J. Math. Fluid Mech. (2023) 25:74

1422-6928/23/030001-29
https://doi.org/10.1007/s00021-023-00814-7

Journal of Mathematical
Fluid Mechanics

Motion of Rigid Bodies of Arbitrary Shape in a Viscous Incompressible Fluid: Wellposedness and Large Time Behaviour

Debayan Maity and Marius Tucsnak

Communicated by G. P. Galdi

Abstract. We investigate the long-time behaviour of a coupled PDE–ODE system that describes the motion of a rigid body of arbitrary shape moving in a viscous incompressible fluid. We assume that the system formed by the rigid body and the fluid fills the entire space $\mathbb{R}^3$. We extend in this way our previous results which were limited to the case when the rigid body was a ball. More precisely, we show that, under appropriate assumptions (in particular smallness ones) on the initial velocity field, the position of the rigid body converges to some final configuration as time goes to infinity. Finally, we show that our methodology can be applied in the case of several rigid bodies of arbitrary shapes moving in a viscous incompressible fluid.

Mathematics Subject Classification. 35Q35, 35B40, 76D03, 76D05.

Keywords. Fluid–structure interaction, Incompressible Navier–Stokes system, Large time behaviour.

1. Introduction

Consider a homogeneous rigid body which occupies at instant $t \geqslant 0$ a smooth bounded domain $\mathcal{S}(t)$ which is moving in a viscous incompressible fluid which fills the remaining part of $\mathbb{R}^3$. The domain occupied by the fluid at instant t is denoted by $\mathcal{F}(t) := \mathbb{R}^3 \backslash \overline{\mathcal{S}(t)}$.

We denote by $h(t)$ and $Q(t)$ the position of the centre of mass at instant t and the orthogonal matrix giving the orientation of the rigid body at instant t, respectively. We thus have

$$\begin{cases} \mathcal{S}(t) = \{h(t) + Q(t)x \mid x \in \mathcal{S}(0)\} & (t \geqslant 0), \\ \dot{Q}(t)Q(t)^{-1}x = \widetilde{\omega}(t) \times x & (t \geqslant 0,\ x \in \mathbb{R}^3), \\ Q(0) = \mathbb{I}_3, \end{cases} \tag{1.1}$$

where $\widetilde{\omega}(t)$ is the angular velocity of the rigid body at instant t. Moreover, the velocity and pressure fields in the fluid are denoted by $\widetilde{u}$ and $\widetilde{\pi}$, respectively. With the above notation, the system describing the coupled motion of the rigid body and of the fluid is completed by the equations

Debayan Maity was partially supported by INSPIRE faculty fellowship (IFA18-MA128) and by Department of Atomic Energy, Government of India, under project no. 12-R & D-TFR-5.01-0520. The work of Marius Tucsnak has been supported by the ANR (Agence Nationale de la Recherche) project TRECOS, Grant ANR-20-CE40-0009.

$$\begin{cases} \partial_t \widetilde{u} + (\widetilde{u}\cdot\nabla)\widetilde{u} - \mu\Delta\widetilde{u} + \nabla\widetilde{\pi} = 0 & (t>0,\ x\in\mathcal{F}(t)),\\ \operatorname{div}\widetilde{u} = 0, & (t>0,\ x\in\mathcal{F}(t)),\\ \widetilde{u}(t,x) = \dot{h}(t) + \widetilde{\omega}(t)\times(x-h(t)) & (t>0,\ x\in\partial\mathcal{F}(t)),\\ m\ddot{h}(t) = -\int_{\partial\mathcal{S}(t)} \sigma(\widetilde{u},\widetilde{\pi})\nu\,\mathrm{d}s & (t>0),\\ J\dot{\widetilde{\omega}}(t) = J\widetilde{\omega}(t)\times\widetilde{\omega}(t) - \int_{\partial\mathcal{S}(t)}(x-h(t))\times\sigma(\widetilde{u},\widetilde{\pi})\nu\,\mathrm{d}s & (t>0),\\ \widetilde{u}(0,x) = u_0(x) & (y\in\mathcal{F}(0)),\\ h(0)=0,\ \dot{h}(0)=\ell_0,\ \widetilde{\omega}(0)=\omega_0. \end{cases} \tag{1.2}$$

In the above equations, the fluid is supposed to be homogeneous with density equal to 1 and of constant viscosity $\mu > 0$. Moreover, the unit vector field normal to $\partial S(t)$ and directed towards the interior of $\mathcal{S}(t)$ is denoted by $\nu(t,\cdot)$. The constant $m > 0$ denotes the mass of the rigid body and the matrix $J(t)$ stands for the inertia tensor of the rigid body at time $t > 0$, whereas the Cauchy stress tensor field in the fluid is given by the constitutive law

$$\sigma(\widetilde{u},\widetilde{\pi})_{k\ell} = -\widetilde{\pi}\delta_{k\ell} + \mu\left(\frac{\partial\widetilde{u}_k}{\partial y_\ell} + \frac{\partial\widetilde{u}_\ell}{\partial y_k}\right) \quad (1 \leqslant k,\ell \leqslant 3),$$

with $\delta_{k\ell}$ standing for the Kronecker symbol. The above equations can be easily adapted to the case of several rigid bodies, situation which will be studied in Sect. 8.

Over the last two decades, there has been considerable interest in the studying of the initial and boundary value problem (1.1)–(1.2). We refer to Serre [17], Takahashi [19], Takahasi and Tucsnak [20], Cumsille and Takahashi [1], Geissert et. al [7] and the references therein regarding wellposedness issues for (1.1)–(1.2), and we discuss the existing results on the large time behaviour of solutions, particularly the trajectory of the rigid body, in more detail here. For the corresponding system in two space dimensions Ervedoza et. al [3] considered the case when the rigid body is a disk. They proved that, under suitable regularity and smallness assumptions on the initial data, the velocity of the mass centre of the rigid body, denoted by $\dot{h}(t)$, decays like t^{-1}, as $t \to \infty$, thus not excluding the possibility of an unbounded trajectory of the rigid disk. Ferriere and Hillairet [5] studied the same system and they were able to remove the smallness assumption on the initial data. As far as we know, describing the large time behavior in the two-dimensional case with solids of arbitrary shapes is still an open question. The large time behaviour, that we are interested in this work, in the case of three space dimensions, has recently been described in [4], provided that the rigid is a ball. The main result in [4] asserts that, given $\varepsilon \in (0, 1/2)$, under suitable regularity and smallness assumptions on the initial data, $\dot{h}(t)$ decays quicker then $t^{-(3/2-\varepsilon)}$. In particular, this means that the position of the centre of the rigid ball converges to some $h_\infty \in \mathbb{R}^3$, as $t \to \infty$. Very recently, Galdi [6] proved that, with a rigid body of arbitrary shape in three space dimensions, the velocity fields of the fluid and of the solid tend to zero (in appropriate norms). The results in [6] contain no decay estimates so that they cannot be directly used to investigate the potential stabilization of the position of the rigid body towards a "final" position.

In this paper we consider rigid bodies of arbitrary shapes and we provide decay rates for fluid velocity $\widetilde{u}$ as well as for the solid velocities $\dot{h}$ and $\widetilde{\omega}$ (see Theorem 2.3 below). More precisely, we show that, given $\varepsilon > 0$ arbitrarily small and under suitable regularity and smallness assumptions on the initial data, $\dot{h}(t)$ and $\widetilde{\omega}(t)$ decay quicker then $t^{-(3/4-\varepsilon)}$ when $t \to \infty$, see Corollary 2.5 below. Due to the necessity of new estimates (involving, in particular, second order derivatives of the velocity field), these estimates provide a slower decay than in the case of a rigid ball. Nevertheless, our decay estimates are sufficient to conclude that the rigid body "asymptotically stops" at some finite distance. More precisely, we show that there exist $p > 1$ and $\eta > \frac{p-1}{p}$ such that the map $t \mapsto (1+t^2)^{\frac{\eta}{2}} \begin{bmatrix} \dot{h}(t) \\ \tilde{\omega}(t) \end{bmatrix}$ lies in $L^p([0,\infty);\mathbb{R}^6)$, which implies, in particular, that $h \in L^\infty([0,\infty);\mathbb{R}^3)$ (see Corollary 2.5 below).

We also show that our methodology adapts to the case of multiple rigid bodies. As far as we know, this is the first result that describes the long-term behaviour of several rigid bodies moving in a three-dimensional viscous incompressible fluid. In fact, our results appear to be the first to establish global existence and uniqueness (for small initial data) in the three-dimensional case with multiple rigid bodies. For global wellposedness in two space dimensions and with several moving disks we refer to Sabbagh [16].

The plan of this article is as follows. In Sect. 2, we introduce some notations that will be used throughout the paper. Our main results are stated in Theorem 2.3 and Corollary 2.5. To this aim, we introduce a change of variables and rewrite system (1.2) in the reference configuration in Sect. 3. The main result in the reference configuration is stated in Theorem 3.4. In Sect. 4 we introduce the *fluid–structure operator* and we recall some of its properties from [4]. Section 5 is devoted to the establishment of the decay estimates of the *fluid–structure semigroup*. We recall, in particular, the $L^q - L^r$ decay estimates already proved in [4] and then we prove some new decay estimates, which are made necessary by new terms appearing in the change of variables. In Sect. 6 we prove infinite time maximal type regularity results for a non homogeneous linear fluid–structure system. Theorems 3.4 and 2.3 are proved in Sect. 7. Finally, in Sect. 8 we explain how our results extend to the case of several rigid bodies.

2. Main Results

We first introduce several function spaces needed to state our main results. Let $G \subset \mathbb{R}^3$ be an open set with smooth boundary. For every $q > 1$ and $k \in \mathbb{N}$, we denote the standard Lebesgue and Sobolev spaces by $L^q(G)$ and by $W^{k,q}(G)$, respectively. The notation $W^{s,q}(G)$, with $s \in \mathbb{R}$ and $q > 1$ stands for the Sobolev–Slobodeckij spaces. The norms on $[L^q(G)]^n$ and $[W^{s,q}(G)]^n$ with $n \in \mathbb{N}$, will be denoted by $\|\cdot\|_{q,G}$ and $\|\cdot\|_{s,q,G}$, respectively. When $G = \mathbb{R}^3$, these norms will be simply denoted by $\|\cdot\|_q$ and $\|\cdot\|_{s,q}$, respectively. Moreover, the space $W_0^{k,q}(G)$ is defined as the completion of $C_0^\infty(G)$ with respect to the $W^{k,q}(G)$ norm. For $k, m \in \mathbb{N}$, $k < m$, and for $1 < p < \infty$, $1 < q < \infty$, we consider the standard definition of the Besov spaces by real interpolation of Sobolev spaces

$$B^s_{q,p}(G) = \left(W^{k,q}(G), W^{m,q}(G)\right)_{\theta,p} \text{ where } s = (1-\theta)k + \theta m,\ \theta \in (0,1).$$

We refer to Triebel [22] for a detailed presentation of the Besov spaces. We denote by $C^k_b(G)$ the set of continuous and bounded functions with derivatives continuous and bounded up to the order k on G. For $\tau \in (0,\infty]$ we set

$$W^{1,2}_{p,q}((0,\tau);G) = L^p((0,\tau);W^{2,q}(G)) \cap W^{1,p}((0,\tau);L^q(G)). \tag{2.1}$$

For $\eta \geqslant 0$, $p \in [1,\infty]$ and for a Banach space $\mathcal{X}$, we set

$$L^p_\eta([0,\infty);\mathcal{X}) := \left\{ f \mid (1+t^2)^{\eta/2} f(t) \in L^p([0,\infty);\mathcal{X}) \right\}, \tag{2.2}$$

$$W^{1,p}_\eta((0,\infty);\mathcal{X}) := \left\{ f \mid (1+t^2)^{\eta/2} \partial_t^m f(t) \in L^p([0,\infty);\mathcal{X}) \ \text{ for } \quad m = 0,1 \right\}. \tag{2.3}$$

With the above notation, we define

$$W^{1,2}_{p,q,\eta}((0,\infty);G) := L^p_\eta([0,\infty);W^{2,q}(G)) \cap W^{1,p}_\eta((0,\infty);L^q(G)). \tag{2.4}$$

To state our main results we need some Banach spaces of functions defined on time variable domains.

