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Energy Conservation for the Generalized Surface Quasi-geostrophic Equation

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Abstract. In this paper, we consider the generalized surface quasi-geostrophic equation with the velocity v determined by $v=\mathcal{R}^{\perp}\Lambda^{\gamma-1}\theta$, $0<\gamma<2$. It is shown that the L^p-norm of weak solutions is conserved provided $\theta\in L^{p+1}\left(0,T;B^{\frac{\gamma}{3}}_{p+1,c(\mathbb{N})}\right)$ for $0<\gamma<\frac{3}{2}$ or $\theta\in L^{p+1}\left(0,T;B^{\alpha}_{p+1,\infty}\right)$ for any $\gamma-1<\alpha<1$ with $\frac{3}{2}\leq\gamma<2$. Therefore, the accurate relationships between the critical regularity for the energy conservation of the weak solutions and the regularity of velocity for the generalized surface quasi-geostrophic equation are presented.

Mathematical subject classification. 35Q35, 35Q86, 76D03.

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1. Introduction

In this paper, we consider the generalized surface quasi-geostrophic (SQG) equation in $(0,T)\times\mathbb{R}^2$ below

$$\begin{cases}\theta_t+v\cdot\nabla\theta=0,\\ v=\mathcal{R}^{\perp}\Lambda^{\gamma-1}\theta=(-\mathcal{R}_2\Lambda^{\gamma-1}\theta,\mathcal{R}_1\Lambda^{\gamma-1}\theta),\ \gamma\in[0,2),\\ \theta|_{t=0}=\theta_0,\end{cases}\tag{1.1}$$

where the unknown function $\theta(x,t)$ is a scalar and $v(x,t)$ is determined by $\theta(x,t)$. The Riesz transforms $\mathcal{R}_j$ are defined by $\widehat{\mathcal{R}_j f}=-\frac{i\xi_j}{|\xi|}\widehat{f}(\xi)$, $j=1,2$, where $\hat{f}(\xi)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}f(x)e^{-i\xi\cdot x}\,dx$. $\Lambda^s f$ is defined via the Fourier transform $\widehat{\Lambda^s f}(\xi)=|\xi|^s\hat{f}(\xi)$. This model was introduced in [10–12,14] and includes many classical hydrodynamic equations. In particular, the case $\gamma=0$ corresponds to the 2-D incompressible Euler equations, where the unknown functions $\theta=\theta(x,t)$ and $v=v(x,t)$ are the vorticity and the velocity field, respectively. The case $\gamma=1$ corresponds to the following standard surface quasi-geostrophic equation

$$\begin{cases}\theta_t+v\cdot\nabla\theta=0,\\ v(x,t)=(-\mathcal{R}_2\theta,\mathcal{R}_1\theta),\\ \theta|_{t=0}=\theta_0,\end{cases}\tag{1.2}$$

which describes a famous approximation model of the nonhomogeneous fluid flow in a rapidly rotating 3-D half-space (see [15]). In this case, the unknown functions $\theta=\theta(x,t)$ and $v=v(x,t)$ represent the potential temperature and the velocity field, respectively.

The generalized surface quasi-geostrophic equation have attracted a lot of attention in recent years and important progress has been made (see e.g. [10–12,14,20–22,30–32]). The goal of this paper is to examine the relationships between critical regularity of weak solutions keeping energy conservation and the regularity of velocity for the generalized surface quasi-geostrophic equation. A classical question involving energy conservation in incompressible fluid is the Onsager conjecture. In [23], Onsager conjectured that the weak solutions with Hölder continuity exponent $\alpha > \frac{1}{3}$ of the 3-D incompressible Euler equations do conserve energy. This conjecture was proved by Constantin-E-Titi [16] in the Besov space $L^3(0,T;B^{\alpha}_{3,\infty}(\mathbb{T}^3))$ with $\alpha > 1/3$. Subsequently, Cheskidov–Constantin–Friedlander–Shvydkoy [13] sharpened the result of [16] by proving that energy is conserved for velocities in the critical space $L^3(0,T;B^{1/3}_{3,c(\mathbb{N})})$, where $B^{1/3}_{3,c(\mathbb{N})} = \{v \in B^{1/3}_{3,\infty} : \lim_{q\to\infty} 2^q\|\Delta_q v\|^3_{L^3} = 0\}$ and $\Delta_q v$ stands for a smooth restriction of v into Fourier modes of order 2^q (see Sect. 2). The space $B^{1/3}_{3,c(\mathbb{N})}$ is usually called as the Onsager's critical space. Along this direction, there are some progress recently, one can refer to [3,18] for details.

We turn our attention back to the persistence of energy for the surface quasi-geostrophic equation. A parallel of Constantin-E-Titi's result for standard surface quasi-geostrophic Eq. (1.2) was obtained by Zhou [34], where the L^2-norm conservation for the weak solutions is established provided $\theta \in L^3(0,T;B^{\alpha}_{3,\infty}(\mathbb{R}^2))$ with $\alpha > \frac{1}{3}$. Chae [9] proved that the L^p-norm of θ is preserved if the weak solutions (θ, v) satisfy

$$v \in L^{r_1}(0,T;\dot{B}^{\alpha}_{p+1,\infty}(\mathbb{R}^2)) \text{ and } \theta \in L^{r_2}(0,T;B^{\alpha}_{p+1,\infty}(\mathbb{R}^2)), \frac{1}{r_1} + \frac{p}{r_2} = 1, \alpha > \frac{1}{3}. \tag{1.3}$$

Very recently, Akramova-Wiedemann [1] presented the following sufficient conditions

$$\theta \in L^{p_1}(0,T;\dot{B}^{\alpha}_{3,\infty}(\mathbb{R}^2)),\ \alpha > \frac{1}{3},\ p_1 \le \frac{6}{2-3\alpha},$$

implying L^p-norm conservation of the weak solutions for system (1.2). We would like to mention that Dai [17] showed that the energy of any viscosity solution of system (1.2) with supercritical dissipation $\Lambda^\alpha\theta$ satisfying $\theta \in L^2(0,T;B^{\frac{1}{2}}_{2,c(\mathbb{N})}(\mathbb{R}^2))$ is invariant. However, we note that all the above results are in Onsager's subcritical space other than the Onsager's critical space, which means the regularity of space is required to satisfy $\alpha > \frac{1}{3}$ rather than $\alpha = \frac{1}{3}$. Hence, our first objective is to obtain sufficient conditions on the regularity of weak solutions to guarantee conservation of the energy for generalized surface quasi-geostrophic Eq. (1.1) in Onsager's critical space. Now, we formulate our first result as follows.

Theorem 1.1. *Suppose $\theta \in C([0,T];L^p(\mathbb{R}^2))$, $p \in [2,\infty)$ is a weak solution of system (1.1) in the sense of Definition 2.1, then the L^p-norm of θ is preserved, that is, for any $t \in [0,T]$,*

$$\|\theta(t)\|_{L^p(\mathbb{R}^2)} = \|\theta_0\|_{L^p(\mathbb{R}^2)},$$

provided one of the following conditions is satisfied

$$\theta \in L^{p+1}(0,T;B^{\frac{\gamma}{3}}_{p+1,c(\mathbb{N})}(\mathbb{R}^2)) \text{ with } 0 < \gamma < \frac{3}{2}; \tag{1.4}$$

or

$$\theta \in L^{p+1}(0,T;B^{\alpha}_{p+1,\infty}(\mathbb{R}^2)) \text{ for any } \gamma - 1 < \alpha < 1 \text{ with } \frac{3}{2} \le \gamma < 2. \tag{1.5}$$

Remark 1.1. Theorem 1.1 extends the result in [13] on the 3-D Euler equations to system (1.1) with $0 < \gamma < \frac{3}{2}$. Besides, a special case of this theorem with $p = 2$ and $\gamma = 1$ is novel and improves the corresponding result in [34]. However, since system (1.1) with $\frac{3}{2} \le \gamma < 2$ is more singular than the case $0 < \gamma < \frac{3}{2}$, we only get the subcritical criterion for energy conservation. It would be an interesting problem to study the persistence of energy for system (1.1) in Onsager's critical space for the case $\frac{3}{2} \le \gamma < 2$.

Remark 1.2. This theorem reveals how the regularity of the velocity field influences the critical regularity of the weak solutions preserving the energy in generalized surface quasi-geostrophic Eq. (1.1).

