu|n'|^2 t}}{\nu|n'|^2}\hat{a}_1(\tfrac{\pi}{2},n') - 2\frac{\tanh\frac{\pi}{2}|n'|}{|n'|}\int_0^t\big(1-e^{-\nu|n'|^2(t-s)}\big)\hat{z}_1(n',s)ds\\ &\quad - \frac{\lambda^\alpha(n')}{\nu|n'|^2}\tanh\frac{\pi}{2}|n'|\int_0^t\big(1-e^{-\nu|n'|^2(t-s)}\big)\hat{h}(n',s)ds.\end{aligned}$$

Substituting (4.12) into the above equation, we get

$$\begin{aligned}\hat{h}(n',t) &= I_0(n',t) + 2\lambda^\alpha(n')|n'|\tanh\frac{\pi}{2}|n'|\bigg(\int_0^t e^{-3\nu|n'|^2 s}\int_0^s e^{3\nu|n'|^2\sigma}\hat{h}(n',\sigma)d\sigma\\ &\quad - e^{-\nu|n'|^2 t}\int_0^t e^{-2\nu|n'|^2 s}\int_0^s e^{3\nu|n'|^2\sigma}\hat{h}(n',\sigma)d\sigma\bigg)\\ &\quad - \frac{\lambda^\alpha(n')}{\nu|n'|^2}\tanh\frac{\pi}{2}|n'|\int_0^t\big(1-e^{-\nu|n'|^2(t-s)}\big)\hat{h}(n',s)ds\\ &= I_0(n',t) + 2\lambda^\alpha(n')|n'|\tanh\frac{\pi}{2}|n'|\bigg(\frac{1}{3\nu|n'|^2}\int_0^t\big(1-e^{-3\nu|n'|^2(t-s)}\big)\hat{h}(n',s)ds\\ &\quad - \frac{1}{2\nu|n'|^2}\int_0^t\big(e^{-\nu|n'|^2(t-s)} - e^{-3\nu|n'|^2(t-s)}\big)\hat{h}(n',s)ds\bigg)\\ &\quad - \frac{\lambda^\alpha(n')}{\nu|n'|^2}\tanh\frac{\pi}{2}|n'|\int_0^t\big(1-e^{-\nu|n'|^2(t-s)}\big)\hat{h}(n',s)ds\\ &= I_0(n',t) - \frac{\lambda^\alpha(n')}{3\nu|n'|}\tanh\frac{\pi}{2}|n'|\int_0^t\big(1-e^{-3\nu|n'|^2(t-s)}\big)\hat{h}(n',s)ds,\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}I_0(n',t) &:= \hat{h}_0(n') + \frac{1-e^{-\nu|n'|^2 t}}{\nu|n'|^2}\hat{a}_1(\tfrac{\pi}{2},n') - \frac{2}{\nu}\frac{\tanh\frac{\pi}{2}|n'|}{|n'|^3}\big(1-e^{-\nu|n'|^2 t}\big)\partial_1\hat{a}_1(\tfrac{\pi}{2},n')\\ &\quad + \frac{2\tanh\frac{\pi}{2}|n'|}{|n'|}te^{-\nu|n'|^2 t}\partial_1\hat{a}_1(\tfrac{\pi}{2},n').\end{aligned}$$

Let

$$\mu^\alpha(n') := \lambda^\alpha(n')|n'|\tanh\frac{\pi}{2}|n'| = \big(g|n'| + \sigma|n'|^{2\alpha+1}\big)\tanh\frac{\pi}{2}|n'|.$$

Differentiating the Eq. (4.14), we obtain that $\hat{h}=\hat{h}(n',t)$ satisfies

$$D_t\hat{h}(n',t) = D_t\hat{I}_0(n',t) - \mu^\alpha(n')e^{-3\nu|n'|^2 t}\int_0^t e^{3\nu|n'|^2 s}\hat{h}(n',s)ds$$

with $D_t = d/dt$. Multiplying the above equation by $e^{3\nu|n'|t}$, we get by differentiation,

$$D_t\big(e^{3\nu|n'|^2 t}D_t\hat{h}\big) = D_t\big(e^{3\nu|n'|^2 t}D_t\hat{I}_0\big) - \mu^\alpha(n')e^{3\nu|n'|^2 t}\hat{h}.$$