Definition 2.1. Let $1 < p, q < \infty$, let m be a non negative integer and let $\eta \geqslant 0$. Let $h \in W^{1,\infty}((0,\infty);\mathbb{R}^3)$, $Q \in W^{1,\infty}((0,\infty);\mathbb{M}_{3\times 3}(\mathbb{R}^3))$, let $\mathcal{S}(\cdot)$ and $\mathcal{F}(\cdot)$ be defined by (1.1) and let $\mathcal{F}(t) := \mathbb{R}^3 \backslash \overline{\mathcal{S}(t)}$ for every $t \geqslant 0$. Let $X \in W^{1,\infty}((0,\infty);C^2_b(\mathbb{R}^3))$ be such that for every $t \geqslant 0$ we have that $X(t,\cdot)$ is a C^∞-diffeomorphism from $\mathcal{F}(0)$ onto $\mathcal{F}(t)$. We say that u belongs to $L^p_\eta([0,\infty);W^{m,q}(\mathcal{F}(\cdot)))$ (respectively to $W^{1,p}_\eta((0,\infty);L^q(\mathcal{F}(\cdot))))$ if v defined by $v(t,y) = u(t,X(t,y))$ is in $L^p_\eta([0,\infty);W^{m,q}(\mathcal{F}(0)))$ (respectively in $W^{1,p}_\eta((0,\infty);L^q(\mathcal{F}(0))))$.

Using the above definition we introduce below the concept of solution of (1.1), (1.2) to be used in the remaining part of this work.

Definition 2.2. We say $(\widetilde{u}, \widetilde{\pi}, h, Q, \widetilde{\omega})$ is a global solution of (1.1)–(1.2) if

$$h \in C([0,\infty);\mathbb{R}^3), \quad Q \in W^{1,\infty}((0,\infty);\mathbb{M}_{3\times 3}(\mathbb{R}^3)), \quad \dot{h}, \widetilde{\omega} \in W^{1,p}((0,\infty);\mathbb{R}^3),$$
$$\widetilde{u} \in L^p([0,\infty);W^{2,q}(\mathcal{F}(\cdot))) \cap W^{1,p}((0,\infty);L^q(\mathcal{F}(\cdot))), \quad \widetilde{\pi} \in L^p([0,\infty),\widehat{W}^{1,q}(\mathcal{F}(\cdot))).$$

for some $p, q \in (1,\infty)$, (1.1) holds in a classical sense, equations $(1.2)_{1,2}$ and $(1.2)_{4,5}$ hold in the distribution sense on $(0,\infty) \times \mathcal{F}(\cdot)$, equation $(1.2)_3$ is satisfied in the sense of traces, and the initial conditions in $(1.2)_{6,7}$ hold in a classical sense.

We are now in a position to state our main result.

Theorem 2.3. *Let $p, q \in (1,\infty)$ and $\eta > 0$ be such that*

$$q \in (2,\infty), \quad 1 < \frac{1}{p} + \frac{3}{2q} \leqslant \frac{3}{2}, \quad 1 - \frac{1}{p} < \eta < \frac{3}{2q}. \tag{2.5}$$

We assume that

$$u_0 \in B^{2(1-1/p)}_{q,p}(\mathcal{F}(0))^3 \cap L^{q/2}(\mathcal{F}(0))^3, \quad \ell_0 \in \mathbb{R}^3, \quad \omega_0 \in \mathbb{R}^3, \tag{2.6}$$

satisfy the compatibility conditions

$$\operatorname{div}\ u_0 = 0 \ \text{in}\ \mathcal{F}(0), \quad u_0(x) = \ell_0 + \omega_0 \times x \ \text{for}\ x \in \partial\mathcal{S}(0). \tag{2.7}$$

Then there exists $\varepsilon_0 > 0$ such that for any u_0, ℓ_0, ω_0 satisfying (2.6), (2.7) and

$$\|u_0\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(0))^3} + \|u_0\|_{L^{q/2}(\mathcal{F}(0))^3} + \|\ell_0\|_{\mathbb{R}^3} + \|\omega_0\|_{\mathbb{R}^3} \leqslant \varepsilon_0, \tag{2.8}$$

the system (1.1)–(1.2) admits a strong solution, in the sense of Definition 2.2. Moreover, this solution, denoted by $(\widetilde{u}, \widetilde{\pi}, h, Q, \widetilde{\omega})$, satisfies

$$\begin{aligned}
&\widetilde{u} \in L^p_\eta([0,\infty);W^{2,q}(\mathcal{F}(\cdot))) \cap W^{1,p}_\eta((0,\infty);L^q(\mathcal{F}(\cdot))) \cap C_b([0,\infty);B^{2(1-1/p)}_{q,p}(\mathcal{F}(\cdot))),\\
&\nabla\widetilde{\pi} \in L^p_\eta([0,\infty);L^q(\mathcal{F}(\cdot))),\\
&h \in C([0,\infty);\mathbb{R}^3), \quad Q \in W^{1,\infty}((0,\infty);\mathbb{M}_{3\times 3}(\mathbb{R})),\\
&\dot{h} \in W^{1,p}_\eta((0,\infty);\mathbb{R}^3), \quad \widetilde{\omega} \in W^{1,p}_\eta((0,\infty);\mathbb{R}^3).
\end{aligned} \tag{2.9}$$

Remark 2.4. Theorem 2.3 makes regularity assumptions on the initial data that differ slightly from those in the main result in [4], where rigid body is assumed to be ball. In [4], u_0 was supposed to be in the Kato [11] type space $L^3 \cap L^r$, with $r \in (1, 3/2)$ and to be small in L^3 norm. This is due to the fact that the method of proof in [4] does not appear to be applicable to the case of a solid with a non-spherical shape. It is an interesting open question whether the global existence and uniqueness of the solutions of (1.1)–(1.2) hold for non-spherical solids under the assumptions of [4].

As a consequence of the Theorem 2.3 we obtain large time decay estimates for the velocities of the rigid and of the fluid.

Corollary 2.5. *With the assumptions and notation in Theorem 2.3 we have*

$$\|\widetilde{u}(t,\cdot)\|_{L^q(\mathcal{F}(t))^3} + \left\|\dot{h}(t)\right\|_{\mathbb{R}^3} + \|\widetilde{\omega}(t)\|_{\mathbb{R}^3} \leqslant C(1+t^2)^{-\eta/2} \quad (t \geqslant 0), \tag{2.10}$$

where C is a constant independent of $t > 0$. Furthermore, if $p' > 1$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$ then $\dot{h} \in L^p_\eta([0,\infty);\mathbb{R}^3)$ and $\eta p' > 1$, we have that $h \in L^\infty(0,\infty;\mathbb{R}^3)$. In particular, the position of the centre of the moving rigid body converges to some point at finite distance $h_\infty \in \mathbb{R}^3$ as $t \to \infty$.

Remark 2.6. The best known decay estimate for a spherical solid (see [4]) asserts that $\|\dot{h}(t)\|_{\mathbb{R}^3} = \mathcal{O}(t^{-3/2+\varepsilon})$ when $t \to \infty$. Establishing a similar estimate in the case of non spherical shape is another interesting open question.

3. Change of Coordinates

One of the difficulties in the study of the system (1.2) is that the Navier–Stokes equations hold in the non cylindrical (and a priori unknown) domain

$$\mathcal{Q} = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in [0,\infty)\times\mathbb{R}^3 \ \ | \ \ t>0 \text{ and } x\in\mathcal{F}(t) \right\},$$

where $\mathcal{F}(t) := \mathbb{R}^3\backslash\overline{\mathcal{S}(t)}$, with $\mathcal{S}(t)$ defined in (1.1). A way to overcome this difficulty is to perform a change of coordinates defined by the rigid transformation mapping, at each instant $t>0$, the set $\mathcal{S}(t)$ onto $\mathcal{S}(0)$. One of the terms in the transformed equations obtained by this natural change of variables involves a coefficient which is unbounded when the norm of the spatial variable tends to infinity. This fact raises several difficulties which we are unable to tackle in the context of the present work. We refers, for instance, to Hishida [8,9] and references therein for a discussion of this methodology when the motion of the solid is a prescribed one. This is why in this work we rewrite the system (1.2) in the cylindrical domain $(0,\infty)\times\mathcal{F}(0)$, by using the diffeomorphism from the reference configuration $\mathcal{F}(0)$ onto $\mathcal{F}(t)$ which has been proposed in Cumsille and Takahashi [1]. This change of variables has the advantage of providing a system equivalent to (1.2) where the involved PDE is written in the cylindrical domain $(0,\infty)\times\mathcal{F}(0)$, without introducing coefficients which blow up when the norm of the spatial variable tends to infinity.

To attain this aim, we begin by denoting

$$E := \mathcal{F}(0) = \mathbb{R}^3\backslash\mathcal{S}(0), \quad \mathcal{O} := \mathcal{S}(0). \tag{3.1}$$

Let $R>0$ and $h,\widetilde{\omega} : [0,\infty)\to\mathbb{R}^3$ be such that

$$\begin{aligned} &\operatorname{diam}(\mathcal{O}) + \|h\|_{L^\infty([0,\infty);\mathbb{R}^3)} < R, \\ &\widetilde{\omega}\in L^\infty([0,\infty);\mathbb{R}^3). \end{aligned} \tag{3.2}$$

It is easy to see that

$$\mathcal{S}(t)\subset B_R \text{ for all } t\geqslant 0,$$

where B_R is the open ball of radius R and centered at origin. Let $\psi\in[C_0^\infty(B_{2R})]^3$ be a cut-off function such that $\psi=1$ on $\overline{B_R}$. We introduce a function ζ defined in $[0,\infty)\times\mathbb{R}^3$ by

$$\zeta(t,x) = \dot{h}(t)\times(x-h(t)) + \frac{|x-h(t)|^2}{2}\widetilde{\omega}(t) \quad (t\geqslant 0,\ x\in\mathbb{R}^3),$$

and $\Lambda : [0,\infty)\times\mathbb{R}^3\to\mathbb{R}^3$ defined, for every $t\geqslant 0$ and $x\in\mathbb{R}^3$, by

$$\Lambda(t,x) = \psi(x)\left(\dot{h}(t)+\widetilde{\omega}(t)\times(x-h(t))\right) + \begin{bmatrix} \dfrac{\partial\psi(x)}{\partial x_2}\zeta_3(t,x) - \dfrac{\partial\psi(x)}{\partial x_3}\zeta_2(t,x) \\ \dfrac{\partial\psi(x)}{\partial x_3}\zeta_1(t,x) - \dfrac{\partial\psi(x)}{\partial x_1}\zeta_3(t,x) \\ \dfrac{\partial\psi(x)}{\partial x_1}\zeta_2(t,x) - \dfrac{\partial\psi(x)}{\partial x_2}\zeta_1(t,x) \end{bmatrix}. \tag{3.3}$$

We have the following result, see Cumsille and Takahashi [1]:

Lemma 3.1. *Assume that $\dot{h},\widetilde{\omega}\in W_\eta^{1,p}(0,\infty)$ and Let Λ be defined by (3.3). Then we have*

(1) $\Lambda=0$ *outside* B_{2R}.
(2) $\operatorname{div}\Lambda(t,x)=0$ *in* $[0,\infty)\times\mathbb{R}^3$.
(3) $\Lambda(t,x)=\dot{h}(t)+\widetilde{\omega}(t)\times(x-h(t))$ *for all* $t\in[0,\infty)$ *and* $x\in\mathcal{S}(t)$.
(4) Λ *is continuous from* $[0,\infty)\times\mathbb{R}^3$ *to* $\mathbb{R}^3$.
(5) *For every* $t\geqslant 0$ *the map* $x\mapsto\Lambda(t,x)$ *lies in* $C_0^\infty(\mathbb{R}^3)$.

Let X be the flow associated to the vector Λ, defined by:

$$\begin{cases} \partial_t X(t,y) = \Lambda(t, X(t,y)) \quad (t>0), \\ X(0,y) = y \in \mathbb{R}^3. \end{cases} \tag{3.4}$$

The following result was proved in [1, Lemma 2.2].

Lemma 3.2. *For every $y \in \mathbb{R}^3$ the initial value problem (3.4) admits a unique solution $X(\cdot, y) : [0,\infty) \mapsto \mathbb{R}^3$, which is a C^1 function in $[0,\infty)$. Furthermore, recalling that E and $\mathcal{O}$ have been defined in (3.1), we have*

(1) *For every $t \geqslant 0$, the mapping $y \mapsto X(t,y)$ is a C^∞-diffeomorphism from $\mathbb{R}^3$ onto itself and from E onto $\mathcal{F}(t)$.*
(2) *For every $t \geqslant 0$, we denote by $Y(t,\cdot) = [X(t,\cdot)]^{-1}$, the inverse of $X(t,\cdot)$. Then for every $x \in \mathbb{R}^3$ the mapping $t \mapsto Y(t,x)$ is a C^1 function on $[0,\infty)$.*
(3) *For every $t > 0$, $X(t,\mathcal{O}) = \mathcal{S}(t)$ (Thus $Y(t,\mathcal{S}(t)) = \mathcal{O}$.).*
(4) *For every $t \geqslant 0$, we have that $X(t,y) = y$ for every $y \in \mathbb{R}^3 \backslash B_{2R}$. Moreover, $Y(t,x) = x$ for every $x \in \mathbb{R}^3 \backslash B_{2R}$.*
(5) *For every $t \geqslant 0$ and $y \in \mathbb{R}^3$ we have that* $\det(\nabla X(t,y)) = 1$.