Moreover, when $p = 2$, the condition $\theta \in L^3(0,T;L^3(\mathbb{R}^2))$ in Theorem 1.1 can be removed. Precisely, we have

Corollary 1.2. *Let* $0 < \gamma < \frac{3}{2}$. *Assume that* $\theta \in C([0,T]; L^2(\mathbb{R}^2))$ *is a weak solution of system (1.1) in the sense of Definition 2.1 satisfying* $\theta \in L^3(0,T; \dot{B}^{\frac{\gamma}{3}}_{3,c(\mathbb{N})}(\mathbb{R}^2))$, *then the* L^2*-norm of* θ *is preserved, that is, for any* $t \in [0,T]$,

$$\|\theta(t)\|_{L^2(\mathbb{R}^2)} = \|\theta_0\|_{L^2(\mathbb{R}^2)}.$$

Inspired by the persistence of energy criterion (1.3), we have

Theorem 1.3. *Let* $p \in [2,\infty)$, $r_1 \in [1,\infty]$ *and* $r_2 \in [p,\infty]$ *be given, satisfying* $\frac{1}{r_1} + \frac{p}{r_2} = 1$. *Assume that* $\theta \in C([0,T]; L^p(\mathbb{R}^2))$ *is a weak solution of system (1.1) in the sense of Definition 2.1 with* $v \in L^{r_1}(0,T; \dot{B}^{\frac{1}{3}}_{p+1,c(\mathbb{N})}(\mathbb{R}^2))$ *and* $\theta \in L^{r_2}(0,T; B^{\frac{1}{3}}_{p+1,\infty}(\mathbb{R}^2))$, *then the* L^p*-norm of* θ *is preserved, that is, for any* $t \in [0,T]$,

$$\|\theta(t)\|_{L^p(\mathbb{R}^2)} = \|\theta_0\|_{L^p(\mathbb{R}^2)}.$$

Remark 1.3. The same result also holds if $v \in L^{r_1}(0,T; \dot{B}^{\frac{1}{3}}_{p+1,\infty}(\mathbb{R}^2))$ and $\theta \in L^{r_2}(0,T; B^{\frac{1}{3}}_{p+1,c(\mathbb{N})}(\mathbb{R}^2))$ by a slightly modification of the proof of Theorem 1.3, which refines criterion (1.3).

Remark 1.4. Owing to the boundedness of Riesz transforms in homogeneous Besov spaces, Theorem 1.3 guarantees that the L^2-norm of weak solutions of system (1.2) satisfying $\theta \in L^3(0,T; \dot{B}^{\frac{1}{3}}_{3,c(\mathbb{N})}(\mathbb{R}^2))$ is constant.

We will provide two approaches to show Theorem 1.1. One is an application of the Littlewood-Paley theory developed by Cheskidov–Constantin–Friedlander–Shvydkoy in [13]. The second one relies on the Constantin-E-Titi type commutator estimates in physical Onsager type spaces (see Lemma 2.3). For the periodic domain $\mathbb{T}^2$, by means of the Constantin-E-Titi type commutator estimates in Besov VMO spaces in Lemma 2.4, one can further relax the spaces $B^{\beta}_{p+1,c(\mathbb{N})}(\mathbb{T}^2)$ to the larger space $\underline{B}^{\beta}_{p+1,VMO}(\mathbb{T}^2)$ in the above sufficient conditions for keeping the energy for the generalized surface quasi-geostrophic equation. We formulate the energy conservation criterion of weak solutions of generalized surface quasi-geostrophic Eq. (1.1) in Besov VMO spaces below.

Theorem 1.4. *Suppose* $\theta \in C([0,T]; L^p(\mathbb{T}^2))$, $p \in [2,\infty)$ *is a weak solution of system (1.1) in the sense of Definition 2.1, then the weak solution* θ *conserves the* L^p*-norm provided one of the following conditions is satisfied*

(1) $0 < \gamma < \frac{3}{2}$,

$$\theta \in L^{p+1}(0,T; \underline{B}^{\frac{\gamma}{3}}_{p+1,VMO}(\mathbb{T}^2)) \cap L^{p+1}(0,T; B^{\frac{\gamma}{3}}_{p+1,\infty}(\mathbb{T}^2));$$

(2) $0 < \gamma < 2$, $\frac{1}{r_1} + \frac{p}{r_2} = 1$, $r_2 \geq 1 + \frac{1}{p}$,

$$v \in L^{r_1}(0,T; \underline{B}^{\frac{1}{3}}_{p+1,VMO}(\mathbb{T}^2)) \quad \text{and } \theta \in L^{r_2}(0,T; B^{\frac{1}{3}}_{p+1,\infty}(\mathbb{T}^2));$$

(3) $0 < \gamma < 2$, $\frac{1}{r_1} + \frac{p}{r_2} = 1$, $r_1 \geq 1 + \frac{1}{p}$,

$$v \in L^{r_1}(0,T; B^{\frac{1}{3}}_{p+1,\infty}(\mathbb{T}^2)) \quad \text{and } \theta \in L^{r_2}(0,T; \underline{B}^{\frac{1}{3}}_{p+1,VMO}(\mathbb{T}^2)).$$

Remark 1.5. Owing to the inclusion relationship $B^{\frac{1}{3}}_{3,c(\mathbb{N})} \subseteq \underline{B}^{\frac{1}{3}}_{3,VMO}$ in [3,18], this theorem is an improvement of corresponding result in Theorem 1.1 for the periodic case.

The rest of the paper is organized as follows. In Sect. 2, we present some notations and auxiliary lemmas which will be frequently used throughout this paper. The energy conservation of weak solutions for the generalized surface quasi-geostrophic equation is considered in Sect. 3. Concluding remarks are given in Sect. 4.

2. Notations and Some Auxiliary Lemmas

Sobolev spaces: First, we introduce some notations used in this paper. For $p \in [1, \infty]$, the notation $L^p(0, T; X)$ stands for the set of measurable functions on the interval $(0, T)$ with values in X and $\|f(t, \cdot)\|_X$ belonging to $L^p(0, T)$. The classical Sobolev space $W^{k,p}(\mathbb{R}^d)$ is equipped with the norm $\|f\|_{W^{k,p}(\mathbb{R}^d)} = \sum_{|\alpha|=0}^{k} \|D^\alpha f\|_{L^p(\mathbb{R}^d)}$.

Besov spaces: We denote $\mathcal{S}$ the Schwartz class of rapidly decreasing functions, $\mathcal{S}'$ the space of tempered distributions and $\mathcal{S}'/\mathcal{P}$ the quotient space of tempered distributions which modulo polynomials. We use $\mathcal{F}f$ or $\widehat{f}$ to denote the Fourier transform of a tempered distribution f. To define Besov spaces, we need the following dyadic unity partition (see e.g. [2]). Choose two nonnegative radial functions $\varrho, \varphi \in C^\infty(\mathbb{R}^d)$ supported respectively in the ball $\mathcal{B} = \{\xi \in \mathbb{R}^d : |\xi| \le \frac{4}{3}\}$ and the shell $\mathcal{C} = \{\xi \in \mathbb{R}^d : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ such that

$$\varrho(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d; \qquad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \ne 0.$$

Write $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\varrho$, then nonhomogeneous dyadic blocks Δ_j are defined by

$$\Delta_j u := 0 \text{ if } j \le -2,\ \Delta_{-1} u := \varrho(D)u = \int_{\mathbb{R}^d} \tilde{h}(y)u(x-y)dy,$$
$$\text{and } \Delta_j u := \varphi\left(2^{-j}D\right)u = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)u(x-y)dy \text{ if } j \ge 0.$$

The nonhomogeneous low-frequency cut-off operator S_j is defined by

$$S_j u := \sum_{k\le j-1} \Delta_k u = \varrho(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} \tilde{h}(2^j y)u(x-y)dy,\ j \in \mathbb{N}\cup 0.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ and homogeneous low-frequency cut-off operators $\dot{S}_j$ are defined for $\forall j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u := \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)u(x-y)dy,\ j \in \mathbb{Z}$$
$$\text{and } \ \dot{S}_j u := \varrho(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} \tilde{h}(2^j y)u(x-y)dy,\ j \in \mathbb{Z}$$

Now we introduce the definition of Besov spaces. Let $(p, r) \in [1, \infty]^2$, $s \in \mathbb{R}$, the nonhomogeneous Besov space

$$B^s_{p,r} := \left\{ f \in \mathcal{S}'\left(\mathbb{R}^d\right); \|f\|_{B^s_{p,r}} := \left\|2^{js}\| \Delta_j f \|_{L^p}\right\|_{\ell^r(\mathbb{Z})} < \infty \right\}$$

and the homogeneous space

$$\dot{B}^s_{p,r} := \left\{ f \in \mathcal{S}'\left(\mathbb{R}^d\right)/\mathcal{P}\left(\mathbb{R}^d\right); \|f\|_{\dot{B}^s_{p,r}} := \left\|2^{js}\| \dot{\Delta}_j f \|_{L^p}\right\|_{\ell^r(\mathbb{Z})} < \infty \right\}.$$

Moreover, for $s > 0$ and $1 \le p, q \le \infty$, we may write the equivalent norm below in the nonhomogeneous Besov norm $\|f\|_{B^s_{p,q}}$ of $f \in \mathcal{S}'$ as

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}.$$

Motivated by [13], we define $\dot{B}^\alpha_{p,c(\mathbb{N})}$ to be the class of all tempered distributions f for which

$$\|f\|_{\dot{B}^\alpha_{p,\infty}} < \infty \text{ and } \lim_{j\to\infty} 2^{j\alpha}\left\|\dot{\Delta}_j f\right\|_{L^p} = 0,\ \text{for any } 1 \le p \le \infty. \tag{2.1}$$

It is clear that the Besov spaces $\dot{B}^\alpha_{p,q}$ are included in $\dot{B}^\alpha_{p,c(\mathbb{N})}$ for any $1 \le q < \infty$. Likewise, one can define the Besov spaces $B^\alpha_{p,c(\mathbb{N})}$ similarly.