Thus we obtain the governing ODE of $\{\hat{h}(n',t)\}_{n'\neq 0, t>0}$:

$$\big(D_t^2 + 3\nu|n'|^2 D_t + \mu^\alpha(n')\big)\hat{h} = D_t(D_t + 3\nu|n'|^2)I_0 =: D_t J_0(n',t) \tag{4.15}$$

with the initial conditions

$$\hat{h}(n',0)=\hat{h}_0(n'), \qquad D_t\hat{h}(n',0)=\hat{z}_0(n',0)=\hat{a}_1(\tfrac{\pi}{2},n'), \tag{4.16}$$

where

$$\begin{aligned} J_0(n',t) &:= \big(D_t+3\nu|n'|^2\big)I_0(n',t) \\ &= 3\nu|n'|^2\hat{h}_0(n')+(3-2e^{-\nu|n'|^2t})\hat{a}_1(\tfrac{\pi}{2},n') \\ &\quad+\frac{2\tanh\frac{\pi}{2}|n'|}{|n'|}\big(e^{-\nu|n'|^2t}+2\nu|n'|^2te^{-\nu|n'|^2t}-3\big)\partial_1\hat{a}_1(\tfrac{\pi}{2},n'). \end{aligned}$$

Since the characteristic equation $\xi^2+3\nu|n'|^2\xi+\mu^\alpha(n')=0$, we set

$$\begin{aligned} \omega_1(n') &:= \frac{3}{2}\nu|n'|^2\big(1-\sqrt{R^\alpha(n')}\big), \\ \omega_2(n') &:= \frac{3}{2}\nu|n'|^2\big(1+\sqrt{R^\alpha(n')}\big) \end{aligned}$$

with

$$R^\alpha(n')=1-\frac{4\mu^\alpha(n')}{9\nu^2|n'|^4}=1-\frac{4\tanh\frac{\pi}{2}|n'|}{9\nu^2}\left(\frac{g}{|n'|^3}+\frac{\sigma}{|n'|^{3-2\alpha}}\right).$$

Then we can find a number $N_0\in\mathbb{N}$, a sum of two squares such that

$$\text{(i) } R^\alpha(n')>0 \quad \text{for } |n'|\ge\sqrt{N_0} \qquad \text{(ii) } R^\alpha(n')\le 0 \quad \text{for } |n'|\le\sqrt{N_0-1}, \tag{4.17}$$

since $\tanh\frac{\pi}{2}|n'|/|n'|^{3-2\alpha}$ is a decreasing function in terms of $|n'|$ for every $\alpha<3/2$. Furthermore, thanks to the discreteness of n', there exists a constant $\varepsilon_0=\varepsilon_0(\nu,g,\sigma,\alpha)\in(0,1)$ independent of n' such that

$$R^\alpha(n')\ge\varepsilon_0>0 \quad \text{for } |n'|\ge\sqrt{N_0}.$$

In the case (i), we set

$$g_1(n',t):=e^{-\omega_1(n')t}, \qquad g_2(n',t):=e^{-\omega_2(n')t}$$

with the Wronskian

$$\begin{aligned} W(n',t) &:= g_1(n',t)D_tg_2(n',t)-D_tg_1(n',t)g_2(n',t) \\ &= -3\nu|n'|^2\sqrt{R^\alpha(n')}e^{-3\nu|n'|^2t}. \end{aligned} \tag{4.18}$$

By the solution formula for the 2nd order linear ODE:

$$\begin{aligned} \hat{h}(n',t) &= \left(C_1-\int D_tJ_0(n',t)\frac{g_2(n',t)}{W(n',t)}dt\right)g_1(n',t) \\ &\quad+\left(C_2+\int D_tJ_0(n',t)\frac{g_1(n',t)}{W(n',t)}dt\right)g_2(n',t) \\ &= \left(C_1+\int J_0(n',t)D_t\left(\frac{g_2(n',t)}{W(n',t)}\right)dt\right)g_1(n',t) \\ &\quad+\left(C_2-\int J_0(n',t)D_t\left(\frac{g_1(n',t)}{W(n',t)}\right)dt\right)g_2(n',t), \end{aligned} \tag{4.19}$$

we obtain the general solution to (4.15) given by

$$\begin{aligned} \hat{h}(n',t) &= \left(C_1(n')-\frac{\omega_1(n')}{3\nu|n'|^2\sqrt{R^\alpha(n')}}\int_0^t e^{\omega_1(n')s}J_0(n',s)ds\right)e^{-\omega_1(n')t} \\ &\quad+\left(C_2(n')+\frac{\omega_2(n')}{3\nu|n'|^2\sqrt{R^\alpha(n')}}\int_0^t e^{\omega_2(n')s}J_0(n',s)ds\right)e^{-\omega_2(n')t}. \end{aligned}$$