We consider the change of coordinates and unknown functions defined by

$$u(t,y) = \operatorname{Cof}\left(\nabla X^\top(t,y)\right)\right) \widetilde{u}(t, X(t,y)), \quad \pi(t,y) = \widetilde{\pi}(t, X(t,y)) \quad (t \geqslant 0, y \in \mathcal{F}), \tag{3.5}$$

$$\ell(t) = Q^{-1}(t)\dot{h}(t), \quad \omega(t) = Q^{-1}(t)\widetilde{\omega}(t) \quad t \geqslant 0, \tag{3.6}$$

where, given a square matrix B, the notation Cof B stands for the cofactor matrix of B and $B^\top$ designs the transposed of the matrix B. According to [1,12], using the above change of variables and denoting

$$a := \operatorname{Cof}(\nabla Y)^\top, \quad b := \operatorname{Cof}(\nabla X)^\top \tag{3.7}$$

so that

$$u(t,y) = b(t,y)\widetilde{u}(t, X(t,y)), \quad \widetilde{u}(t,x) = a(t,x)u(t, Y(t,x)) \quad (t \geqslant 0, x, y \in \mathbb{R}^3), \tag{3.8}$$

the system (1.1)–(1.2) can be rewritten

$$\begin{cases} \partial_t u - \operatorname{div} \sigma(u,\pi) = \mathfrak{F}(u,\pi,\ell,\omega) & (t>0, y \in E), \\ \operatorname{div} u = 0 & t>0, y \in E, \\ u(t,y) = \ell(t) + \omega(t) \times y & (t>0, y \in \partial\mathcal{O}), \\ m\dot{\ell} = -\displaystyle\int_{\partial\mathcal{O}} \sigma(u,\pi)\nu \, \mathrm{d}s + \mathcal{G}_1(\ell,\omega) & (t>0), \\ J(0)\dot{\omega} = -\displaystyle\int_{\partial\mathcal{O}} y \times \sigma(u,\pi)\nu \, \mathrm{d}s + \mathcal{G}_2(\ell,\omega) & (t>0), \\ v(0,y) = u_0(y) & (y \in E), \\ \ell(0) = \ell_0, \quad \omega(0) = \omega_0, \end{cases} \tag{3.9}$$

$$\begin{cases} \dot{Q}(t)y = Q\omega \times y & (t \geqslant 0, \ y \in \mathbb{R}^3), \\ \dot{h}(t) = Q(t)\ell(t) & (t \geqslant 0), \\ Q(0) = \mathbb{I}_3, \ h(0) = 0, \end{cases} \tag{3.10}$$

with $\mathcal{F}, \mathcal{G}_1$ and $\mathcal{G}_2$ in (3.9) given by

$$\begin{aligned}\mathcal{F}_\alpha(u,\pi,\ell,\omega) &= \nu\sum_{i,j,k} b_{\alpha i}\frac{\partial^2 a_{ik}}{\partial x_j^2}(X)u_k + 2\nu\sum_{i,j,k,m} b_{\alpha i}\frac{\partial a_{ik}}{\partial x_j}(X)\frac{\partial u_k}{\partial y_m}\frac{\partial Y_m}{\partial x_j}(X)\\ &+\nu\sum_{i,j,k}\frac{\partial^2 u_\alpha}{\partial y_j\partial y_k}\left(\frac{\partial Y_j}{\partial x_i}(X)\frac{\partial Y_k}{\partial x_i}(X)-\delta_{j,i}\delta_{k,i}\right)+\nu\sum_{i,j}\frac{\partial u_\alpha}{\partial y_j}\frac{\partial^2 Y_j}{\partial x_i^2}(X)\\ &-\sum_{i,j}\frac{\partial \pi}{\partial y_i}\left(\frac{\partial Y_\alpha}{\partial x_j}(X)\frac{\partial Y_i}{\partial x_j}(X)-\delta_{\alpha,j}\delta_{i,j}\right)\\ &-\sum_{i,j,k,m} b_{\alpha i}\frac{\partial a_{ik}}{\partial x_j}(X)a_{jm}(X)u_k u_m - [(u\cdot\nabla)u]_\alpha\\ &-[b(\partial_t a)(X)u]_\alpha - [(\nabla u)(\partial_t Y)(X)]_\alpha, \quad \alpha\in\{1,2,3\},\end{aligned} \tag{3.11}$$

$$\mathcal{G}_1(\ell,\omega) = -m(\omega\times\ell), \quad \mathcal{G}_2(\ell,\omega) = J(0)\omega\times\omega. \tag{3.12}$$

Definition 3.3. We say (u,π,ℓ,ω,h,Q) is a global solution to the system (3.9)–(3.12), if

$$u\in L^p((0,\infty);W^{2,q}(E))\cap W^{1,p}((0,\infty);L^q(E)), \quad \pi\in L^p((0,\infty);\widehat{W}^{1,q}(E)),$$
$$\ell,\omega\in W^{1,p}((0,\infty);\mathbb{R}^3), \quad h\in W^{1,\infty}((0,\infty);\mathbb{R}^3), \quad Q\in W^{1,\infty}((0,\infty);\mathbb{M}_{3\times 3}(\mathbb{R}^3))$$

for some $p,q\in(1,\infty)$, equations $(3.9)_{1,2}$ holds in the sense of distribution in $(0,\tau)\times\mathcal{F}$, equations $(3.9)_{4,5}$ holds in the sense of distribution in $(0,\tau)$, equation $(3.9)_3$ is satisfied the sense of traces, (3.10) holds in classical sense, and the initial conditions in $(3.9)_{6,7}$ are satisfied.

We prove the following existence and uniqueness result for the system (3.9)–(3.12).

Theorem 3.4. *Let $p\in(1,\infty)$ and $\eta>0$ satisfy the assumptions in Theorem 2.3. Then there exists $\varepsilon_0>0$ such that for any u_0, ℓ_0, ω_0 satisfying (2.6), (2.7) and (2.8), the system (3.9)–(3.12) admits a unique solution (u,π,ℓ,ω,h,Q) in the sense of Definition 3.3. Moreover, this solution satisfy*

$$\begin{aligned}&u\in L^p_\eta([0,\infty);W^{2,q}(E))\cap W^{1,p}_\eta((0,\infty);L^q(E))\cap C_b([0,\infty);B^{2(1-1/p)}_{q,p}(E)),\\ &\nabla\pi\in L^p_\eta([0,\infty);L^q(E)),\\ &\ell\in W^{1,p}_\eta(0,\infty;\mathbb{R}^3),\ \omega\in W^{1,p}_\eta(0,\infty;\mathbb{R}^3),\\ &h\in W^{1,\infty}((0,\infty);\mathbb{R}^3), \quad Q\in W^{1,\infty}((0,\infty);\mathbb{M}_{3\times3}(\mathbb{R})).\end{aligned}\tag{3.13}$$

In fact, the equivalence between Theorems 2.3 and 3.4, follows from the following proposition, whose proof is obvious.

Proposition 3.5. *A quintuplet $(\widetilde{u},\widetilde{\pi},h,Q,\widetilde{\omega})$ satisfying (2.9) is a solution of (1.1)–(1.2) in the sense of definition Definition 2.2 if and only if (u,π,ℓ,ω,h,Q), defined by (3.5)–(3.6), satisfies (3.13) and is a solution to (3.9), in the sense of Definition 3.3.*

The remaining part of this work is devoted to the proof of Theorem 3.4. The proof relies on a fixed point theorem and a linearization. The idea is to replace in the above system, the nonlinear terms $\mathcal{F},\mathcal{G}_1$ and $\mathcal{G}_2$ by given source terms f, g_1 and g_2, More precisely, we obtain the following linear system

$$\begin{cases}\partial_t u-\mu\Delta u+\nabla\pi=f, \quad \operatorname{div} u=0 & (t\geqslant 0, y\in E),\\ u=\ell+\omega\times y & (t\geqslant 0, y\in\partial\mathcal{O}),\\ m\dot{\ell}+\displaystyle\int_{\partial\mathcal{O}}\sigma(u,\pi)\nu\,\mathrm{d}s=g_1 & (t\geqslant 0),\\ \mathcal{J}\dot{\omega}+\int_{\partial\mathcal{O}} y\times\sigma(u,\pi)\nu\,\mathrm{d}s=g_2 & (t\geqslant 0),\\ u(0)=u_0 & (y\in E),\\ \ell(0)=\ell_0, \quad \omega(0)=\omega_0. \end{cases}\tag{3.14}$$

We refer to the above equations as the *linearized fluid–structure* system. We can say the considered system is “monolithic”, in the sense that this linearization preserves the coupling between the equations describing the fluid and those describing the solid. In the sections below, we will study the regularity and decay properties of this linear system.

Let us remark that, when the fluid-rigid body system fills a bounded cavity Ω, i.e. $E = \Omega\backslash\overline{\mathcal{O}}$, regularity as well as decay properties of the above linear system were studied in [14,19].

4. Some Background on the Fluid–Structure Semigroup

In this section we introduce the *fluid structure semigroup*, which plays an important role in the remaining part of this work. We first recall, following Ervedoza et al. [4], the definition and some important properties of this semigroup. Note that these properties have been proved in [4] for a solid of arbitrary shape, so that they are directly applicable in the context of the present paper. Nevertheless, in order to tackle the nonlinear problem for arbitrary shape solids, these estimates have to be completed with new ones, see the forthcoming sections. In the second part of this section we state a maximal regularity property for this semigroup, which is an adaptation to the context of this paper of the proof of the maximal regularity property which has been given in Maity and Tucsnak [14] for the similar semigroup in a bounded domain.

Throughout this section we use the notation $\mathcal{O}$ for an open bounded set of $\mathbb{R}^3$ with smooth boundary. Moreover, we denote by ν the unit normal vector on $\partial\mathcal{O}$ oriented towards the interior of $\mathcal{O}$. Given $q > 1$ we define the space

$$\mathbb{X}^q = \left\{\Phi \in L^q_\sigma \mid D(\Phi) = 0 \text{ in } \mathcal{O}\right\}, \tag{4.1}$$

where

$$L^q_\sigma = \overline{\left\{\varphi \in [C_0^\infty(\mathbb{R})]^3 \mid \operatorname{div}\, \varphi = 0\right\}}^{\|\cdot\|_{q,\mathbb{R}^3}},$$

and the strain rate tensor field $D(\varphi)$ is defined by

$$D(\varphi)_{ij} = \frac{1}{2}\left(\frac{\partial\varphi_i}{\partial x_j} + \frac{\partial\varphi_j}{\partial x_i}\right) \quad \left(\varphi \in \left[W^{1,q}(\mathbb{R}^3)\right]^3, i,\ j \in \{1,2,3\}\right). \tag{4.2}$$

Note that, for every $q \in (1,\infty)$ the dual $(\mathbb{X}^q)^*$ of $\mathbb{X}^q$ can be identified with $\mathbb{X}^{q'}$, where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, with the duality pairing

$$\langle f, g\rangle_{\mathbb{X}^{q'},\mathbb{X}^q} = \int_{\mathcal{O}} \rho f\cdot g \,\mathrm{d}x + \int_E f\cdot g\,\mathrm{d}x \quad (f \in \mathbb{X}^{q'},\ g \in \mathbb{X}^q),$$

where $\rho > 0$ is a constant (standing for the density of the rigid body). Since every Φ in $\mathbb{X}^q$ satisfies $D(\Phi) = 0$ in $\mathcal{O}$, there exists a unique couple $\begin{bmatrix}\ell\\ \omega\end{bmatrix} \in \mathbb{C}^3\times\mathbb{C}^3$ such that

$$\Phi(y) = \varphi(y)\mathbb{1}_E(y) + (\ell + \omega\times y)\mathbb{1}_{\mathcal{O}}(y) \quad (y \in \mathbb{R}^3),$$

where $\mathbb{1}_U$ stands for the characteristic function of the set U (see, for instance, Temam [21, Lemma 1.1]). Denoting $E = \mathbb{R}^3\backslash\overline{\mathcal{O}}$ we can thus use the identification:

$$\mathbb{X}^q \simeq \left\{\begin{bmatrix}\varphi\\ \ell\\ \omega\end{bmatrix} \in [L^q(E)]^3\times\mathbb{C}^3\times\mathbb{C}^3,\ \text{with } \operatorname{div}(\varphi) = 0 \text{ in } E,\right.$$
$$\left.\varphi(y)\cdot\nu(y) = (\ell + \omega\times y)\cdot\nu(y) \text{ for } y \in \partial\mathcal{O}\right\}. \tag{4.3}$$