A function f belongs to the Besov-VMO space $L^p(0,T;\underline{B}^{\alpha}_{q,VMO}(\mathbb{T}^d))$ if it satisfies

$$\|f\|_{L^p(0,T;L^q(\mathbb{T}^d))}<\infty,$$

and

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{\alpha}}\left(\int_0^T\Big[\int_{\mathbb{T}^d}\fint_{B_\varepsilon(x)}|f(x)-f(y)|^q dydx\Big]^{\frac{p}{q}}dt\right)^{\frac{1}{p}}\\&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{\alpha}}\left(\int_0^T\Big[\int_{\mathbb{T}^d}\fint_{B_\varepsilon(0)}|f(x)-f(x-y)|^q dydx\Big]^{\frac{p}{q}}dt\right)^{\frac{1}{p}}=0.\end{aligned}$$

Mollifier kernel: Let $\eta_\varepsilon:\mathbb{R}^d\to\mathbb{R}$ be a standard mollifier.i.e. $\eta(x)=C_0e^{-\frac{1}{1-|x|^2}}$ for $|x|<1$ and $\eta(x)=0$ for $|x|\geq 1$, where C_0 is a constant such that $\int_{\mathbb{R}^d}\eta(x)dx=1$. For $\varepsilon>0$, we define the rescaled mollifier $\eta_\varepsilon(x)=\frac{1}{\varepsilon^d}\eta(\frac{x}{\varepsilon})$ and for any function $f\in L^1_{loc}(\mathbb{R}^d)$, its mollified version is defined as

$$f^\varepsilon(x)=(f*\eta_\varepsilon)(x)=\int_{\mathbb{R}^d}f(x-y)\eta_\varepsilon(y)dy,\quad x\in\mathbb{R}^d.$$

Next, we collect some Lemmas which will be used in the present paper.

Lemma 2.1 (Bernstein inequality [2]). *Let $\mathcal{B}$ be a ball of $\mathbb{R}^d$, and $\mathcal{C}$ be a ring of $\mathbb{R}^d$. There exists a positive constant C such that for all integer $k\geq 0$, all $1\leq a\leq b\leq\infty$ and $u\in L^a(\mathbb{R}^d)$, the following estimates are satisfied:*

$$\begin{aligned}\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^b(\mathbb{R}^d)}&\leq C^{k+1}\lambda^{k+d\left(\frac{1}{a}-\frac{1}{b}\right)}\|u\|_{L^a(\mathbb{R}^d)},\quad \operatorname{supp}\hat{u}\subset\lambda\mathcal{B},\\ C^{-(k+1)}\lambda^k\|u\|_{L^a(\mathbb{R}^d)}\leq\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^a(\mathbb{R}^d)}&\leq C^{k+1}\lambda^k\|u\|_{L^a(\mathbb{R}^d)},\quad \operatorname{supp}\hat{u}\subset\lambda\mathcal{C}.\end{aligned}$$

Lemma 2.2 ([26]). *Let Ω denote the whole space $\mathbb{R}^d$ or the periodic domain $\mathbb{T}^d$. Suppose that $f\in L^p(0,T;\dot{B}^{\alpha}_{q,\infty}(\Omega))$, $g\in L^p(0,T;\dot{B}^{\beta}_{q,c(\mathbb{N})}(\Omega))$ with $\alpha,\beta\in(0,1)$, $p,q\in[1,\infty]$, then there holds that, for any $k\in\mathbb{N}^+$, as $\varepsilon\to 0$,*

(1) $\|f^\varepsilon-f\|_{L^p(0,T;L^q(\Omega))}\leq CO(\varepsilon^\alpha)\|f\|_{L^p(0,T;\dot{B}^{\alpha}_{q,\infty}(\Omega))}$;
(2) $\|\nabla^k f^\varepsilon\|_{L^p(0,T;L^q(\Omega))}\leq CO(\varepsilon^{\alpha-k})\|f\|_{L^p(0,T;\dot{B}^{\alpha}_{q,\infty}(\Omega))}$;
(3) $\|g^\varepsilon-g\|_{L^p(0,T;L^q(\Omega))}\leq Co(\varepsilon^\beta)\|g\|_{L^p(0,T;\dot{B}^{\beta}_{q,c(\mathbb{N})}(\Omega))}$;
(4) $\|\nabla^k g^\varepsilon\|_{L^p(0,T;L^q(\Omega))}\leq Co(\varepsilon^{\beta-k})\|g\|_{L^p(0,T;\dot{B}^{\beta}_{q,c(\mathbb{N})}(\Omega))}$;

Remark 2.1. The results still hold for $g\in L^p(0,T;\underline{B}^{\beta}_{q,VMO}(\mathbb{T}^d))$, whose proof is proposed in [3,27,28].

Next, we will state the Constantin-E-Titi type commutator estimates in physical Onsager type spaces (see also [33]).

Lemma 2.3. ([26]) *Let Ω denote the whole space $\mathbb{R}^d$ or the periodic domain $\mathbb{T}^d$. Assume that $0<\alpha,\beta<1$, $1\leq p,q,p_1,p_2\leq\infty$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, then as $\varepsilon\to 0$, there holds,*

$$\|(fg)^\varepsilon-f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(\Omega))}\leq Co(\varepsilon^{\alpha+\beta}),\tag{2.2}$$

provided one of the following three conditions holds

(1) $f\in L^{p_1}(0,T;\dot{B}^{\alpha}_{q_1,c(\mathbb{N})}(\Omega))$, $g\in L^{p_2}(0,T;\dot{B}^{\beta}_{q_2,\infty}(\Omega))$,$1\leq q_1,q_2\leq\infty$ $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$;
(2) $\nabla f\in L^{p_1}(0,T;B^{\alpha}_{q_1,c(\mathbb{N})}(\Omega))$, $\nabla g\in L^{p_2}(0,T;B^{\beta}_{q_2,\infty}(\Omega))$, $\frac{2}{d}+\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$,$1\leq q_1,q_2<d$;
(3) $f\in L^{p_1}(0,T;B^{\alpha}_{q_1,c(\mathbb{N})}(\Omega))$, $\nabla g\in L^{p_2}(0,T;B^{\beta}_{q_2,\infty}(\Omega))$, $\frac{1}{d}+\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$,$1\leq q_2<d$, $1\leq q_1\leq\infty$.

The Constantin-E-Titi type commutator estimates in Besov VMO spaces was initiated by Bardos, Gwiazda, Świerczewska-Gwiazda, Titi and Wiedemann in [3]. The readers may refer to [27,28] for the proof of the following version.

Lemma 2.4. ([3,27,28]) *Assume that* $0<\alpha,\beta<1$, $1\le p,q,p_1,p_2\le\infty$ *and* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. *Then, there holds*

$$\|(fg)^\varepsilon-f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))}\le o(\varepsilon^{\alpha+\beta}),\ \ as\ \varepsilon\to 0, \tag{2.3}$$

provided that one of the following conditions is satisfied,

(1) $f\in L^{p_1}(0,T;\underline{B}^{\alpha}_{q_1,VMO}(\mathbb{T}^d))$, $g\in L^{p_2}(0,T;\underline{B}^{\beta}_{q_2,VMO}(\mathbb{T}^d))$, $1\le q_1,q_2\le\infty$, $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$;

(2) $f\in L^{p_1}(0,T;\underline{B}^{\alpha}_{q_1,VMO}(\mathbb{T}^d))$, $g\in L^{p_2}(0,T;\dot{B}^{\beta}_{q_2,\infty}(\mathbb{T}^d))$, $1\le q_1,q_2\le\infty$, $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}, q_2\ge\frac{q_1}{q_1-1}$ *and* $p_2\ge\frac{q_1}{q_1-1}$.