We also have that the initial conditions of $\hat{h}(n',t)$ imply $C_1+C_2=\hat{h}_0(n')$ and $-\omega_1(n')C_1-\omega_2(n')C_2+J_0(n',0)=\hat{a}_1(\frac{\pi}{2},n')$, that is,

$$\begin{pmatrix}C_1\\C_2\end{pmatrix}=\frac{1}{\omega_2-\omega_1}\begin{pmatrix}\omega_2&-1\\-\omega_1&1\end{pmatrix}\begin{pmatrix}\hat{h}_0(n')\\J_0(n',0)-\hat{a}_1(\frac{\pi}{2},n')\end{pmatrix}$$
$$=\frac{\hat{h}_0(n')}{2}\begin{pmatrix}1+1/\sqrt{R^\alpha(n')}\\1-1/\sqrt{R^\alpha(n')}\end{pmatrix}+\frac{\hat{a}_1(\frac{\pi}{2},n')-J_0(n',0)}{3\nu|n'|^2\sqrt{R^\alpha(n')}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Since

$$|J_0(n',t)|\le 3\nu|n'|^2|\hat{h}_0(n')|+3|\hat{a}_1(\tfrac{\pi}{2},n')|+\frac{6}{|n'|}|\partial_1\hat{a}_1(\tfrac{\pi}{2},n')|$$

and

$$|C_1(n')|+|C_2(n')|\le\left(1+\frac{1}{\sqrt{\varepsilon_0}}\right)|\hat{h}_0(n')|+\frac{2}{3\nu|n'|^2\sqrt{\varepsilon_0}}|\hat{a}_1(\tfrac{\pi}{2},n')-J_0(n',0)|$$
$$\le\left(1+\frac{3}{\sqrt{\varepsilon_0}}\right)|\hat{h}_0(n')|+\frac{3}{\nu|n'|^3\sqrt{\varepsilon_0}}|\partial_1\hat{a}_1(\tfrac{\pi}{2},n')|,$$

we deduce from (4.19) that

$$\begin{aligned}|\hat{h}(n',t)|&\le|C_1(n')|+|C_2(n')|+\frac{2-e^{\omega_1(n')t}-e^{\omega_2(n')t}}{3\nu|n'|^2\sqrt{\varepsilon_0}}\sup_{0<s<t}|J_0(n',s)|\\&\le\left(1+\frac{3}{\sqrt{\varepsilon_0}}\right)|\hat{h}_0(n')|+\frac{1}{\nu\sqrt{\varepsilon_0}}\left(2\frac{|\hat{a}_1(\frac{\pi}{2},n')|}{|n'|^2}+7\frac{|\partial_1\hat{a}_1(\frac{\pi}{2},n')|}{|n'|^3}\right)\end{aligned}\tag{4.20}$$

for all $|n'|\ge\sqrt{N_0}$. Therefore, it follows from the above estimate that

$$\begin{aligned}&\left\||n'|^{m+1}|\hat{h}(n',t)|\right\|_{l^2(\{|n'|\ge\sqrt{N_0}\})}\\&\le C_3\Big(\left\||n'|^{m+1}|\hat{h}_0(n')|\right\|_{l^2(\{|n'|\ge\sqrt{N_0}\})}+\left\||n'|^{m-1}|\hat{a}_1(n')|\right\|_{l^2(\{|n'|\ge\sqrt{N_0}\})}\\&\quad+\left\||n'|^{m-2}|\partial_1\hat{a}_1(n')|\right\|_{l^2(\{|n'|\ge\sqrt{N_0}\})}\Big)\end{aligned}\tag{4.21}$$

with a constant $C_3=C_3(\nu,g,\sigma,\alpha)>0$. If $N_0=1$, then we have

$$\|h(t)\|_{H^{m+1}(\mathbb{T}^2)}\le C_3(1+T)\big(1+\|\eta_0\|_{H^{m+1}(\mathbb{T}^2)}+\|a_1\|_{W^m_{s_1}(\Omega)}\big),\tag{4.22}$$

since (4.13). On the other hand, if $N_0>1$, we must divide the case (ii) in (4.17) into the two cases separately:

(a) $R^\alpha(n')<0$ for $|n'|\le\sqrt{N_0-1}$ (b) $R^\alpha(n')=0$ for $|n'|=\sqrt{N_0-1}$.