We recall from [4, Section 3] that the *fluid–structure operator* $\mathbb{A}_q : \mathcal{D}(\mathbb{A}_q) \to \mathbb{X}^q$ is defined, for every $q > 1$, by

$$\mathcal{D}(\mathbb{A}_q) = \left\{ \varphi \in \left[W^{1,q}(\mathbb{R}^3)\right]^3 \cap \mathbb{X}^q \;\middle|\; \varphi_{|E} \in \left[W^{2,q}(E)\right]^3 \right\}, \tag{4.4}$$

$$\mathbb{A}_q\varphi = \mathbb{P}_q\mathcal{A}_q\varphi \quad (\varphi \in \mathcal{D}(\mathbb{A}_q)), \tag{4.5}$$

where $\mathbb{P}_q$ is the projection operator from $\left[L^q(\mathbb{R}^3)\right]^3$ onto $\mathbb{X}^q$ and $\mathcal{A}_q : \mathcal{D}(\mathcal{A}_q) \to \left[L^q(\Omega)\right]^3$ is defined by $\mathcal{D}(\mathcal{A}_q) = \mathcal{D}(\mathbb{A}_q)$ and for every $\varphi \in \mathcal{D}(\mathcal{A}_q)$,

$$(\mathcal{A}_q\varphi) = \begin{cases} \mu\Delta\varphi & \text{in } E, \\ -2\mu m^{-1}\displaystyle\int_{\partial\mathcal{O}} D(\varphi)\nu \,\mathrm{d}\gamma - \left(2\mu\mathcal{J}^{-1}\displaystyle\int_{\partial\mathcal{O}} y \times D(\varphi)\nu \,\mathrm{d}\gamma\right) \times y & \text{in } \mathcal{O}. \end{cases} \tag{4.6}$$

The mass of the rigid body m and its inertia tensor $\mathcal{J}$ appearing in the above equations are defined in terms of the constant density ρ of the solid by

$$m = \int_{\mathcal{O}} \rho \,\mathrm{d}y, \quad \mathcal{J} = (\mathcal{J}_{k,\ell})_{k,\ell\in\{1,2,3\}} \text{ with } \mathcal{J}_{k,\ell} = \int_{\mathcal{O}} \rho \left(\delta_{k,\ell}|y|^2 - y_k y_\ell\right) \mathrm{d}x. \tag{4.7}$$

Recalling the function space defined in (2.1) and by slightly adapting the methodology used in [19,20], we can obtain the following equivalence:

Lemma 4.1. *Let $1 < p, q < \infty$, and let $\tau \leqslant \infty$. Assume*

$$u \in W^{1,2}_{p,q}((0,\tau); E), \quad \pi \in L^p((0,\tau); \widehat{W}^{1,q}(E)), \quad \begin{bmatrix}\ell\\ \omega\end{bmatrix} \in W^{1,p}((0,\tau);\mathbb{R}^6), \tag{4.8}$$

is a solution of (3.14), then

$$\dot{U}(t) = \mathbb{A}_q U(t) + F(t), \quad U(0) = U_0, \tag{4.9}$$

where

$$\begin{cases} U(t,y) = u(t,y)\mathbb{1}_E(y) + (\ell(t) + \omega(t) \times y)\mathbb{1}_{\mathcal{O}}(y) & (t \in [0,\tau),\ y \in \mathbb{R}^3), \\ F(t,y) = \mathbb{P}_q\left(f(t,y)\mathbb{1}_E(y) + (m^{-1}g_1(t) + \mathcal{J}^{-1})\mathbb{1}_{\mathcal{O}}(y)\right) & (t \in [0,\tau),\ y \in \mathbb{R}^3), \\ U_0(y) = u_0(y)\mathbb{1}_E(y) + (\ell_0 + \omega_0 \times y)\mathbb{1}_{\mathcal{O}}(y) & (y \in \mathbb{R}^3). \end{cases} \tag{4.10}$$

Conversely, assume that $U \in L^p([0,\tau); \mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}((0,\tau); \mathbb{X}^q)$ satisfy (4.9). Then there exists $\pi \in L^p((0,\tau); \widehat{W}^{1,q}(E))$ such that (u,π,ℓ,ω) satisfies the system (3.14), where

$$\begin{cases} u = U|_E, \quad \ell = \frac{1}{m}\displaystyle\int_{\mathcal{O}} U \,\mathrm{d}y, \quad \omega = -\mathcal{J}^{-1}\displaystyle\int_{\mathcal{O}} U \times y \,\mathrm{d}y, \\ f = F|_E, \quad g_1 = \frac{1}{m}\displaystyle\int_{\mathcal{O}} F \,\mathrm{d}y, \quad g_2 = -\mathcal{J}^{-1}\displaystyle\int_{\mathcal{O}} F \times y \,\mathrm{d}y, \\ u_0 = U_0|_E, \quad \ell_0 = \frac{1}{m}\displaystyle\int_{\mathcal{O}} U_0 \,\mathrm{d}y, \quad \omega_0 = -\mathcal{J}^{-1}\displaystyle\int_{\mathcal{O}} U_0 \times y \,\mathrm{d}y. \end{cases} \tag{4.11}$$

We describe next some properties the operator $\mathbb{A}_q$ which will be essential in the remaining part of this work.

Theorem 4.2. *For every $1 < q < \infty$ and $\theta \in \left(\frac{\pi}{2}, \pi\right)$ there exists $M_{q,\theta} > 0$ such that the operator $\mathbb{A}_q$ satisfies*

$$\left\|\lambda(\lambda I - \mathbb{A}_q)^{-1}\right\|_{\mathcal{L}(\mathbb{X}^q)} \leqslant M_{q,\theta} \quad (\lambda \in \Sigma_\theta). \tag{4.12}$$

Consequently, $\mathbb{A}_q$ generates a bounded analytic semigroup $\mathbb{T}^q = (\mathbb{T}^q_t)_{t\geqslant 0}$ on $\mathbb{X}^q$.

For the proof of the above theorem we refer to [4, Theorem 6.1]. In the remaining part of this work $\mathbb{T}^q$ will be designed as *the fluid–structure semigroup*. A second important property of $\mathbb{A}_q$ and of the generated semigroup is given in the result below.

Theorem 4.3. *Let $1 < p, q < \infty$. Then $\mathbb{A}_q$ has the maximal L^p regularity property, which means that for every $\tau > 0$ the maps*

$$F \mapsto \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathbb{T}^q_{t-s} F(s)\,\mathrm{d}s \quad (t \in [0, \tau],\ F \in L^p([0, \tau]; \mathbb{X}^q),$$
$$F \mapsto \mathbb{A}_q \int_0^t \mathbb{T}^q_{t-s} F(s)\,\mathrm{d}s \quad (t \in [0, \tau], F \in L^p([0, \tau]; \mathbb{X}^q),$$

are bounded from $L^p([0, \tau]; \mathbb{X}^q)$ into $L^p([0, \tau]; \mathbb{X}^q)$.

Proof. For $F \in L^p([0, \tau]; \mathbb{X}^q)$ we denote

$$f = F|_E, \quad g_1 = \frac{1}{m} \int_{\mathcal{O}} F\,\mathrm{d}y, \quad g_2 = -\mathcal{J}^{-1} \int_{\mathcal{O}} F \times y\,\mathrm{d}y.$$

Then $\begin{bmatrix} f \\ g_1 \\ g_2 \end{bmatrix} \in L^p([0, \tau]; [L^q(E)]^3 \times \mathbb{R}^3 \times \mathbb{R}^3)$ and, according to [7, Theorem 4.1], there exists a unique triple $\begin{bmatrix} u \\ l \\ \omega \end{bmatrix}$ with

$$u \in W^{1,p}((0, \tau); [L^q(E)]^3) \cap L^p([0, \tau]; \left[W^{2,q}(E)\right]^3), \tag{4.13}$$
$$\pi \in L^p([0, \tau]; \widehat{W}^{1,q}(E)), \ \begin{bmatrix} \ell \\ \omega \end{bmatrix} \in W^{1,p}((0, \tau); \mathbb{R}^6), \tag{4.14}$$

satisfying the system (3.14) with $(u_0, \ell_0, \omega_0) = (0, 0, 0)$.

We set

$$U(t, x) = u(t, x)\mathbb{1}_E(x) + (\ell(t) + \omega(t) \times x)\mathbb{1}_{\mathcal{O}}(x) \quad (t \in [0, \tau],\ x \in \mathbb{R}^3).$$

From (4.13)–(4.14) it follows that

$$U \in L^p([0, \tau]; \mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}((0, \tau); \mathbb{X}^q)),$$

and, according to Lemma 4.1, U solves (4.9) with $U_0 = 0$.

Since the unique solution of (4.9) with $U_0 = 0$ is

$$U(t) = \int_0^t \mathbb{T}^q_{t-s} F(s)\,\mathrm{d}s \quad (t \in [0, \tau], F \in L^p([0, \tau]; \mathbb{X}^q),$$

we obtain the conclusion of the theorem. □

Remark 4.4. Alternatively, one can directly show that the operator $\mathbb{A}_q$ is an $\mathcal{R}$-sectorial operator on $\mathbb{X}^q$ of angle $\theta > \pi/2$. For the fluid–structure semigroup on a bounded spatial domain, $\mathcal{R}$-sectoriality was proved in [13, Theorem 3.11]. By slightly modifying the arguments of [13], it can be shown that the operator $\mathbb{A}_q$ considered here is also an $\mathcal{R}$-sectorial operator. Then according to [23, Theorem 4.3], the operator $\mathbb{A}_q$ has maximal L^p regularity property.

By combining Theorems 4.2, 4.3 and [2, Theorem 2.4] we obtain:

Corollary 4.5. *With the notation and assumptions in Theorem 4.2, for every $\lambda > 0$ the operator $\mathbb{A}_q - \lambda$ generates and exponentially stable C^0 semigroup $\mathbb{T}^{q,\lambda}$ on $\mathbb{X}^q$. Moreover, $\mathbb{A}_q - \lambda$ has the infinite time maximal regularity property, which means that the maps*

$$f \mapsto \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathbb{T}^{q,\lambda}_{t-s} f(s)\,\mathrm{d}s \quad (t \geqslant 0,\ f \in L^p([0, \infty); \mathbb{X}^q),$$
$$f \mapsto \mathbb{A}_q \int_0^t \mathbb{T}^{q,\lambda}_{t-s} f(s)\,\mathrm{d}s \quad (t \geqslant 0, f \in L^p([0, \infty); \mathbb{X}^q),$$

are bounded from $L^p([0,\infty);\mathbb{X}^q)$ *into* $L^p([0,\infty);\mathbb{X}^q)$.

We end this section by recalling the following estimate from [4], which gives a characterization of the norm in $\mathcal{D}(\mathbb{A}_q^m)$.

Proposition 4.6. [4, Proposition 7.3 and 7.4] *Let* $1<q<\infty$.

(1) *For every* $m\in\mathbb{N}$, *if* $U_0\in\mathcal{D}(\mathbb{A}_q^m)$, *then* $U_0|_E\in W^{2\,m,q}(E)$ *and*

$$\|U_0\|_{2m,q,E}\leqslant C\left(\left\|\mathbb{A}_q^m U_0\right\|_{\mathbb{X}^q}+\|U_0\|_{\mathbb{X}^q}\right)\quad (U_0\in\mathcal{D}(\mathbb{A}_q^m)). \tag{4.15}$$

(2) *For any* $m\in\mathbb{N}$, *there exists a positive constant* $C_m>0$ *such that*

$$\left\|\mathbb{A}_q^m U_0\right\|_{\mathbb{X}^q}\leqslant C_m\left(\|U_0\|_{2m,q,E}+\|U_0\|_{\mathbb{X}^q}\right)\quad \left(U_0\in\mathcal{D}(\mathbb{A}_q^m)\right). \tag{4.16}$$

5. Decay Estimates for the Fluid–Structure Semigroup

In this section we continue to use the notation introduced at the beginning of the previous one, namely for the spaces $\mathbb{X}_q$, the fluid–structure semigroup $\mathbb{T}_q$ and its generator $\mathbb{A}_q$. Moreover, we refer to the beginning of Sect. 2 for the definition of the various function spaces and norms appearing in the estimates below.

We begin by recalling that in [4, Theorem 7.1], we have obtained several decay estimates for the fluid–structure semigroup that were sufficient to tackle the nonlinear problem for a rigid body of spherical shape. However, at least within the methodology proposed in this paper, these estimates have to be completed in order to study the case of a rigid body of arbitrary shape. More precisely, we need decay estimates for the second order derivatives with respect to the space variables, the derivative with respect to time and of the associated pressure term.