For the convenience of readers, we present the definition of the weak solutions of the surface quasi-geostrophic Eq. (1.1).

Definition 2.1. A function $\theta\in C_{\text{weak}}([0,T];L^p(\mathbb{R}^2))$ is called a weak solution of the 2-D quasi-geostrophic equation with initial data $\theta_0\in L^p(\mathbb{R}^2)$ with $p\in[2,\infty)$ and $v\in L^{\frac{p}{p-1}}((0,T)\times(\mathbb{R}^2))$ if there holds

$$\int_{\mathbb{R}^2}[\theta(x,t)\varphi(x,t)-\theta(x,0)\varphi(x,0)]dx=\int_0^t\int_{\mathbb{R}^2}\theta(x,s)\big(\partial_t\varphi(x,s)+v(x,s)\cdot\nabla\varphi(x,s)\big)dxds \tag{2.4}$$

and

$$v(x,t)=\mathcal{R}^{\perp}\Lambda^{\gamma-1}\theta, \tag{2.5}$$

for any test function $\varphi\in C_0^\infty([0,T];C^\infty(\mathbb{R}^2))$.

3. Energy Conservation of Weak Solutions for the Surface Quasi-Geostrophic Equation

3.1. Energy Conservation in Besov Spaces

In this subsection, our main task is to prove Theorem 1.1. Two different approaches will be provided. One is Littlewood-Paley theory developed by Cheskidov–Constantin–Friedlander-Shvydkoy in [13] and the other is mainly to use Constantin-E-Titi type commutator estimates in Onsager type spaces (see Lemma 2.3).

Proof of Theorem 1.1.

Approach 1: Littlewood-Paley theory.

Multiplying the first equation of system (1.1) by $S_N(S_N\theta|S_N\theta|^{p-2})$ with $p\ge 2$ (see the notations in Sect. 2), together with the incompressible condition and using integration by parts, we see that

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|S_N\theta|^pdx=(p-1)\int_{\mathbb{R}^2}S_N(v_\ell\theta)\partial_\ell S_N\theta|S_N\theta|^{p-2}dx.$$

Since the divergence-free condition of the velocity field $v(x,t)$ helps us to derive that

$$\int_{\mathbb{R}^2}S_Nv_\ell\partial_\ell S_N\theta S_N\theta|S_N\theta|^{p-2}dx=0,$$

we conclude that

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|S_N\theta|^pdx=(p-1)\int_{\mathbb{R}^2}\left[S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\right]\partial_\ell S_N\theta|S_N\theta|^{p-2}dx.$$

Taking advantage of the Hölder inequality, we discover that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}\left[S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\right]\partial_\ell S_N\theta|S_N\theta|^{p-2}dx\Big|\\&\quad\le C\|S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\|_{L^{\frac{p+1}{2}}(\mathbb{R}^2)}\|\partial_\ell S_N\theta\|_{L^{p+1}(\mathbb{R}^2)}\||S_N\theta|^{p-2}\|_{L^{\frac{p+1}{p-2}}(\mathbb{R}^2)}\end{aligned}$$

$$\leq C\|S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\|_{L^{\frac{p+1}{2}}(\mathbb{R}^2)}\|\partial_\ell S_N\theta\|_{L^{p+1}(\mathbb{R}^2)}\|S_N\theta\|^{p-2}_{L^{p+1}(\mathbb{R}^2)}. \tag{3.1}$$

Note that

$$\begin{aligned}&S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\\&=2^{2N}\int_{\mathbb{R}^2}\tilde h(2^Ny)[v_\ell(x-y)-v_\ell(x)][\theta(x-y)-\theta(x)]dy-(v_\ell-S_Nv_\ell)(\theta-S_N\theta),\end{aligned}\tag{3.2}$$

where we used $2^{2N}\int_{\mathbb{R}^2}\tilde h(2^Ny)dy=\mathcal{F}(\tilde h(\cdot))|_{\xi=0}=1$. By the Minkowski inequality, we get

$$\begin{aligned}&\|S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\|_{L^{\frac{p+1}{2}}(\mathbb{R}^2)}\\&\leq 2^{2N}\int_{\mathbb{R}^2}|\tilde h(2^Ny)|\|v_\ell(x-y)-v_\ell(x)\|_{L^{p+1}(\mathbb{R}^2)}\|\theta(x-y)-\theta(x)\|_{L^{p+1}(\mathbb{R}^2)}dy\\&\quad+\|v_\ell-S_Nv_\ell\|_{L^{p+1}(\mathbb{R}^2)}\|\theta-S_N\theta\|_{L^{p+1}(\mathbb{R}^2)}\\&=I+II.\end{aligned}$$

Now, we estimate I. In view of the mean value theorem and the Bernstein inequality in Lemma 2.1, we know that

$$\|v_\ell(x-y)-v_\ell(x)\|_{L^{p+1}(\mathbb{R}^2)}\leq C\Big(\sum_{j<N}2^j|y|\|\dot\Delta_jv\|_{L^{p+1}(\mathbb{R}^2)}+\sum_{j\geq N}\|\dot\Delta_jv\|_{L^{p+1}(\mathbb{R}^2)}\Big).\tag{3.3}$$

Furthermore, using the Bernstein inequality again and the boundedness of Riesz transforms on Lebesgue spaces yields that

$$\|\dot\Delta_jv\|_{L^{p+1}(\mathbb{R}^2)}=\|\mathcal{R}^\perp\Lambda^{\gamma-1}\dot\Delta_j\theta\|_{L^{p+1}(\mathbb{R}^2)}\leq C2^{j(\gamma-1)}\|\dot\Delta_j\theta\|_{L^{p+1}(\mathbb{R}^2)},\ \text{for } 0<p<\infty.$$

This together with (3.3) means that

$$\begin{aligned}&\|v_\ell(x-y)-v_\ell(x)\|_{L^{p+1}(\mathbb{R}^2)}\\&\leq C\Big(2^{N(\gamma-\alpha)}|y|\sum_{j<N}2^{-(N-j)(\gamma-\alpha)}2^{j\alpha}\|\dot\Delta_j\theta\|_{L^{p+1}(\mathbb{R}^2)}\\&\quad+2^{(\gamma-1-\alpha)N}\sum_{j\geq N}2^{(N-j)(\alpha+1-\gamma)}2^{j\alpha}\|\dot\Delta_j\theta\|_{L^{p+1}(\mathbb{R}^2)}\Big).\end{aligned}\tag{3.4}$$

Before going further, in the spirit of [13], we set the following localized kernel

$$K_1(j)=\begin{cases}2^{j(\alpha+1-\gamma)}, \text{ if } j\leq 0,\\ 2^{-(\gamma-\alpha)j}, \text{ if } j>0,\end{cases}\tag{3.5}$$

and we denote $\dot d_j=2^{j\alpha}\|\dot\Delta_j\theta\|_{L^{p+1}(\mathbb{R}^2)}$. As a consequence, we get

$$\begin{aligned}\|v_\ell(x-y)-v_\ell(x)\|_{L^{p+1}(\mathbb{R}^2)}&\leq C\left[2^{N(\gamma-\alpha)}|y|+2^{(\gamma-1-\alpha)N}\right]\left(K_1*\dot d_j\right)(N)\\&\leq C(2^N|y|+1)2^{(\gamma-1-\alpha)N}\left(K_1*\dot d_j\right)(N).\end{aligned}$$

To bound $\|\theta(x-y)-\theta(x)\|_{L^{p+1}(\mathbb{R}^2)}$, we denote

$$K_2(j)=\begin{cases}2^{j\alpha}, \text{ if } j\leq 0,\\ 2^{-(1-\alpha)j}, \text{ if } j>0.\end{cases}\tag{3.6}$$

Using the same procedure to obtain (3.3) and (3.4) yields

$$\begin{aligned}&\|\theta(x-y)-\theta(x)\|_{L^{p+1}(\mathbb{R}^2)}\\&\leq C\Big(\sum_{j<N}2^j|y|\|\dot\Delta_j\theta\|_{L^{p+1}(\mathbb{R}^2)}+\sum_{j\geq N}\|\dot\Delta_j\theta\|_{L^{p+1}(\mathbb{R}^2)}\Big)\end{aligned}$$

$$\le C\Big(2^{N(1-\alpha)}|y|\sum_{j<N}2^{-(N-j)(1-\alpha)}2^{j\alpha}\|\dot\Delta_j\theta\|_{L^{p+1}(\mathbb{R}^2)}+2^{-\alpha N}\sum_{j\ge N}2^{(N-j)\alpha}2^{j\alpha}\|\dot\Delta_j\theta\|_{L^{p+1}(\mathbb{R}^2)}\Big)$$
$$\le C(2^N|y|+1)2^{-\alpha N}\left(K_2*\dot d_j\right)(N). \tag{3.7}$$