In the case (a), a set of solutions $\{\hat{h}(n',t)\}_{1\le|n'|\le\sqrt{N_0-1},t>0}$ is obtained by the solution formula (4.19) with

$$g_1(n',t):=e^{-\frac{3}{2}\nu|n'|^2t}\cos\left(\frac{3}{2}\nu\sqrt{-R^\alpha(n')}|n'|^2t\right),$$
$$g_2(n',t):=e^{-\frac{3}{2}\nu|n'|^2t}\sin\left(\frac{3}{2}\nu\sqrt{-R^\alpha(n')}|n'|^2t\right),$$

associated with the Wronskian $W(n',t)$. Similarly as above, since there exists a constant $\varepsilon_1=\varepsilon_1(\nu,g,\sigma,\alpha)>0$ independent of n' such that

$$-R^\alpha(n')\ge\varepsilon_1>0\quad\text{for }|n'|\le\sqrt{N_0-1},$$

we can find a constant $C_4 = C_4(\nu, g, \sigma, \alpha) > 0$ such that

$$\begin{aligned} &\big\| |n'|^{m+1} |\hat{h}(n',t)| \big\|_{l^2(\{|n'|\le\sqrt{N_0-1}\})} \\ &\quad \le C_4 e^{-\frac{3}{2}\nu|n'|^2 t} \Big(\big\| |n'|^{m+1} |\hat{h}_0(n')| \big\|_{l^2(\{|n'|\le\sqrt{N_0-1}\})} + \big\| |n'|^{m-1} |\hat{a}_1(n')| \big\|_{l^2(\{|n'|\le\sqrt{N_0-1}\})} \\ &\qquad + \big\| |n'|^{m-2} |\partial_1 \hat{a}_1(n')| \big\|_{l^2(\{|n'|\le\sqrt{N_0-1}\})} \Big). \end{aligned} \tag{4.23}$$

In the case (b), a set of solutions $\{\hat{h}(n',t)\}_{|n'|=\sqrt{N_0-1}, t>0}$ is obtained by the solution formula (4.19) with $g_1(n',t) := e^{-\frac{3}{2}\nu|n'|^2 t}$ and $g_2(n',t) := te^{-\frac{3}{2}\nu|n'|^2 t}$. We also have

$$\begin{aligned} &\big\| |n'|^{m+1} |\hat{h}(n',t)| \big\|_{l^2(\{|n'|=\sqrt{N_0-1}\})} \\ &\quad \le C_5(1+t) e^{-\frac{3}{2}\nu|n'|^2 t} \Big(\big\| |n'|^{m+1} |\hat{h}_0(n')| \big\|_{l^2(\{|n'|=\sqrt{N_0-1}\})} + \big\| |n'|^{m-1} |\hat{a}_1(n')| \big\|_{l^2(\{|n'|=\sqrt{N_0-1}\})} \\ &\qquad + \big\| |n'|^{m-2} |\partial_1 \hat{a}_1(n')| \big\|_{l^2(\{|n'|=\sqrt{N_0-1}\})} \Big) \end{aligned} \tag{4.24}$$

with a constant $C_5 = C_5(\nu, g, \sigma, \alpha) > 0$. Therefore, we get

$$\begin{aligned} \big\| |n'|^{m+1} |\hat{h}(n',t)| \big\|_{l^2(\mathbb{Z}^2)} &= \big\| |n'|^{m+1} |\hat{h}(n',t)| \big\|_{l^2(\{|n'|<\sqrt{N_0-1}\})} + \big\| |n'|^{m+1} |\hat{h}(n',t)| \big\|_{l^2(\{|n'|=\sqrt{N_0-1}\})} \\ &\quad + \big\| |n'|^{m+1} |\hat{h}(n',t)| \big\|_{l^2(\{|n'|\ge\sqrt{N_0}\})} \\ &\le C_6(1+T) \big(\big\| |n'|^{m+1} |\hat{h}_0(n')| \big\|_{l^2(\mathbb{Z}^2)} + \big\| |n'|^{m-1} |\hat{a}_1(\tfrac{\pi}{2}, n')| \big\|_{l^2(\mathbb{Z}^2)} \\ &\quad + \big\| |n'|^{m-2} |\partial_1 \hat{a}_1(\tfrac{\pi}{2}, n')| \big\|_{l^2(\mathbb{Z}^2)} \big) \end{aligned}$$

with $C_6 = C_6(\nu, g, \sigma, \alpha) := \max\{C_3, C_4, C_5\}$. Hence, it follows that

$$\|h(t)\|_{H^{m+1}(\mathbb{T}^2)} \le C_6(1+T) \big(1 + \|\eta_0\|_{H^{m+1}(\mathbb{T}^2)} + \|a_1\|_{W^m_{s_1}(\Omega)}\big) \tag{4.25}$$

for all $0 < t < T$.