We first remark that we have the following estimates on the fluid–structure semigroup which follow from classical properties of analytic semigroups, together with Theorem 4.2 and Proposition 4.6.

Proposition 5.1. *Let* $1<q<\infty$. *Then for any* $\tau\in(0,\infty)$, *there exists a constant* $C>0$, *depending on* τ *and* q, *such that*

$$\|\mathbb{T}_t^q U_0\|_{\mathbb{X}^q}\leqslant C\,\|U_0\|_{\mathbb{X}^q}\quad (t>0, U_0\in\mathbb{X}^q), \tag{5.1}$$

$$\left\|\mathbb{A}_q^k\mathbb{T}_t^q U_0\right\|_{\mathbb{X}^q}\leqslant Ct^{-k}\,\|U_0\|_{\mathbb{X}^q}\quad (t>0, k\in\mathbb{N}, U_0\in\mathbb{X}^q), \tag{5.2}$$

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{T}_t^q U_0\right)\right\|_{\mathbb{X}^q}+\|\mathbb{T}_t^q U_0\|_{2,q,E}\leqslant C\left(\|U_0\|_{2,q,E}+\|U_0\|_{\mathbb{X}^q}\right)\quad (t\in[0,\tau], U_0\in\mathcal{D}(\mathbb{A}_q)). \tag{5.3}$$

The main result of this section is the following:

Theorem 5.2. *Let* $1<q\leqslant r<\infty$ *and* $\sigma=\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)$. *Then there exists a constant* $C>0$, *depending on* q *and on* r, *such that:*

$$\|\mathbb{T}_t^q U_0\|_{\mathbb{X}^r}\leqslant Ct^{-\sigma}\,\|U_0\|_{\mathbb{X}^q}\quad (t>0,\ U_0\in\mathbb{X}^q). \tag{5.4}$$

$$\|\nabla\left(\mathbb{T}_t^q U_0\right)\|_{r,E}\leqslant Ct^{-\gamma_1(t,q,r)}\,\|U_0\|_{\mathbb{X}^q}\quad (t>0,\ U_0\in\mathbb{X}^q), \tag{5.5}$$

$$\left\|\nabla^2\left(\mathbb{T}_t U_0\right)\right\|_{r,E}\leqslant Ct^{-\gamma_2(t,q,r)}\,\|U_0\|_{\mathbb{X}^q}\quad (t>0,\ U_0\in\mathbb{X}^q), \tag{5.6}$$

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{T}_t U_0\right)\right\|_{\mathbb{X}^r}\leqslant Ct^{-\sigma-1}\,\|U_0\|_{\mathbb{X}^q}\quad (t>0,\ U_0\in\mathbb{X}^q). \tag{5.7}$$

where

$$\gamma_1(t,q,r)=\begin{cases}\dfrac{1}{2}+\sigma & \text{if } t\leqslant 1,\\ \min\left\{\dfrac{1}{2}+\sigma,\dfrac{3}{2q}\right\} & \text{if } t>1,\end{cases} \tag{5.8}$$

$\nabla^2 f$ *denotes the hessian of the function* f *and*

$$\gamma_2(t,q,r) = \begin{cases} 1+\sigma & \text{if } t \leqslant 1, \\ \min\left\{1+\sigma, \dfrac{3}{2q}\right\} & \text{if } t > 1. \end{cases} \tag{5.9}$$

Moreover, estimate (5.4) also holds for $r=\infty$*, in the sense that there exists a constant* $C>0$*, depending on* r*, such that*

$$\|\mathbb{T}_t^q U_0\|_{\mathbb{X}^\infty} \leqslant C t^{-\frac{3}{2q}} \|U_0\|_{\mathbb{X}^q} \quad (t>0,\ U_0\in\mathbb{X}^q). \tag{5.10}$$

The remaining part of this section is devoted to the proof of Theorem 5.2. We first show that the estimates in Theorem 5.2 hold for small time. More precisely, we have:

Proposition 5.3. *Let* $1<q\leqslant r<\infty$ *and* $\sigma = \frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)$*. Then for each* $\tau\in(0,\infty)$ *there exists a constant* $C>0$*, depending on* τ*,* q *and* r*, such that*

$$\|\mathbb{T}_t^q U_0\|_{\mathbb{X}^r} \leqslant C t^{-\sigma} \|U_0\|_{\mathbb{X}^q} \quad (t\in[0,\tau],\ U_0\in\mathbb{X}^q), \tag{5.11}$$

$$\|\nabla\mathbb{T}_t^q U_0\|_{r,E} \leqslant C t^{-\sigma-\frac{1}{2}} \|U_0\|_{\mathbb{X}^q} \quad (t\in[0,\tau],\ U_0\in\mathbb{X}^q). \tag{5.12}$$

$$\left\|\nabla^2\mathbb{T}_t^q U_0\right\|_{r,E} \leqslant C t^{-\sigma-1} \|U_0\|_{\mathbb{X}^q} \quad (t\in[0,\tau],\ U_0\in\mathbb{X}^q), \tag{5.13}$$

$$\|\mathbb{A}_q\mathbb{T}_t^q U_0\|_{\mathbb{X}^r} \leqslant C t^{-\sigma-1} \|U_0\|_{\mathbb{X}^q} \quad (t\in[0,\tau],\ U_0\in\mathbb{X}^q), \tag{5.14}$$

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{T}_t^q U_0\right)\right\|_{\mathbb{X}^r} \leqslant C t^{-\sigma-1} \|U_0\|_{\mathbb{X}^q} \quad (t\in[0,\tau],\ U_0\in\mathbb{X}^q), \tag{5.15}$$

Proof. Let $N=[2\sigma]$, where $[2\sigma]$ denotes the integer part of 2σ and assume that N is even. Using Proposition 4.6, (5.3) and (5.2), it follows that there exists a constant $C>0$, depending on τ, q and r, such that, recalling from the beginning of Sect. 2 that $\|\cdot\|_{N,q,E}$ stands for the standard norm in $W^{N,q}(E)$,

$$\begin{aligned} &\|\mathbb{T}_t^q U_0\|_{N,q,E} + |\ell(t)| + |\omega(t)| \leqslant C\left(\left\|\mathbb{A}_q^{N/2}\mathbb{T}_t^q U_0\right\|_{\mathbb{X}^q} + \|\mathbb{T}_t^q U_0\|_{\mathbb{X}^q}\right) \\ &\leqslant C\left(t^{-\frac{N}{2}}\|U_0\|_{\mathbb{X}^q} + \tau^{\frac{N}{2}} t^{-\frac{N}{2}}\|U_0\|_{\mathbb{X}^q}\right) \leqslant C t^{-\frac{N}{2}}\|U_0\|_{\mathbb{X}^q} \quad (t\in(0,\tau], U_0\in\mathbb{X}^q). \end{aligned} \tag{5.16}$$

In a similar manner, we obtain that

$$\|\mathbb{T}_t^q U_0\|_{N+4,q,E} + |\ell(t)| + |\omega(t)| \leqslant C t^{-\frac{N+4}{2}} \|U_0\|_{\mathbb{X}^q} \quad (t\in(0,\tau], U_0\in\mathbb{X}^q). \tag{5.17}$$

The above estimates, combined with standard Sobolev embeddings and interpolation inequalities, imply that for every $j\in\{0,1,2\}$ we have

$$\begin{aligned} \left\|\nabla^j\mathbb{T}_t^q U_0\right\|_{\mathbb{X}^r} &\leqslant C\left(\|\mathbb{T}_t^q U_0\|_{j,r,E} + |\ell(t)| + |\omega(t)|\right) \\ &\leqslant C\left(\|\mathbb{T}_t^q U_0\|_{2\sigma+j,q,E} + |\ell(t)| + |\omega(t)|\right) \\ &\leqslant C\left(\|\mathbb{T}_t^q U_0\|_{N+4,q,E}^{\frac{2\sigma+j-N}{4}} \|\mathbb{T}_t^q U_0\|_{N,q,E}^{\frac{N+4-j-2\sigma}{4}} + |\ell(t)| + |\omega(t)|\right) \\ &\leqslant C t^{-\sigma-j/2}\|U_0\|_{\mathbb{X}^q} \quad (t\in(0,\tau], U_0\in\mathbb{X}^q). \end{aligned}$$

This proves the estimates (5.11), (5.12) and (5.13). We next combine (5.11)–(5.13) and Proposition 4.6 to obtain that

$$\|\mathbb{A}_q\mathbb{T}_t^q U_0\|_{\mathbb{X}^r} \leqslant C\left(\|\mathbb{T}_t^q U_0\|_{2,r,E} + \|\mathbb{T}_t^q U_0\|_{\mathbb{X}^r}\right) \leqslant C t^{-\sigma-1}\|U_0\|_{\mathbb{X}^q} \quad (t\in(0,\tau], U_0\in\mathbb{X}^q).$$

This proves (5.14), and the estimate (5.15) also follows. □

We next recall the following local decay estimates from [4, Theorem 7.10]:

Proposition 5.4. *With the notation and assumptions in Theorem 5.2, let $R_0 > 0$ be such that $\overline{\mathcal{O}} \subset B_{R_0}$, $d > R_0 + 5$, $m \in \mathbb{N}$, and $E_d = \{y \in E \mid |y| < d\}$. Then there exists a positive constant C, depending only on E, d, m and q, such that, for every $t \geqslant 0$ and every $U_0 \in \operatorname{Ran}(\mathbb{T}_1^q)$, denoting $U(t) = \mathbb{T}_t^q U_0$, we have*

$$\|u(t,\cdot)\|_{2m,q,E_d} + |\ell(t)| + |\omega(t)| \leqslant C(1+t)^{-\frac{3}{2q}} \left(\|u_0\|_{[3/q]+2m+2,q,E} + |\ell_0| + |\omega_0|\right), \tag{5.18}$$

$$\|\partial_t u(t,\cdot)\|_{2m,q,E_d} + |\dot{\ell}(t)| + |\dot{\omega}(t)| \leqslant C(1+t)^{-\frac{3}{2q}} \left(\|u_0\|_{[3/q]+2m+4,q,E} + |\ell_0| + |\omega_0|\right), \tag{5.19}$$

where $[s]$ denotes the integer part of $s \in \mathbb{R}$, u_0, l_0, ω_0 and u, l, ω are defined from U_0, respectively from U, by (4.11). Moreover,

$$\|\pi(t,\cdot)\|_{2m+1,q,E_d} \leqslant C(1+t)^{-\frac{3}{2q}} \left(\|u_0\|_{[3/q]+2m+4,q,E} + |\ell_0| + |\omega_0|\right), \tag{5.20}$$

where π is the pressure field associated to (u, ℓ, ω) via the system (3.14) with $(f, g_1, g_2) = (0, 0, 0)$.

The result below provides decay estimates for the restriction of $u(t,\cdot)$ to the exterior of the bounded set E_d.

Proposition 5.5. *With the notation and assumptions in Theorem 5.2 and Proposition 5.4, for every $q \in (1,\infty)$ and $r \in [q,\infty)$ there exists a positive constant C, depending only on E, d and q, such that*

$$\|u(t,\cdot)\|_{r,\{|y|>d\}} \leqslant C(1+t)^{-\sigma} \left(\|u_0\|_{[3/q]+[2\sigma]+7,q,E} + |\ell_0| + |\omega_0|\right), \tag{5.21}$$

$$\|\nabla u(t,\cdot)\|_{r,\{|y|>d\}} \leqslant C(1+t)^{-\min\{(1/2+\sigma),3/2q\}} \left(\|u_0\|_{[3/q]+[2\sigma]+9,q,E} + |\ell_0| + |\omega_0|\right), \tag{5.22}$$

$$\left\|\nabla^2 u(t,\cdot)\right\|_{r,\{|y|>d\}} \leqslant C(1+t)^{-\min\{(1+\sigma),3/2q\}} \left(\|u_0\|_{[3/q]+[2\sigma]+10,q,E} + |\ell_0| + |\omega_0|\right), \tag{5.23}$$

for every $t \geqslant 0$ and $U_0 \in \operatorname{Ran} \mathbb{T}_1^q$.

Proof. The proof of the estimate (5.21) follows from [4, Proposition 7.11]. We focus on the remaining two estimates.