Notice that

$$\sup_N 2^{2N}\int_{\mathbb{R}^2}|\tilde h(2^Ny)|(2^N|y|+1)^2dy<\infty.$$

Hence, we deduce from (3.4) and (3.7) that

$$I\le C2^{(\gamma-1-\alpha)N}\left(K_1*\dot d_j\right)(N)2^{-\alpha N}\left(K_2*\dot d_j\right)(N).$$

In light of the Bernstein inequality, we infer that

$$\|v_\ell-S_Nv_\ell\|_{L^{p+1}(\mathbb{R}^2)}\le\sum_{j\ge N}\|\dot\Delta_jv\|_{L^{p+1}}\le C2^{(\gamma-1-\alpha)N}\left(K_1*\dot d_j\right)(N),$$

where we used $N>0$. Likewise,

$$\|\theta-S_N\theta\|_{L^{p+1}(\mathbb{R}^2)}\le C2^{-\alpha N}\left(K_2*\dot d_j\right)(N),$$

from which it follows that

$$II\le C2^{(\gamma-1-\alpha)N}\left(K_1*\dot d_j\right)(N)2^{-\alpha N}\left(K_2*\dot d_j\right)(N).$$

Consequently, we know that

$$\|S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\|_{L^{\frac{p+1}{2}}(\mathbb{R}^2)}\le C2^{(\gamma-1-\alpha)N}\left(K_1*\dot d_j\right)(N)2^{-\alpha N}\left(K_2*\dot d_j\right)(N). \tag{3.8}$$

We conclude by some straightforward calculations that

$$\|\partial_\ell S_N\theta\|_{L^{p+1}(\mathbb{R}^2)}\le\sum_{j\le N}2^j\|\Delta_j\theta\|_{L^{p+1}(\mathbb{R}^2)}\le 2^{N(1-\alpha)}\left(K_2*d_j\right)(N), \tag{3.9}$$

where $d_j=2^{j\alpha}\|\Delta_j\theta\|_{L^{p+1}(\mathbb{R}^2)}$. Inserting (3.8) and (3.9) into (3.1) gives

$$\Big|\int_{\mathbb{R}^2}\big[S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\big]\partial_\ell S_N\theta|S_N\theta|^{p-2}dx\Big|$$
$$\le C2^{(\gamma-3\alpha)N}\left(K_1*\dot d_j\right)(N)\left(K_2*\dot d_j\right)(N)\left(K_2*d_j\right)(N)\|S_N\theta\|^{p-2}_{L^{p+1}(\mathbb{R}^2)}$$
$$\le C2^{(\gamma-3\alpha)N}\left(K_1*\dot d_j\right)(N)\left(K_2*\dot d_j\right)(N)\left(K_2*d_j\right)(N)\|\theta\|^{p-2}_{L^{p+1}(\mathbb{R}^2)}. \tag{3.10}$$

By choosing α satisfying

$$\begin{cases}\gamma-1<\alpha<\gamma,\\ 0<\alpha<1,\end{cases} \tag{3.11}$$

we know that $K_1,K_2\in l^1(\mathbb{Z})$.

Case 1: If we choose $\alpha=\frac{\gamma}{3}$, then by (3.11), $0<\gamma<\frac{3}{2}$. It follows from (3.10) that

$$\Big|\int_{\mathbb{R}^2}\big[S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\big]\partial_\ell S_N\theta|S_N\theta|^{p-2}dx$$
$$\le C\left(K_1*\dot d_j\right)(N)\left(K_2*\dot d_j\right)(N)\|\theta\|^{p-1}_{B^{\alpha}_{p+1,\infty}(\mathbb{R}^2)}. \tag{3.12}$$

Since $\theta \in L^{p+1}(0,T;B^{\frac{\gamma}{3}}_{p+1,c(\mathbb{N})}(\mathbb{R}^2))$, we have

$$\begin{aligned}&\int_0^T \Big|\int_{\mathbb{R}^2}\big[S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\big]\partial_\ell S_N\theta|S_N\theta|^{p-2}dx\Big|dt\\ &\quad\le C\int_0^T\left(K_1*\dot{d}_j\right)(N)\left(K_2*\dot{d}_j\right)(N)\|\theta\|^{p-1}_{B^{\alpha}_{p+1,\infty}(\mathbb{R}^2)}dt\\ &\quad\le C\int_0^T\|\theta(t)\|^{p+1}_{B^{\alpha}_{p+1,\infty}(\mathbb{R}^2)}dt<\infty.\end{aligned}$$

Hence, we conclude by the the dominated convergence theorem that

$$\begin{aligned}&\int_0^T \Big|\int_{\mathbb{R}^2}\big[S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\big]\partial_\ell S_N\theta|S_N\theta|^{p-2}dx\Big|dt\\ &\quad\le C\int_0^T\left(K_1*\dot{d}_j\right)(N)\left(K_2*\dot{d}_j\right)(N)\|\theta(t)\|^{p-1}_{B^{\alpha}_{p+1,\infty}(\mathbb{R}^2)}dt\to 0,\ \text{as } N\to+\infty\end{aligned}$$

Case 2: When $\frac{3}{2}\le\gamma<2$, we choose α satisfying $\frac{\gamma}{3}\le\gamma-1<\alpha<1$. From (3.10), we get

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}\big[S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\big]\partial_\ell S_N\theta|S_N\theta|^{p-2}dx\Big|\\ &\quad\le C2^{(\gamma-3\alpha)N}\|\theta\|^{p+1}_{B^{\alpha}_{p+1,\infty}(\mathbb{R}^2)},\end{aligned}$$

which gives

$$\begin{aligned}&\int_0^T \Big|\int_{\mathbb{R}^2}\big[S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\big]\partial_\ell S_N\theta|S_N\theta|^{p-2}dx\Big|dt\\ &\quad\le C2^{(\gamma-3\alpha)N}\int_0^T\|\theta(t)\|^{p+1}_{B^{\alpha}_{p+1,\infty}(\mathbb{R}^2)}dt\to 0,\ \text{as } N\to+\infty.\end{aligned}$$

Hence, no matter in which case, we have

$$\Big|\int_0^T\int_{\mathbb{R}^2}\big[S_N(v_\ell\theta)-S_Nv_\ell S_N\theta\big]\partial_\ell S_N\theta|S_N\theta|^{p-2}dxdt\Big|\to 0,\ \text{as } N\to+\infty.$$

Then we can complete the proof of Theorem 1.1.

Approach 2: Constantin-E-Titi type commutator estimates in Onsager type spaces.

Mollifying system (1.1) in spatial direction (see the notations in Sect. 2) and using the divergence-free condition, we know that

$$\theta^{\varepsilon}_t+\operatorname{div}(v\theta)^{\varepsilon}=0,$$

which yields that

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\theta^{\varepsilon}|^p dx=(p-1)\int_{\mathbb{R}^2}(v_\ell\theta)^{\varepsilon}\partial_\ell\theta^{\varepsilon}|\theta^{\varepsilon}|^{p-2}dx.$$

The incompressible condition allows us to formulate the above equation as

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\theta^{\varepsilon}|^p dx=(p-1)\int_{\mathbb{R}^2}\big[(v_\ell\theta)^{\varepsilon}-v_\ell^{\varepsilon}\theta^{\varepsilon}\big]\partial_\ell\theta^{\varepsilon}|\theta^{\varepsilon}|^{p-2}dx,$$

which immediately means

$$\frac{1}{p}\left(\|\theta^{\varepsilon}(x,t)\|_{L^p(\mathbb{R}^2)}-\|\theta^{\varepsilon}(x,0)\|_{L^p(\mathbb{R}^2)}\right)=(p-1)\int_0^t\int_{\mathbb{R}^2}\left((v_\ell\theta)^{\varepsilon}-v_\ell^{\varepsilon}\theta^{\varepsilon}\right)\partial_\ell\theta^{\varepsilon}|\theta^{\varepsilon}|^{p-2}dxds. \tag{3.13}$$

The Hölder inequality enables us to get

$$\Big|\int_0^t\int_{\mathbb{R}^2}\big[(v_\ell\theta)^{\varepsilon}-v_\ell^{\varepsilon}\theta^{\varepsilon}\big]\partial_\ell\theta^{\varepsilon}|\theta^{\varepsilon}|^{p-2}dxds\Big|$$

$$\leq C\|(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\|_{L^{\frac{p+1}{2}}(0,T;L^{\frac{p+1}{2}}(\mathbb{R}^2))}\|\partial_\ell\theta^\varepsilon\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^2))}\||\theta^\varepsilon|^{p-2}\|_{L^{\frac{p+1}{p-2}}(0,T;L^{\frac{p+1}{p-2}}(\mathbb{R}^2))}. \quad (3.14)$$