For the Eq. (4.12), we have

$$|\hat{z}_1(n',t)| \le |\partial_1 \hat{a}_1(\tfrac{\pi}{2}, n')| + (g+\sigma)|n'|^{2\alpha-1} \int_0^t e^{-3\nu|n'|^2(t-s)} ds \sup_{0<s<t} (|n'|^3 |\hat{h}(n',s)|),$$

which yields that for all $0 < t < T$,

$$\begin{aligned} \big\| |n'|^{m-2} |\hat{z}_1(n',t)| \big\|_{l^2(\mathbb{Z}^2)} &\le \big\| |n'|^{m-2} |\partial_1 \hat{a}_1(\tfrac{\pi}{2}, n')| \big\|_{l^2(\mathbb{Z}^2)} \\ &\quad + \frac{g+\sigma}{(3\nu)^{\alpha-\frac{1}{2}}} \int_0^1 \frac{d\tau}{(1-\tau)^{\alpha-\frac{1}{2}}}\, t^{\frac{3}{2}-\alpha} \sup_{0<t<T} \big\| |n'|^{m+1} |\hat{h}(n',t)| \big\|_{l^2(\mathbb{Z}^2)}. \end{aligned}$$

Therefore, we deduce from (4.25) with $\hat{z}_1(0,t) = \partial_1 \hat{a}_1(\frac{\pi}{2}, 0)$ that

$$\|z_1(t)\|_{H^{m-2}(\mathbb{T}^2)} \le C_7(1 + T^{\frac{5}{2}-\alpha}) \big(1 + \|\eta_0\|_{H^{m+1}(\mathbb{T}^2)} + \|a_1\|_{W^m_{s_1}(\Omega)}\big) \tag{4.26}$$

for all $0 < t < T$ with a constant $C_7 = C_7(\nu, g, \sigma, \alpha) > 0$. Furthermore, one can see that such a solution z_1 lies in $C([0,T), H^{m-2}(\mathbb{T}^2))$.

Substituting (4.12) into (4.11), we have that for every $n' \ne 0$,

$$\begin{aligned} \hat{z}_0(n',t) &= e^{-\nu|n'|^2 t} \hat{a}_1(\tfrac{\pi}{2}, n') - 2\nu|n'| \tanh \frac{\pi}{2}|n'|\, te^{-\nu|n'|^2 t} \partial_1 \hat{a}_1(\tfrac{\pi}{2}, n') \\ &\quad - \mu^\alpha(n') \int_0^t e^{-3\nu|n'|^2(t-s)} \hat{h}(n',s) ds, \end{aligned} \tag{4.27}$$

which implies

$$|\hat{z}_0(n',t)| \le |\hat{a}_1(\tfrac{\pi}{2},n')| + \frac{2|\partial_1\hat{a}_1(\frac{\pi}{2},n')|}{|n'|} + (g+\sigma)|n'|^{2\alpha-1}\int_0^t e^{-3\nu|n'|^2(t-s)}ds \sup_{0<s<t}(|n'|^2|\hat{h}(n',s)|)$$

for all $0<t<T$. Thus,

$$\begin{aligned}\big\||n'|^{m-1}\hat{z}_0(n',t)\big\|_{l^2(\mathbb{Z}^2)} &\le \big\||n'|^{m-1}\hat{a}_1(\tfrac{\pi}{2},n')\big\|_{l^2(\mathbb{Z}^2)} + 2\big\||n'|^{m-2}\partial_1\hat{a}_1(n',s)\big\|_{l^2(\mathbb{Z}^2)}\\ &\quad + \frac{g+\sigma}{(3\nu)^{\alpha-\frac12}}\int_0^1 \frac{d\tau}{(1-\tau)^{\alpha-\frac12}}\, t^{\frac32-\alpha}\sup_{0<s<t}\big\||n'|^{m+1}\hat{h}(n',s)\big\|_{l^2(\mathbb{Z}^2)}.\end{aligned}$$

Therefore, we have that for all $0<t<T$,

$$\|z_0(t)\|_{H^{m-2}(\mathbb{T}^2)} \le C_8\big(1+T^{\frac52-\alpha}\big)\big(1+\|\eta_0\|_{H^{m+1}(\mathbb{T}^2)}+\|a_1\|_{W^m_{s_1}(\Omega)}\big) \tag{4.28}$$

with a constant $C_8 = C_8(\nu,g,\sigma,\alpha)>0$, since $\hat{z}_0(0,t)=\hat{a}_1(\frac{\pi}{2},0)$. Furthermore, one can verify that z_0 belongs to $C([0,T),H^{m-1}(\mathbb{T}^2))$.