Let $\chi \in C^\infty(\mathbb{R}^3)$ be such that $\chi(y) = 1$ for $|y| > d$ and $\chi(y) = 0$ for $|y| < d-1$. It follows that for every $t \geqslant 0$ we have that $\operatorname{supp} \operatorname{div}(\chi u(t,\cdot)) \subset \{d-1 < |y| < d\}$. Then there exists $v_3(t,\cdot)$ such that $\operatorname{div} v_3 = \operatorname{div}(\chi u)$, $\operatorname{supp} v_3(t,\cdot) \subset \{d-1 < |y| < d\}$ and for every $m \in \mathbb{N}$, we have

$$\|v_3(t,\cdot)\|_{m,q} \leqslant C(1+t)^{-\frac{3}{2q}} \left(\|u_0\|_{[3/q]+m+2,q,E} + |\ell_0| + |\omega_0|\right), \tag{5.24}$$

$$\|\partial_t v_3(t,\cdot)\|_{m,q} \leqslant C(1+t)^{-\frac{3}{2q}} \left(\|u_0\|_{[3/q]+m+4,q,E} + |\ell_0| + |\omega_0|\right), \tag{5.25}$$

for some constant $C > 0$ depending on m and q. We refer to [4, Proposition 7.11] for a proof of the existence of v_3 satisfying the above two estimates.

We now define

$$v_4(t,y) = \chi(y)u(t,y) - v_3(t,y) \quad (t > 0,\ y \in \mathbb{R}^3). \tag{5.26}$$

Note that $\operatorname{div} v_4 = 0$ so that v_4 satisfies

$$\begin{cases} \partial_t v_4 - \mu\Delta v_4 + \nabla(\chi\pi) = h, \quad \operatorname{div} v_4 = 0, & (t > 0, y \in \mathbb{R}^3), \\ v_4(0,x) = v_{40}(x) & (y \in \mathbb{R}^3), \end{cases} \tag{5.27}$$

where

$$h = -2(\nabla\chi \cdot \nabla)u(t) - \mu(\Delta\chi)u + \partial_t v_3 - \mu\Delta v_3 + \pi\nabla\chi, \tag{5.28}$$

and

$$v_{40}(y) = \chi(x)U_0(y) - v_3(0,y) \quad (y \in \mathbb{R}^3). \tag{5.29}$$

Moreover, the function $y \mapsto h(t,y)$ is supported in the annulus $\{d-1 \leqslant |y| \leqslant d\}$ and from (5.24), (5.25) we obtain

$$\|v_{40}\|_{m,q} \leqslant C\Big(\|u_0\|_{2m,q,E} + |\ell_0| + |\omega_0|\Big), \quad (m \in \mathbb{N}), \tag{5.30}$$

$$\|h(t)\|_{m,q} \leqslant C(1+t)^{-3/2q}\Big(\|u_0\|_{[3/q]+m+4,q} + |\ell_0| + |\omega_0|\Big) \quad (m \in \mathbb{N}). \tag{5.31}$$

Let $\mathbb{S}$ be the Stokes semigroup in $\mathbb{R}^3$. According to well known estimates of the heat kernel (see, for instance, [10, Lemma 5.1]), for any $m \in \mathbb{Z}_+$, there exists a constant $C_m > 0$, depending on q and r, such that

$$\|\nabla^m \mathbb{S}_t f\|_r \leqslant C_m(1+t)^{-\sigma-m/2}\,\|f\|_{[2\sigma]+m+1,q} \quad (t \geqslant 0). \tag{5.32}$$

With the above notation $v_4(t,\cdot)$ can be written as

$$v_4(t,\cdot) = \mathbb{S}_t v_{40} + \int_0^t \mathbb{S}_{t-s}h(s)\,\mathrm{d}s := v_{41}(t,\cdot) + v_{42}(t,\cdot). \tag{5.33}$$

From (5.32) we have, for $j = 1,2$

$$\left\|\nabla^j v_{41}(t,\cdot)\right\|_r \leqslant C(1+t)^{-(j/2+\sigma)}\,\|v_{40}\|_{[2\sigma]+j+1,q}\,. \tag{5.34}$$

In order to estimate $v_{42}(t,\cdot)$ we consider three cases.

Case 1: $3/2q > 1$. Using (5.32) and (5.31) it follows that

$$\left\|\nabla^j v_{42}(t,\cdot)\right\|_r \leqslant C\Big(\|u_0\|_{[3/q]+[2\sigma]+j+5,q} + |\ell_0| + |\omega_0|\Big)\int_0^t (1+(t-s))^{-(j/2+\sigma)}(1+s)^{-3/2q}\,\mathrm{d}s.$$

Noting that

$$\int_0^{t/2}(1+(t-s))^{-(\frac{j}{2}+\sigma)}(1+s)^{-3/2q}\,\mathrm{d}s \leqslant C(1+t)^{-(\frac{j}{2}+\sigma)}\int_0^{t/2}(1+s)^{-3/2q}\,\mathrm{d}s \leqslant C(1+t)^{-(\frac{j}{2}+\sigma)},$$

and that

$$\begin{aligned}
\int_{t/2}^t (1+(t-s))^{-(\frac{j}{2}+\sigma)}(1+s)^{-3/2q}\,\mathrm{d}s &\leqslant C(1+t)^{-3/2q}\int_{t/2}^t(1+(t-s))^{-(\frac{j}{2}+\sigma)}\,\mathrm{d}s\\
&\leqslant \begin{cases} C(1+t)^{-3/2q} & \text{if } j/2+\sigma > 1,\\ C(1+t)^{-(1/2+\sigma)} & \text{if } j = 1, \sigma \leqslant 1/2,\\ C(1+t)^{-1} & \text{if } j=2, \sigma = 0 (q=r).\end{cases}
\end{aligned}$$

it follows that for $j \in \{1,2\}$ we have

$$\left\|\nabla^2 v_{42}(t,\cdot)\right\|_r \leqslant C(1+t)^{-\min\{\frac{j}{2}+\sigma,3/2q\}}\Big(\|u_0\|_{[3/q]+[2\sigma]+j+5,q} + |\ell_0| + |\omega_0|\Big). \tag{5.35}$$

Case 2: $3/2q \leqslant 1$ and $r > 3/2$ if $j=1$. We take $1 < q_0 < 3/2$, and let $\sigma_0 = 3/2(1/q_0 - 1/r)$. Thus for $j \in \{1,2\}$ we have $j/2 + \sigma_0 > 1$, so that, using (5.32), (5.31) and the fact that h is compactly supported, it follows that

$$\begin{aligned}
\left\|\nabla^j v_{42}(t,\cdot)\right\|_r &\leqslant C\int_0^t (1+(t-s))^{-(j/2+\sigma_0)}\,\|h(s)\|_{[2\sigma_0]+j+1,q_0}\,\mathrm{d}s\\
&\leqslant C\int_0^t (1+(t-s))^{-(j/2+\sigma_0)}\,\|h(s)\|_{[2\sigma_0]+j+1,q}\,\mathrm{d}s\\
&\leqslant C\Big(\|u_0\|_{[3/q]+j+8,q} + |\ell_0| + |\omega_0|\Big)\int_0^t(1+(t-s))^{-(j/2+\sigma_0)}(1+s)^{-3/2q}\,\mathrm{d}s.
\end{aligned}$$

We next note that

$$\int_0^{t/2} (1+(t-s))^{-(j/2+\sigma_0)}(1+s)^{-3/2q}\mathrm{d}s \leqslant C(1+t)^{-(j/2+\sigma_0)}\int_0^{t/2}(1+s)^{-3/2q}$$
$$\leqslant C\begin{cases}(1+t)^{-(j/2+\sigma_0)}\ln(1+t) \leqslant (1+t)^{-(j/2+\sigma)} & \text{if } \frac{3}{2q}=1,\\ (1+t)^{-(j/2+\sigma_0)}(1+t)^{1-3/2q} \leqslant (1+t)^{-(1+\sigma)} & \text{if } \frac{3}{2q}<1,\end{cases}$$

and

$$\int_{t/2}^{t} (1+(t-s))^{-(j/2+\sigma_0)}(1+s)^{-3/2q}\,\mathrm{d}s \leqslant C(1+t)^{-3/2q}.$$

Consequently,

$$\left\|\nabla^j v_{42}(t,\cdot)\right\|_r \leqslant C(1+t)^{-\min\{(j/2+\sigma),3/2q\}}\Big(\|u_0\|_{[3/q]+j+8,q}+|\ell_0|+|\omega_0|\Big) \quad (j\in\{1,2\}). \tag{5.36}$$

Case 3: It remains to consider the case $j=1$ and $q=r=3/2$. We take $q_0<3/2$. Then

$$\begin{aligned}\|\nabla v_{42}(t,\cdot)\|_{3/2} &\leqslant C\int_0^t (1+(t-s))^{-3/2q_0+1/2}\,\|h(s)\|_{2,q_0}\;\mathrm{d}s\\ &\leqslant C\int_0^t (1+(t-s))^{-3/2q_0+1/2}\,\|h(s)\|_{2,q}\;\mathrm{d}s\\ &\leqslant C\Big(\|u_0\|_{[3/q]+6,q}+|\ell_0|+|\omega_0|\Big)\int_0^t(1+(t-s))^{-3/2q_0+1/2}(1+s)^{-1}\;\mathrm{d}s\\ &\leqslant C(1+t)^{-1/2}\Big(\|u_0\|_{[3/q]+6,q}+|\ell_0|+|\omega_0|\Big).\end{aligned}$$

Combining the above estimate with (5.30), (5.31), (5.34), (5.35) and (5.36), we deduce that

$$\left\|\nabla^j v_4(t,\cdot)\right\|_r \leqslant C(1+t)^{-\min\{(j/2+\sigma),3/2q\}}\Big(\|u_0\|_{[3/q]+[2\sigma]+j+8,q}+|\ell_0|+|\omega_0|\Big).$$

We can thus conclude the proof of the proposition by combining the above estimate and (5.24). □

We are now in a position to prove the main result in this section.

Proof of Theorem 5.2. For every $U_0\in\mathbb{X}^q$, we have $\mathbb{T}_1U_0\in\mathcal{D}(\mathbb{A}_q^k)$ for all $k\in\mathbb{N}$. Thus combining (5.12), (5.13), (5.22) and (5.23) we obtain (5.5) and (5.6).

Finally, we apply (5.2) and (5.4) to obtain that for every $t>0$ and $U_0\in X^q$ we have

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{T}_tU_0\right)\right\|_{\mathbb{X}^r} = \|\mathbb{A}_q\mathbb{T}_tU_0\|_{\mathbb{X}^r} \leqslant Ct^{-1}\left\|\mathbb{T}_{t/2}U_0\right\|_{\mathbb{X}^r} \leqslant Ct^{-(1+\sigma)}\|U_0\|_{\mathbb{X}^q}.$$

This proves the estimate (5.7), and the proof the theorem is complete. □

6. Regularity of the Linear Non Homogeneous System

In this section, we study the regularity and the decay properties of the solutions of the non homogeneous linear system

$$\frac{\mathrm{d}U}{\mathrm{d}t}(t)=\mathbb{A}_qU(t)+F(t)\quad t>0,\quad U(0)=U_0. \tag{6.1}$$

We continue here to use the definitions of time weighted Sobolev spaces from Sect. 2 and we recall that the operator $\mathbb{A}_q$ has been introduced in (4.4), (4.5). We essentially study maximal regularity (in infinite time) properties of these system. Let us recall the time weighted spaces from (2.2)–(2.4),

The main result in this section is:

Theorem 6.1. *Let $p, q_0, q \in (1,\infty)$ and $\eta > 0$ satisfy*

$$q_0 < q, \tag{6.2a}$$

$$1 < \eta + \frac{1}{p} < \frac{3}{2}\left(\frac{1}{q_0} - \frac{1}{q}\right). \tag{6.2b}$$

Then for every $U_0 \in \mathbb{X}^{q_0} \cap (\mathbb{X}^q, \mathcal{D}(\mathbb{A}_q))_{1-1/p,p}$ and for every $F \in L^p_\eta([0,\infty); \mathbb{X}^{q_0} \cap \mathbb{X}^q)$, the system (6.1) admits a unique solution $U \in L^p_\eta([0,\infty); \mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}_\eta([0,\infty); \mathbb{X}^q)$. Moreover, there exists a positive constant C_L, (the index L coming from "Linear") depending only on p, q_0, η, E and $\mathcal{O}$ such that

$$\|U\|_{L^p_\eta([0,\infty);\mathcal{D}(\mathbb{A}_q))\cap W^{1,p}_\eta([0,\infty);\mathbb{X}^q)} \leqslant C_L \left(\|U_0\|_{\mathbb{X}^{q_0}\cap(\mathbb{X}^q,\mathcal{D}(\mathbb{A}_q))_{1-1/p,p}} + \|F\|_{L^p_\eta([0,\infty);\mathbb{X}^{q_0}\cap\mathbb{X}^q)}\right). \tag{6.3}$$

The guiding idea in proving the above result is borrowed from Shibata [18] (see also Murata and Shibata [15]). More precisely, we look for a solution to the system (6.1) in the form

$$U = U_1 + U_2, \tag{6.4}$$

where U_1 satisfies

$$\frac{\mathrm{d}U_1}{\mathrm{d}t}(t) = (\mathbb{A}_q - \mathbb{I})U_1(t) + F(t), \quad U_1(0) = U_0, \tag{6.5}$$

and U_2 satisfies

$$\frac{\mathrm{d}U_2}{\mathrm{d}t}(t) = \mathbb{A}_q U_2(t) + U_1(t), \quad U_2(0) = 0. \tag{6.6}$$

From Corollary 4.5, we know that, $\mathbb{A}_q - \mathbb{I}$ has the infinite time maximal L^p regularity property and generates an exponentially stable semigroup. Thus, for any $p, q \in (1,\infty)$ and $\eta \geqslant 0$, and for every $U_0 \in (\mathbb{X}^q, \mathcal{D}(\mathbb{A}_q))_{1-1/p,p}$, $F \in L^p_\eta([0,\infty); \mathbb{X}^q)$, we have that U_1 belongs to $L^p_\eta([0,\infty); \mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}_\eta((0,\infty); \mathbb{X}^q)$ (see Theorem 6.2 below). Using Duhmael's principle we have that

$$U_2(t) = \int_0^t \mathbb{T}_{t-s} U_1(s)\, \mathrm{d}s \quad (t \geqslant 0).$$

In order to estimate U_2 and its derivatives with respect to the space and time variables, we are going to use $L^{q_0} - L^q$, with $q_0 < q$, decay estimates of the fluid–structure semigroup (see Theorem 5.2). To do this, we need to have $U_1 \in L^p_\eta([0,\infty); \mathbb{X}^{q_0})$, which can be ensured by taking $U_0 \in \mathbb{X}^{q_0}$ and $F \in L^p_\eta([0,\infty); \mathbb{X}^{q_0})$.