Since $B^s_{p,q} = \dot{B}^s_{p,q} \cap L^p$ for $s > 0$, the hypothesis $\theta \in L^{p+1}(0,T;B^\alpha_{p+1,c(\mathbb{N})}(\mathbb{R}^2))$ means $\theta \in L^{p+1}(0,T;\dot{B}^\alpha_{p+1,c(\mathbb{N})}(\mathbb{R}^2))$. This together with the boundedness of Riesz transforms in homogeneous Besov spaces gives

$$v = \mathcal{R}^\perp\Lambda^{\gamma-1}\theta \in L^{p+1}(0,T;\dot{B}^{\alpha-\gamma+1}_{p+1,c(\mathbb{N})}(\mathbb{R}^2)).$$

Combining $\theta \in L^{p+1}(0,T;\dot{B}^\alpha_{p+1,c(\mathbb{N})}(\mathbb{R}^2))$ with $v \in L^{p+1}(0,T;\dot{B}^{\alpha-\gamma+1}_{p+1,c(\mathbb{N})}(\mathbb{R}^2))$ and invoking Lemma 2.3, we see that, as $\varepsilon \to 0$,

$$\|(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\|_{L^{\frac{p+1}{2}}(0,T;L^{\frac{p+1}{2}}(\mathbb{R}^2))} \leq o(\varepsilon^{2\alpha-\gamma+1}), \quad (3.15)$$

where α is selected to satisfy $0 < \alpha < 1$ and $0 < \alpha - \gamma + 1 < 1$. Using Lemma 2.2, we know that, as $\varepsilon \to 0$,

$$\|\partial_\ell\theta^\varepsilon\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^2))} \leq o(\varepsilon^{\alpha-1}). \quad (3.16)$$

Moreover, in view of the definition of Besov spaces, we have

$$\||\theta^\varepsilon|^{p-2}\|_{L^{\frac{p+1}{p-2}}(0,T;L^{\frac{p+1}{p-2}}(\mathbb{R}^2))} \leq C\|\theta^\varepsilon\|^{p-2}_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^2))} \leq C\|\theta\|^{p-2}_{L^{p+1}(0,T;B^\alpha_{p+1,c(\mathbb{N})}(\mathbb{R}^2))}. \quad (3.17)$$

Then substituting (3.15)–(3.17) into (3.14), setting $\alpha = \frac{\gamma}{3}$ with $0 < \gamma < \frac{3}{2}$, we have

$$\left|\int_0^t\int_{\mathbb{R}^2}\left[(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\right]\partial_i\theta^\varepsilon|\theta^\varepsilon|^{p-2}dxds\right| \leq o(\varepsilon^{3\alpha-\gamma})\|\theta\|^{p-2}_{L^{p+1}(0,T;B^\alpha_{p+1,c(\mathbb{N})}(\mathbb{R}^2))} \to 0, \text{ as } \varepsilon \to 0.$$

Then we have completed the proof of the first part of Theorem 1.1. By using a similar argument to get (3.15)-(3.17), we can conclude the second part of Theorem 1.1 for $\theta \in L^{p+1}(0,T;B^\alpha_{p+1,\infty}(\mathbb{R}^2))$. □

Proof of Corollary 1.2. By a slight variant of the above proof, one can show this corollary. Indeed, we conclude by (3.13) with $p = 2$ that

$$\|\theta^\varepsilon(x,t)\|_{L^2(\mathbb{R}^2)} - \|\theta^\varepsilon(x,0)\|_{L^2(\mathbb{R}^2)} = 2\int_0^t\int_{\mathbb{R}^2}\left((v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\right)\partial_\ell\theta^\varepsilon dxds. \quad (3.18)$$

With the help of Hölder's inequality, we discover that

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}^2}\left[(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\right]\partial_\ell\theta^\varepsilon dxds\right| \\ &\quad \leq C\|(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\|_{L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(\mathbb{R}^2))}\|\partial_\ell\theta^\varepsilon\|_{L^3(0,T;L^3(\mathbb{R}^2))}.\end{aligned} \quad (3.19)$$

A combination of $\theta \in L^3(0,T;\dot{B}^\alpha_{3,c(\mathbb{N})}(\mathbb{R}^2))$ and the boundedness of Riesz transforms in homogeneous Besov spaces yield $v = \mathcal{R}^\perp\Lambda^{\gamma-1}\theta \in L^3(0,T;\dot{B}^{\alpha-\gamma+1}_{3,c(\mathbb{N})}(\mathbb{R}^2))$. Hence, by applying Lemma 2.3 to $\theta \in L^3(0,T;\dot{B}^\alpha_{3,c(\mathbb{N})}(\mathbb{R}^2))$ and $v \in L^3(0,T;\dot{B}^{\alpha-\gamma+1}_{3,c(\mathbb{N})}(\mathbb{R}^2))$, we derive that, as $\varepsilon \to 0$,

$$\|(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\|_{L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(\mathbb{R}^2))} \leq o(\varepsilon^{2\alpha-\gamma+1}), \quad (3.20)$$

where $0 < \alpha < 1$ and $0 < \alpha - \gamma + 1 < 1$ are required. Moreover, according to Lemma 2.2, we observe that, as $\varepsilon \to 0$,

$$\|\partial_\ell\theta^\varepsilon\|_{L^3(0,T;L^3(\mathbb{R}^2))} \leq o(\varepsilon^{\alpha-1}). \quad (3.21)$$

By plugging (3.20)-(3.21) into (3.19) and taking $\alpha = \frac{\gamma}{3}$ with $0 < \gamma < \frac{3}{2}$, we end up with

$$\left|\int_0^t\int_{\mathbb{R}^2}\left[(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\right]\partial_i\theta^\varepsilon dxds\right| \leq o(\varepsilon^{3\alpha-\gamma}) \to 0, \text{ as } \varepsilon \to 0.$$

At this stage, this corollary is proved. □

Next, we present the proof of Theorem 1.3. To prove Theorem 1.3, it suffices to replace (3.4) by

$$\begin{aligned}&\|v_\ell(x-y)-v_\ell(x)\|_{L^{p+1}(\mathbb{R}^2)}\\&\quad\leq C\left(2^{N(1-\alpha)}|y|\sum_{j\leq N}2^{-(N-j)(1-\alpha)}2^{j\alpha}\|\dot\Delta_j v\|_{L^{p+1}(\mathbb{R}^2)}+2^{-\alpha N}\sum_{j>N}2^{(N-j)(\alpha)}2^{j\alpha}\|\dot\Delta_j v\|_{L^{p+1}(\mathbb{R}^2)}\right)\\&\quad\leq C\left[2^{N(1-\alpha)}|y|+2^{-\alpha N}\right]\left(K_1*\dot d_j\right)(N)\\&\quad\leq C(2^N|y|+1)2^{-\alpha N}\left(K_1*\dot d_j\right)(N),\end{aligned}$$

where

$$K_1(j)=\begin{cases}2^{j\alpha}, \text{ if } j\leq 0,\\ 2^{-(1-\alpha)j}, \text{ if } j>0,\end{cases}$$

and $\dot d_j=2^{j\alpha}\|\dot\Delta_j v\|_{L^{p+1}}$. We omit the details here. We only outline its proof by Constantin-E-Titi type commutator estimates in physical Onsager type spaces in the following.

Proof of Theorem 1.3. Based on the second proof of Theorem 1.1, we just give the key estimates. It follows from the Hölder inequality that

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{R}^2}\left[(v_\ell\theta)^\varepsilon-v_\ell^\varepsilon\theta^\varepsilon\right]\partial_i\theta^\varepsilon|\theta^\varepsilon|^{p-2}dxds\Big|\\&\quad\leq C\|(v_\ell\theta)^\varepsilon-v_\ell^\varepsilon\theta^\varepsilon\|_{L^{\frac{r_1r_2}{r_1+r_2}}(0,T;L^{\frac{p+1}{2}}(\mathbb{R}^2))}\|\partial_\ell\theta^\varepsilon\|_{L^{r_2}(0,T;L^{p+1}(\mathbb{R}^2))}\||\theta^\varepsilon|^{p-2}\|_{L^{p_4}(0,T;L^{\frac{p+1}{p-2}}(\mathbb{R}^2))},\end{aligned}\tag{3.22}$$

where $\frac{r_1+r_2}{r_1r_2}+\frac{1}{r_2}+\frac{1}{p_4}=1$.