As for Q_1 in (3.9), we estimate for any $\rho<3/2$,

$$\begin{aligned}&\big\||\partial_1|^{\rho}|\nabla'|^{m-2}Q_1(t)\big\|_{L^2(\mathbb{T}^3)}\\ &\quad= \big\|(2n_1-1)^{\rho}|n'|^{m-2}\mathcal{F}_{s_1}[Q_1](n,t)\big\|_{l^2(\mathbb{N}\times\mathbb{Z}^2)}\\ &\quad\le \frac{8}{\pi}\left(\sum_{n_1\in\mathbb{N}}\frac{1}{(2n_1-1)^{2(2-\rho)}}\left\|\frac{|n'|^{m+2}}{\kappa(n)^2}(1-e^{-\nu\kappa(n)t})\sup_{0<s<t}|z_1(n',s)|\right\|^2_{l^2(n'\in\mathbb{Z}^2)}\right)^{\frac12}\\ &\qquad+\frac{8}{\pi}\frac{g+\sigma}{\nu}\left(\sum_{n_1\in\mathbb{N}}\frac{1}{(2n_1-1)^{2(2-\rho)}}\left\|\frac{|n'|^{m+2+2\alpha}}{\kappa(n)^2}(1-e^{-\nu\kappa(n)t})\sup_{0<s<t}|h(n',s)|\right\|^2_{l^2(n'\in\mathbb{Z}^2)}\right)^{\frac12}\\ &\quad\le C_\rho\left(1+\frac{g+\sigma}{\nu}\right)\left(\sup_{0<s<t}\big\||n'|^{m-2}\hat{z}_1(n',s)\big\|_{l^2(\mathbb{Z}^2)}+\sup_{0<s<t}\big\||n'|^{m+1}\hat{h}(n',s)\big\|_{l^2(\mathbb{Z}^2)}\right)\\ &\quad\le C_\rho\left(1+\frac{g+\sigma}{\nu}\right)\left(\sup_{0<t<T}\|z_1(n',t)\|_{H^{m-2}(\mathbb{T}^2)}+\sup_{0<t<T}\|h(n',t)\|_{H^{m+1}(\mathbb{T}^2)}\right)\\ &\quad\le C'_\rho\left(1+\frac{g+\sigma}{\nu}\right)\big(1+T^{\frac52-\alpha}\big)\big(1+\|\eta_0\|_{H^{m+1}(\mathbb{T}^2)}+\|a_1\|_{W^m_{s_1}(\Omega)}\big)\end{aligned}$$

for all $0<t<T$, where C_ρ, C'_ρ are positive constants depending only on ρ,ν,g,σ,α. Similarly, we can see that that for any $\rho'<1/2$, Q_i for $i=2,3$ satisfies

$$\begin{aligned}&\big\||\partial_1|^{\rho'}|\nabla'|^{m-2}Q_i(t)\big\|_{L^2(\mathbb{T}^3)}\\ &\quad\le C_{\rho'}\left(1+\frac{g+\sigma}{\nu}\right)\left(\sup_{0<t<T}\|z_1(n',t)\|_{H^{m-2}(\mathbb{T}^2)}+\sup_{0<t<T}\|h(n',t)\|_{H^{m+1}(\mathbb{T}^2)}\right)\\ &\quad\le C'_{\rho'}\left(1+\frac{g+\sigma}{\nu}\right)\big(1+T^{\frac52-\alpha}\big)\big(1+\|\eta_0\|_{H^{m+1}(\mathbb{T}^2)}+\|a_1\|_{W^m_{s_1}(\Omega)}\big)\end{aligned}$$

for all $0<t<T$, where $C_{\rho'}, C'_{\rho'}$ are positive constants depending only on $\rho',\nu,g,\sigma,\alpha$. Hence, we complete the proof of Theorem 4.1. □

Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author declares that he has no conflict of interest.

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