We pass now to the detailed proof of Theorem 6.1. We first prove a maximal L^p type regularity of the system (6.5).

Theorem 6.2. *Assume $1 < p, q < \infty$, and let $\eta \geqslant 0$. Then for any $U_0 \in (\mathbb{X}^q, \mathcal{D}(\mathbb{A}_q))_{1-1/p,p}$, and for every $F \in L^p_\eta([0,\infty); \mathbb{X}^q)$, the system (6.5) admits a unique strong solution*

$$U_1 \in L^p_\eta([0,\infty); \mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}_\eta(0,\infty); \mathbb{X}^q).$$

Moreover, there exists a constant $C > 0$ such that

$$\|U_1\|_{L^p_\eta([0,\infty);\mathcal{D}(\mathbb{A}_q))\cap W^{1,p}_\eta((0,\infty);\mathbb{X}^q)} \leqslant C\Big(\|U_0\|_{(\mathbb{X}^q,\mathcal{D}(\mathbb{A}_q))_{1-1/p,p}} + \|F\|_{L^p_\eta([0,\infty);\mathbb{X}^q)}\Big). \tag{6.7}$$

Proof. We first note that, since $\eta \geqslant 0$, we have $F \in L^p([0,\infty); \mathbb{X}^q)$. Therefore, by Corollary 4.5 and [13, Theorem 2.7], the system (6.5) admits a unique solution

$$U_1 \in L^p([0,\infty); \mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}((0,\infty); \mathbb{X}^q). \tag{6.8}$$

In particular, this proves the theorem for $\eta = 0$. Let us now assume that $\eta \in (0,1]$ and set

$$V(t) = (1+t^2)^{\eta/2} U_1(t) \quad (t \geqslant 0).$$

Then V satisfies

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) = (\mathbb{A}_q - \mathbb{I})V(t) + G(t), \quad V(0) = U_0, \tag{6.9}$$

where

$$G(t) = (1+t^2)^{\eta/2}F(t) - \left(\frac{\mathrm{d}}{\mathrm{d}t}(1+t^2)^{\eta/2}\right)U_1(t).$$

Therefore, using the hypothesis of the theorem, (6.8) and the fact that $0 < \eta \leqslant 1$, we obtain $G \in L^p([0,\infty);\mathbb{X}^q)$. Therefore,

$$(1+t^2)^{\eta/2}U_1 = V \in L^p([0,\infty);\mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}((0,\infty);\mathbb{X}^q). \tag{6.10}$$

The result for $\eta > 1$ can be obtained by repeatedly using the above argument. This completes the proof of the theorem. □

As a consequence of the above theorem, we have the following result.

Theorem 6.3. *With the assumptions in Theorem 6.1, for every $U_0 \in \mathbb{X}^{q_0} \cap (\mathbb{X}^q, \mathcal{D}(\mathbb{A}_q))_{1-1/p,p}$ and for every $F \in L^p_\eta([0,\infty);\mathbb{X}^{q_0} \cap \mathbb{X}^q)$ the system (6.5) admits a unique strong solution*

$$U_1 \in L^p_\eta([0,\infty);\mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}_\eta((0,\infty);\mathbb{X}^q) \cap L^p_\eta([0,\infty);\mathbb{X}^{q_0}).$$

Moreover, there exists a constant $C > 0$ (possibly depending on η) such that

$$\begin{aligned}&\|U_1\|_{L^p_\eta([0,\infty);\mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}_\eta((0,\infty);\mathbb{X}^q) \cap L^p_\eta([0,\infty);\mathbb{X}^{q_0})}\\ &\quad\leqslant C\Big(\|U_0\|_{\mathbb{X}^{q_0} \cap (\mathbb{X}^q,\mathcal{D}(\mathbb{A}_q))_{1-1/p,p}} + \|F\|_{L^p_\eta([0,\infty);\mathbb{X}^q\cap\mathbb{X}^{q_0})}\Big),\end{aligned} \tag{6.11}$$

for every $U_0 \in \mathbb{X}^{q_0} \cap (\mathbb{X}^q, \mathcal{D}(\mathbb{A}_q))_{1-1/p,p}$ and $F \in L^p_\eta([0,\infty);\mathbb{X}^q \cap \mathbb{X}^{q_0})$.

Proof. It only remains to show $U_1 \in L^p_\eta([0,\infty);\mathbb{X}^{q_0})$. We note that $U_1 = U_{11} + U_{12}$, with

$$\frac{\mathrm{d}U_{11}}{\mathrm{d}t}(t) = (\mathbb{A}_q - \mathbb{I})U_{11}(t), \quad U_1(0) = U_0,$$

and

$$\frac{\mathrm{d}U_{12}}{\mathrm{d}t}(t) = (\mathbb{A}_q - \mathbb{I})U_{12}(t) + F(t), \quad U_{12}(0) = 0.$$

Recall that for every $q > 1$ the operator $\mathbb{A}_q - \mathbb{I}$ generates an exponentially stable semigroup on $\mathbb{X}^q$ (see Corollary 4.5). Thus there exists $C > 0$ and $\lambda_1 > 0$ such that

$$\|U_{11}(t)\|_{\mathbb{X}^{q_0}} \leqslant Ce^{-\lambda_1 t}\|U_0\|_{\mathbb{X}^{q_0}} \quad (t > 0, U_0 \in \mathbb{X}^{q_0}).$$

Thus we clearly have $U_{11} \in L^p_\eta([0,\infty);\mathbb{X}^{q_0})$. On the other hand, since $F \in L^p_\eta([0,\infty);\mathbb{X}^{q_0})$, we can argue similarly to the proof of Theorem 6.2 to obtain that $(1+t^2)^{\eta/2}U_{12} \in L^p([0,\infty);\mathcal{D}(\mathbb{A}_{q_0})) \cap W^{1,p}((0,\infty);\mathbb{X}^{q_0})$. In particular, $U_{12} \in L^p_\eta([0,\infty);\mathbb{X}^{q_0})$. This completes the proof of the theorem. □

We are now going to show that, under the assumptions in Theorem 6.1, the solution U_2 of the system (6.6) lies in $L^p_\eta([0,\infty);\mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}_\eta([0,\infty);\mathbb{X}^q)$.

Consequently, the conclusion of Theorem 6.1 follows from Theorems 6.3 and 6.4 below.

Theorem 6.4. *With the assumptions in Theorem 6.1, let U_1 be the solution constructed in Theorem 6.3. Then the solution U_2 of (6.6) belongs to $L^p_\eta([0,\infty);\mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}_\eta([0,\infty);\mathbb{X}^q)$. Moreover, there exists a constant $C > 0$ such that*

$$\|U_2\|_{L^p_\eta([0,\infty);\mathcal{D}(\mathbb{A}_q))\cap W^{1,p}_\eta([0,\infty);\mathbb{X}^q)} \leqslant C\Big(\|U_0\|_{\mathbb{X}^{q_0}\cap(\mathbb{X}^q,\mathcal{D}(\mathbb{A}_q))_{1-1/p,p}} + \|F\|_{L^p_\eta([0,\infty);\mathbb{X}^q\cap\mathbb{X}^{q_0})}\Big). \tag{6.12}$$

Proof. To simplify the notation and when there is no risk of confusion, the fluid–structure semigroup will be simply denoted by $\mathbb{T}$. In what follows, we will set

$$\mathcal{J} = \|U_0\|_{\mathbb{X}^{q_0}\cap(\mathbb{X}^q,\mathcal{D}(\mathbb{A}_q))_{1-1/p,p}} + \|F\|_{L^p_\eta([0,\infty);\mathbb{X}^q\cap\mathbb{X}^{q_0})}. \tag{6.13}$$

The constants C appearing in this proof depend only on p, q, q_0, η and $\mathcal{O}$, and may change from line to line. In some cases it will be precisely stated that C also depends on a fixed time τ. The proof is divided into several parts.

Step 1: Reformulation of the problem

By virtue of Duhamael's principle, U_2 can be written as

$$U_2(t) = \int_0^t \mathbb{T}_{t-s} U_1(s)\ \mathrm{d}s \quad (t \geqslant 0). \tag{6.14}$$

Moreover, for $t > 2$, we express U_2 as

$$\begin{aligned} U_2(t) = \int_0^{t/2} \mathbb{T}_{t-s} U_1(s)\ \mathrm{d}s + \int_{t/2}^{t-1} \mathbb{T}_{t-s} U_1(s)\ \mathrm{d}s + \int_{t-1}^{t} \mathbb{T}_{t-s} U_1(s)\ \mathrm{d}s \\ := U_{21}(t) + U_{22}(t) + U_{23}(t). \end{aligned} \tag{6.15}$$

Step 2: Small time estimates Let $\tau \in (0, \infty)$. According to Proposition 5.1, there exists a constant $C > 0$, depending on τ, such that for every $t \in (0, \tau]$ we have

$$\|U_2(t)\|_{\mathbb{X}^q} \leqslant C \|U_1(t)\|_{\mathbb{X}^q}, \tag{6.16}$$

$$\left\| \frac{\mathrm{d}U_2}{\mathrm{d}t}(t) \right\|_{\mathbb{X}^q} + \|U_2(t,\cdot)\|_{2,q,E} \leqslant C\Big(\|U_1(t,\cdot)\|_{2,q,E} + \|U_1(t)\|_{\mathbb{X}^q} \Big). \tag{6.17}$$

Combining the above estimates and (6.11), we obtain that

$$\left(\|U_2\|_{L^\infty_\eta([0,\tau];\mathbb{X}^q)} + \|U_2\|_{L^p_\eta([0,\tau];\mathbb{X}^q)} + \left\| \frac{\mathrm{d}U_2}{\mathrm{d}t} \right\|_{L^p_\eta([0,\tau];\mathbb{X}^q)} + \|U_2\|_{L^p_\eta([0,\tau];W^{2,q}(E))} \right) \leqslant C\mathcal{J}. \tag{6.18}$$

Step 3: Estimates of U_2 *in* $L^p(L^q)$ *norm* For $t > 2$ we consider the decomposition of U_2 given in (6.15). In view of (6.2b), there exists $\beta > 0$ such that

$$1 < \eta + \frac{1}{p} < \beta \leqslant \min\left\{ \frac{3}{2}\left(\frac{1}{q_0} - \frac{1}{q} \right), \gamma_1(t, q_0, q), \gamma_2(t, q_0, q) \right\}, \tag{6.19}$$

where $\gamma_1(t, q_0, q)$ and $\gamma_2(t, q_0, q)$ are defined in (5.8), (5.9). Note that, the condition (6.2b) implies that $\eta p' > 1$, where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