From $v\in L^{r_1}(0,T;\dot B^{\frac13}_{p+1,c(\mathbb{N})})$ and $\theta\in L^{r_2}(0,T;\dot B^{\frac13}_{p+1,\infty})$, we deduce from Lemma 2.3 that, as $\varepsilon\to 0$,

$$\|(v_\ell\theta)^\varepsilon-v_\ell^\varepsilon\theta^\varepsilon\|_{L^{\frac{r_1r_2}{r_1+r_2}}(0,T;L^{\frac{p+1}{2}}(\mathbb{R}^2))}\leq C\mathrm{o}(\varepsilon^{\frac23}).\tag{3.23}$$

From Lemma 2.2, we infer that, as $\varepsilon\to 0$,

$$\|\partial_\ell\theta^\varepsilon\|_{L^{r_2}(0,T;L^{p+1}(\mathbb{R}^2))}\leq C\mathrm{O}(\varepsilon^{-\frac23}).\tag{3.24}$$

According to the definition of Besov spaces, we have

$$\||\theta^\varepsilon|^{p-2}\|_{L^{p_4}(0,T;L^{\frac{p+1}{p-2}}(\mathbb{R}^2))}\leq C\|\theta^\varepsilon\|^{p-2}_{L^{p_4(p-2)}(0,T;L^{p+1}(\mathbb{R}^2))}\leq C\|\theta\|^{p-2}_{L^{r_2}(0,T;B^{\frac13}_{p+1,\infty})},\tag{3.25}$$

where we used $p_4(p-2)=r_2$, which means $\frac{p}{r_2}+\frac{1}{r_1}=1$ and $p\geq 2$.

Then substituting (3.23)–(3.25) into (3.22) and letting $\varepsilon\to 0$, we have

$$\Big|\int_0^t\int_{\mathbb{R}^2}\left[(v_\ell\theta)^\varepsilon-v_\ell^\varepsilon\theta^\varepsilon\right]\partial_\ell\theta^\varepsilon|\theta^\varepsilon|^{p-2}dxds\Big|\leq C\mathrm{o}(1)\|\theta\|^{p-2}_{L^{p_4(p-2)}(0,T;B^{\frac13}_{p+1,\infty})}\to 0.$$

Then we have completed the proof of Theorem 1.3. □

3.2. Energy Conservation in Besov VMO Spaces

We address the energy conservation of weak solutions of the generalized surface quasi-geostrophic Eq. (1.1) in Besov VMO spaces in this subsection.

Proof of Theorem 1.4. (1) With (3.13) in hand, it suffices to show that $\int_0^t\int_{\mathbb{T}^2}\left[(v_\ell\theta)^\varepsilon-v_\ell^\varepsilon\theta^\varepsilon\right]\partial_\ell\theta^\varepsilon|\theta^\varepsilon|^{p-2}dxds$ converges to 0 as $\varepsilon\to 0$. We deduce from (3.14) that

$$\Big|\int_0^t\int_{\mathbb{T}^2}\left[(v_\ell\theta)^\varepsilon-v_\ell^\varepsilon\theta^\varepsilon\right]\partial_\ell\theta^\varepsilon|\theta^\varepsilon|^{p-2}dxds\Big|$$

$$\leq C\|(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\|_{L^{\frac{p+1}{2}}(0,T;L^{\frac{p+1}{2}}(\mathbb{T}^2))}\|\partial_\ell\theta^\varepsilon\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{T}^2))}\||\theta^\varepsilon|^{p-2}\|_{L^{\frac{p+1}{p-2}}(0,T;L^{\frac{p+1}{p-2}}(\mathbb{T}^2))}. \quad (3.26)$$

We conclude from $\theta \in L^{p+1}(0,T;B^{\frac{\gamma}{3}}_{p+1,\infty}(\mathbb{T}^2))$ that $\theta \in L^{p+1}(0,T;\dot{B}^{\frac{\gamma}{3}}_{p+1,\infty}(\mathbb{T}^2))$. A combination of this and $v = \mathcal{R}^\perp\Lambda^{\gamma-1}\theta$ implies that $v \in L^{p+1}(0,T;\dot{B}^{1-\frac{2\gamma}{3}}_{p+1,\infty}(\mathbb{T}^2))$.

According to $\theta \in L^{p+1}(0,T;\underline{B}^{\frac{\gamma}{3}}_{p+1,VMO}(\mathbb{T}^2))$, $v \in L^{p+1}(0,T;\dot{B}^{1-\frac{2\gamma}{3}}_{p+1,\infty}(\mathbb{T}^2))$ and Lemma 2.4, we obtain, as $\varepsilon \to 0$,

$$\|(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\|_{L^{\frac{p+1}{2}}(0,T;L^{\frac{p+1}{2}}(\mathbb{T}^2))} \leq o(\varepsilon^{1-\frac{\gamma}{3}}). \quad (3.27)$$

From Remark 2.1, we know that, as $\varepsilon \to 0$,

$$\|\partial_\ell\theta^\varepsilon\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{T}^2))} \leq o(\varepsilon^{\frac{\gamma}{3}-1}). \quad (3.28)$$

Inserting (3.27), (3.28) and (3.17) into (3.26), we end up with

$$\left|\int_0^t\int_{\mathbb{T}^2}\left[(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\right]\partial_\ell\theta^\varepsilon|\theta^\varepsilon|^{p-2}dxds\right| \leq o(1)\|\theta\|^{p-2}_{L^{p+1}(0,T;B^{\frac{\gamma}{3}}_{p+1,\infty}(\mathbb{T}^2))} \to 0,\ \varepsilon \to 0.$$

This yields the desired energy balance.

(2) $v \in L^{r_1}(0,T;\underline{B}^{\frac{1}{3}}_{p+1,VMO}(\mathbb{T}^2))$, $\theta \in L^{r_2}(0,T;B^{\frac{1}{3}}_{p+1,\infty}(\mathbb{T}^2))$ and Lemma 2.4 guarantee that

$$\|(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\|_{L^{\frac{r_1r_2}{r_1+r_2}}(0,T;L^{\frac{p+1}{2}}(\mathbb{T}^2))} \leq C\mathrm{o}(\varepsilon^{\frac{2}{3}}),\ \text{as } \varepsilon \to 0, \quad (3.29)$$

where we have to require $r_2 \geq 1 + \frac{1}{p}$.

Plugging (3.29), (3.24) and (3.25) into (3.22), we arrive at, as $\varepsilon \to 0$,

$$\left|\int_0^t\int_{\mathbb{T}^2}\left[(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\right]\partial_\ell\theta^\varepsilon|\theta^\varepsilon|^{p-2}dxds\right| \leq \mathrm{o}(1)\|\theta\|^{p-2}_{L^{r_2}(0,T;B^{\frac{1}{3}}_{p+1,\infty}(\mathbb{T}^2))}.$$

This implies the desired energy law.

(3) It follows from $v \in L^{r_1}(0,T;B^{\frac{1}{3}}_{p+1,\infty}(\mathbb{T}^2))$, $\theta \in L^{r_2}(0,T;\underline{B}^{\frac{1}{3}}_{p+1,VMO}(\mathbb{T}^2))$ and Lemma 2.4 that

$$\|(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\|_{L^{\frac{r_1r_2}{r_1+r_2}}(0,T;L^{\frac{p+1}{2}}(\mathbb{R}^2))} \leq C\mathrm{o}(\varepsilon^{\frac{2}{3}}),\ \text{as } \varepsilon \to 0, \quad (3.30)$$

where we need $r_1 \geq 1 + \frac{1}{p}$.

Thanks to Remark 2.1, we discover that

$$\|\partial_\ell\theta^\varepsilon\|_{L^{r_2}(0,T;L^{p+1}(\mathbb{R}^2))} \leq C\mathrm{o}(\varepsilon^{-\frac{2}{3}}). \quad (3.31)$$

Making use of the definition of Besov VMO space, we calculate

$$\||\theta^\varepsilon|^{p-2}\|_{L^{p_4}(0,T;L^{\frac{p+1}{p-2}}(\mathbb{T}^2))} \leq C\|\theta^\varepsilon\|^{p-2}_{L^{p_4(p-2)}(0,T;L^{p+1}(\mathbb{T}^2))} \leq C\|\theta\|^{p-2}_{L^{r_2}(0,T;\underline{B}^{\frac{1}{3}}_{p+1,VMO}(\mathbb{T}^2))}, \quad (3.32)$$

and $\frac{p}{r_2} + \frac{1}{r_1} = 1$. Substituting (3.30)–(3.32) into (3.22), we infer that, as $\varepsilon \to 0$,

$$\left|\int_0^t\int_{\mathbb{T}^2}\left[(v_\ell\theta)^\varepsilon - v_\ell^\varepsilon\theta^\varepsilon\right]\partial_\ell\theta^\varepsilon|\theta^\varepsilon|^{p-2}dxds\right| \leq \mathrm{o}(1)\|\theta\|^{p-2}_{L^{r_2}(0,T;\underline{B}^{\frac{1}{3}}_{p+1,VMO}(\mathbb{T}^2))}.$$

This means the desired energy relation. The theorem is thus proved. □

4. Conclusion

We apply the Littlewood-Paley theory as [13] and the Constantin-E-Titi type commutator estimates in Onsager type spaces to study the energy conservation of weak solutions for the generalized surface quasi-geostrophic equation with the velocity v determined by $v = \mathcal{R}^{\perp}\Lambda^{\gamma-1}\theta$ with $0 < \gamma < 2$, respectively. For the case $0 < \gamma < \frac{3}{2}$, the sufficient conditions for the energy conservation of weak solutions of this equation in Onsager's critical space are derived. For the more singular case $\frac{3}{2} \leq \gamma < 2$, we obtain the corresponding results in subcritical spaces. For periodic domain, we consider the energy conservation of weak solutions in Besov VMO space recently introduced by Fjordholm-Wiedemann in [18]. As pointed in [3], the space $\underline{B}^{\frac{1}{3}}_{3,VMO}$ is an almost optimal regularity class for the conservation of energy. It is worth remarking that the sufficient conditions for implying the conservation of L^p-norm for $p \in (1, 2)$ are unknown in the generalized surface quasi-geostrophic equation.

A natural question is to extend our results to other models which modify the velocity field. A possible candidate is the inviscid Leary-α or Euler-α system. After we completed the main part of this paper, we learned the energy conservation of these models recently studied by Beekie-Novack in [4] and Boutros-Titi in [6]. Compared with their results, the results here give how the critical regularity for the energy conservation of the weak solutions depends on the the parameter γ of the velocity. It seems that the arguments in this paper can be applicable to other fluid models such as the surface growth model without dissipation

$$h_t + \partial_{xx}(h_x)^2 = 0, \tag{4.1}$$

where h stands for the height of a crystalline layer. The background of the surface growth model (4.1) can be found in [5,24,25,29]. The energy conservation in the Besov space $L^3(0, T; B^{\alpha}_{3,\infty}(\mathbb{T}^3))$ with $\alpha > 1/3$ was considered in [29]. One can establish the persistence of energy criterion in the Onsager's critical spaces for the inviscid surface growth model (4.1).

The non-uniqueness of weak solutions to the standard surface quasi-geostrophic Eq. (1.2) can be found in [7,19]. It would be interesting to show that the weak solutions to the generalized quasi-geostrophic Eq. (1.1) are not unique.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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References

[1] Akramov, I., Wiedemann, E.: Renormalization of active scalar equations. Nonlinear Anal. **179**, 254–269 (2019)
[2] Bahouri, H., Chemin, J.-Y., Danchin, R.: Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehren dermathematischen Wissenschaften 343, Springer-Verlag, (2011)
[3] Bardos, C., Gwiazda, P., Świerczewska-Gwiazda, A., Titi, E.S., Wiedemann, E.: Onsager's conjecture in bounded domains for the conservation of entropy and other companion laws. Proc. R. Soc. A. **475**, 18 (2019)
[4] Beekie, R., Novack, M.: Non-conservative solutions of the Euler-α equations. J. Math. Fluid Mech. **25**, 22 (2023)
[5] Blömker, D., Romito, M.: Regularity and blow up in a surface growth model. Dyn. Partial Differ. Equ. **6**, 227–252 (2009)
[6] Boutros, D.W., Titi, E.S.: Onsager's conjecture for subgrid scale α-models of turbulence. Phys. D. **443**, 133553 (2023)
[7] Buckmaster, T., Shkoller, S., Vicol, V.: Nonuniqueness of weak solutions to the SQG equation. Commun. Pure Appl. Math. **72**, 1809–1874 (2019)
[8] Chae, D.: Remarks on the helicity of the 3-D incompressible Euler equations. Commun. Math. Phys. **240**, 501–507 (2003)
[9] Chae, D.: On the conserved quantities for the weak solutions of the Euler equations and the Quasi-geostrophic equations. Commun. Math. Phys. **266**, 197–210 (2006)
[10] Chae, D., Constantin, P., Wu, J.: Inviscid models generalizing the two-dimensional Euler and the surface quasi-geostrophic equations. Arch. Ration. Mech. Anal. **202**, 35–62 (2011)
[11] Chae, D., Constantin, P., Wu, J.: Dissipative models generalizing the 2D Navier-Stokes and surface quasi-geostrophic equations. Indiana Univ. Math. J. **61**, 1997–2018 (2012)
[12] Chae, D., Constantin, P., Córdoba, D., Gancedo, F., Wu, J.: Generalized surface quasi-geostrophic equations with singular velocities. Comm. Pure Appl. Math. **65**, 1037–1066 (2012)
[13] Cheskidov, A., Constantin, P., Friedlander, S., Shvydkoy, R.: Energy conservation and Onsager's conjecture for the Euler equations. Nonlinearity **21**, 1233–52 (2008)
[14] Constantin, P., Iyer, G., Wu, J.: Global regularity for a modified critical dissipative quasi-geostrophic equation. Indiana Univ. Math. J. **57**, 2681–2692 (2008)
[15] Constantin, P., Majda, A., Tabak, E.: Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar. Nonlinearity **7**, 1495–1533 (1994)
[16] Constantin, P., Weinan, E., Titi, E.S.: Onsager's conjecture on the energy conservation for solutions of Euler's equation. Commun. Math. Phys. **165**, 207–209 (1994)
[17] Dai, M.: Regularity criterion and energy conservation for the supercritical Quasi-geostrophic equation. J. Math. Fluid Mech. **19**, 191–202 (2017)
[18] Fjordholm, U.S., Wiedemann, E.: Statistical solutions and Onsager's conjecture. Phys. D. **376–377**, 259–265 (2018)
[19] Isett, P., Ma, A.: A direct approach to nonuniqueness and failure of compactness for the SQG equation. Nonlinearity **34**, 3122–3162 (2021)
[20] Kiselev, A.: Nonlocal maximum principles for active scalars. Adv. Math. **227**, 1806–1826 (2011)
[21] Miao, C., Xue, L.: On the regularity of a class of generalized quasi-geostrophic equations. J. Differ. Equ. **251**, 2789–2821 (2011)
[22] Miao, C., Xue, L.: Global wellposedness for a modified critical dissipative quasi-geostrophic equation. J. Differ. Equ. **252**, 792–818 (2012)
[23] Onsager, L.: Statistical hydrodynamics. Nuovo Cim. **6**, 279–287 (1949)
[24] Ożański, W.S.: A sufficient integral condition for local regularity of solutions to the surface growth model. J. Funct. Anal. **276**, 2990–3013 (2019)
[25] Ożański, W.S., Robinson, J.C.: Partial regularity for a surface growth model. SIAM J. Math. Anal. **51**, 228–255 (2019)
[26] Wang, Y., Wei, W., Ye, Y.: Analytical validation of the helicity conservation for the isentropic compressible Euler equations. arXiv:2208.05715, (2022)
[27] Wang, Y., Ye, Y., Liu, J.: Energy and helicity conservation in the incompressible ideal flows. submitted. (2022)
[28] Wang, Y., Yang, J., Ye, Y.: On two conserved quantities in the inviscid electron and Hall magnetohydrodynamic equations. arXiv:2303.12248
[29] Wei, W., Ye, Y., Mei, X.: Energy conservation and Onsager's conjecture for a surface growth model. Dyn. Partial Differ. Equ., (2023)
[30] Xue, L., Zheng, X.: Note on the well-posedness of a slightly supercritical surface quasi-geostrophic equation. J. Differ. Equ. **253**(2), 795–813 (2012). https://doi.org/10.1016/j.jde.2012.04.003
[31] Ye, Z.: Long-time asymptotic behavior of the generalized two-dimensional quasi-geostrophic equation. J. Funct. Anal. **283**, 109669 (2022)
[32] Yu, H., Zheng, X., Jiu, Q.: Remarks on well-posedness of the generalized surface quasi-geostrophic equation. Arch. Ration. Mech. Anal. **232**, 265–301 (2019)
[33] Yu, X.: A note on the energy conservation of the ideal MHD equations. Nonlinearity **22**, 913–922 (2009)
[34] Zhou, Y.: On the energy and helicity conservations for the 2-D quasi-geostrophic equation. Ann. Henri Poincaré **6**, 791–799 (2005)

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