To estimate U_{21} we note that for every $t > 2$ we have

$$\begin{aligned} &\|U_{21}(t)\|_{\mathbb{X}^q} + \sum_{j=1}^{2} \left\| \nabla^j U_{21}(t) \right\|_{q,E} \\ &\underset{\text{Eq.}(5.4),(5.5),(5.6)}{\leqslant} C \int_0^{t/2} \left((t-s)^{-\frac{3}{2}\left(\frac{1}{q_0} - \frac{1}{q}\right)} + (t-s)^{-\gamma_1(q_0,q)} + (t-s)^{-\gamma_2(q_0,q)} \right) \|U_1(s)\|_{\mathbb{X}^{q_0}} \\ &\underset{(6.19)}{\leqslant} Ct^{-\beta} \int_0^{t/2} (1+s^2)^{-\eta/2} (1+s^2)^{\eta/2} \|U_1(s)\|_{\mathbb{X}^{q_0}}\ \mathrm{d}s \\ &\leqslant Ct^{-\beta} \|U_1\|_{L^p_\eta([0,\infty);\mathbb{X}^{q_0})} \left(\int_0^\infty (1+t^2)^{-\eta p'/2}\ \mathrm{d}t \right)^{1/p'} \\ &\underset{(\eta p' > 1)}{\leqslant} Ct^{-\beta} \|U_1\|_{L^p_\eta([0,\infty);\mathbb{X}^{q_0})} \leqslant C(1+t^2)^{-\beta/2} \|U_1\|_{L^p_\eta([0,\infty);\mathbb{X}^{q_0})}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_2^\infty (1+t^2)^{\frac{p\eta}{2}} \left(\|U_{21}(t)\|^p_{\mathbb{X}^q} + \sum_{j=1,2} \left\| \nabla^j U_{21}(t) \right\|^p_{q,E} \right)\ \mathrm{d}t \\ &\qquad \leqslant C \|U_1\|^p_{L^p_\eta([0,\infty);\mathbb{X}^{q_0})} \int_2^\infty (1+t^2)^{\frac{p}{2}(\eta-\beta)}\ \mathrm{d}t \leqslant C \|U_1\|^p_{L^p_\eta([0,\infty);\mathbb{X}^{q_0})}. \end{aligned} \tag{6.20}$$

In order to estimate U_{22}, we note that for every $t>2$ we have

$$\begin{aligned}
&\|U_{22}(t)\|_{\mathbb{X}^q}+\sum_{j=1}^{2}\left\|\nabla^j U_{22}(t)\right\|_{q,E}\\
&\underset{\text{Eq.(5.4),(5.5),(5.6)}}{\leqslant} C\int_{t/2}^{t-1}\left((t-s)^{-\frac{3}{2}\left(\frac{1}{q_0}-\frac{1}{q}\right)}+(t-s)^{-\gamma_1(t,q_0,q)}+(t-s)^{-\gamma_2(t,q_0,q)}\right)\|U_1(s)\|_{\mathbb{X}^{q_0}}\\
&\underset{(6.19)}{\leqslant} C\int_{t/2}^{t-1}(t-s)^{-\beta}(1+s^2)^{-\eta/2}(1+s^2)^{\eta/2}\|U_1(s)\|_{\mathbb{X}^{q_0}}\ \mathrm{d}s\\
&\leqslant C(1+t^2)^{-\eta/2}\int_{t/2}^{t-1}(t-s)^{-\beta(1/p+1/p')}(1+s^2)^{\eta/2}\|U_1(s)\|_{\mathbb{X}^{q_0}}\ \mathrm{d}s\quad(\text{ as } 4(1+s^2)>(1+t^2))\\
&\leqslant C(1+t^2)^{-\eta/2}\left(\int_{t/2}^{t-1}(t-s)^{-\beta}\ \mathrm{d}s\right)^{1/p'}\left(\int_{t/2}^{t-1}(t-s)^{-\beta}(1+s^2)^{p\eta/2}\|U_1(s)\|_{\mathbb{X}^{q_0}}^p\ \mathrm{d}s\right)^{1/p}\\
&\leqslant C(1+t^2)^{-\eta/2}\left(\int_{t/2}^{t-1}(t-s)^{-\beta}(1+s^2)^{p\eta/2}\|U_1(s)\|_{\mathbb{X}^{q_0}}^p\ \mathrm{d}s\right)^{1/p}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_2^\infty(1+t^2)^{\frac{p\eta}{2}}\left(\|U_{22}(t)\|_{\mathbb{X}^q}^p+\sum_{j=1,2}\left\|\nabla^j U_{22}(t)\right\|_{q,E}^p\right)\ \mathrm{d}t\\
&\leqslant C\int_2^\infty\int_{t/2}^{t-1}(t-s)^{-\beta}(1+s^2)^{p\eta/2}\|U_1(s)\|_{\mathbb{X}^{q_0}}^p\ \mathrm{d}s\mathrm{d}t\\
&\leqslant C\int_1^\infty(1+s^2)^{p\eta/2}\|U_1(s)\|_{\mathbb{X}^{q_0}}^p\left(\int_{s+1}^{2s}(t-s)^{-\beta}\mathrm{d}t\right)\mathrm{d}s\leqslant C\|U_1\|_{L^p_\eta([0,\infty);\mathbb{X}^{q_0})}^p.
\end{aligned}\tag{6.21}$$

To estimate U_{23}, we first note that for $j\in\{0,1,2\}$ and $t>2$ we have

$$\begin{aligned}
&\|U_{23}(t)\|_{\mathbb{X}^q}+\left\|\nabla^j U_{23}(t)\right\|_{q,E}\\
&\underset{\text{Eq.5.3}}{\leqslant} C\int_{t-1}^{t}\left(\|U_1(s)\|_{2,q,E}+\|U_1(s)\|_{\mathbb{X}^q}\right)\ \mathrm{d}s\\
&\leqslant C(1+t^2)^{-\eta/2}\int_{t-1}^{t}(1+s^2)^{\eta/2}\left(\|U_1(s)\|_{2,q,E}+\|U_1(s)\|_{\mathbb{X}^q}\right)\ \mathrm{d}s,
\end{aligned}$$

where the last inequality follows from the fact that $3(1+s^2)>(1+t^2)$ for every $s\in[t-1,t]$.

Proceeding similarly as in the estimate of U_{22} above we obtain

$$\begin{aligned}
&\int_2^\infty(1+t^2)^{\frac{p\eta}{2}}\left(\|U_{21}(t)\|_{\mathbb{X}^q}^p+\sum_{j=1,2}\left\|\nabla^j U_{21}(t)\right\|_{q,E}^p\right)\ \mathrm{d}t\\
&\qquad\leqslant C\left(\|U_1\|_{L^p_\eta([0,\infty);W^{2,q_2}(E))}^p+\|U_1\|_{L^p_\eta([0,\infty);\mathbb{X}^q)}^p\right).
\end{aligned}\tag{6.22}$$

Combining the estimates (6.20)–(6.22), (6.18) (for some fixed $\tau\geqslant 2$) and (6.11), we obtain

$$\|U_2\|_{L^p_\eta([0,\infty);W^{2,q_2}(E))}^p+\|U_2\|_{L^p_\eta([0,\infty);\mathbb{X}^q)}^p\leqslant C\mathcal{J}.\tag{6.23}$$

Step 4: Estimates of the derivative with respect to time in $L^p(L^q)$ norm From (6.14), we have

$$\frac{\mathrm{d}U_2}{\mathrm{d}t}(t)=U_1(t)+\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{T}_{t-s}U_1(s)\right)\ \mathrm{d}s\quad(t>0).$$

Then using the similar arguments as in step 3 above, we obtain

$$\left\| \frac{\mathrm{d}U_2}{\mathrm{d}t} \right\|_{L^p_\eta([0,\infty);\mathbb{X}^q)} \leqslant C\mathcal{J}. \tag{6.24}$$

Final step Finally, combining the estimates (6.23), (6.24), we get the estimate (6.12). This completes the proof of Theorem 6.4. □

7. Proof of the Main Result

The aim of this section is to prove Theorems 3.4 and 2.3. The main ingredients of the proofs are the estimates on the linearized problem which we proved in the previous sections, combined with a fixed point argument. Due to the fact that the estimates on the linearized system include second order derivatives with respect to the space variables and pressure terms we can use a maximal regularity based fixed point argument which is slightly simpler than the Kato type one used in [4].

We begin by replacing the nonlinear terms F, G_1 and G_2 in (3.9) by given source terms f, g_1 and g_2, and study the regularity and the decay properties of this linear system. More precisely, we first consider the system:

$$\begin{cases} \partial_t u - \mu\Delta u + \nabla\pi = f, \quad \operatorname{div} u = 0 & (t \geqslant 0, x \in E), \\ u = \ell + \omega \times x & (t \geqslant 0, x \in \partial\mathcal{O}), \\ m\dot{\ell} + \int_{\partial\mathcal{O}} \sigma(u,\pi)\nu \,\mathrm{d}s = g_1 & (t \geqslant 0), \\ \mathcal{J}\dot{\omega} + \int_{\partial\mathcal{O}} x \times \sigma(u,\pi)\nu \,\mathrm{d}s = g_2 & (t \geqslant 0), \\ u(0) = u_0 & (x \in E), \\ \ell(0) = \ell_0, \quad \omega(0) = \omega_0. \end{cases} \tag{7.1}$$

In the spirit of Lemma 4.1, the above system can be seen as an alternative manner of writing the Eq. 6.1 studied in the previous section. The reason for which we prefer to write the linearized system in the form (7.1) instead of (6.1) is that our fixed point procedure requires estimates on terms explicitly involving the pressure. Due to this fact it is important for our approach to rephrase the results in Theorem 6.1 in terms of the solution (u, π, l, ω) of (7.1).

We are going to apply Theorem 6.1 to the system (7.1) with suitable choice of exponents. To this aim, let us take $p, q \in (1,\infty)$ and $\eta > 0$ such that

$$q \in (2,\infty), \quad 1 < \frac{1}{p} + \frac{3}{2q} \leqslant \frac{3}{2}, \quad 1 - \frac{1}{p} < \eta < \frac{3}{2q}. \tag{7.2}$$

For $p, q_0, q \in (1,\infty)$ and $\eta > 0$, satisfying the above conditions, we define

$$\begin{aligned} \mathcal{W}_\eta = \Big\{ &(u,\pi,\ell,\omega) \mid u \in L^p_\eta([0,\infty); W^{2,q}(E)) \cap W^{1,p}_\eta((0,\infty); L^q(E)), \\ &\pi \in L^p_\eta([0,\infty); \widehat{W}^{1,q}(E)), \quad \ell,\omega \in W^{1,p}_\eta([0,\infty);\mathbb{R}^3) \Big\}, \end{aligned} \tag{7.3}$$

equipped with the norm

$$\begin{aligned} \|(u,\pi,\ell,\omega)\|_{\mathcal{W}_\eta} &:= \|u\|_{L^p_\eta([0,\infty);W^{2,q}(E))^3} + \|\partial_t u\|_{L^p_\eta([0,\infty);L^q(E))^3} + \|u\|_{L^\infty_\eta((0,\infty);B^{2(1-1/p)}_{q,p}(E))^3} \\ &+ \|\nabla\pi\|_{L^p_\eta([0,\infty);L^q(E))^3} + \|\ell\|_{L^\infty_\eta([0,\infty);\mathbb{R}^3)} + \|\ell\|_{W^{1,p}_\eta((0,\infty);\mathbb{R}^3)} \\ &+ \|\omega\|_{L^\infty_\eta([0,\infty);\mathbb{R}^3)} + \|\omega\|_{W^{1,p}_\eta((0,\infty);\mathbb{R}^3)}\,. \end{aligned} \tag{7.4}$$

We introduce the set of initial data

$$\mathcal{I} = \Big\{(u_0, \ell_0, \omega_0) \mid u_0 \in L^{q/2}(E)^3 \cap B^{2(1-1/p)}_{q,p}(E)^3,\ \ell_0 \in \mathbb{R}^3,\ \omega_0 \in \mathbb{R}^3,$$
$$\operatorname{div}\ u_0 = 0 \text{ in } E, \quad u_0 = \ell_0 + \omega_0 \times x \text{ on } \partial\mathcal{O}\Big\}, \tag{7.5}$$

equipped with the norm

$$\|(u_0, \ell_0, \omega_0)\|_{\mathcal{I}} := \|u_0\|_{B^{2(1-1/p)}_{q,p}(E)^3} + \|u_0\|_{L^{q/2}(E)^3} + \|\ell_0\|_{\mathbb{R}^3} + \|\omega_0\|_{\mathbb{R}^3}. \tag{7.6}$$

Finally, we introduce the space of source terms

$$\mathcal{R}_\eta = \Big\{(f, g_1, g_2) \mid f \in L^p_\eta([0,\infty); L^{q/2}(E)^3) \cap L^p_\eta([0,\infty); L^q(E)^3)$$
$$g_1 \in L^p_\eta([0,\infty); \mathbb{R}^3),\ g_2 \in L^p_\eta([0,\infty); \mathbb{R}^3)\Big\}, \tag{7.7}$$

equipped with the norm

$$\|(f, g_1, g_2)\|_{\mathcal{R}_\eta} := \|f\|_{L^p_\eta([0,\infty);L^q(E))^3} + \|f\|_{L^p_\eta([0,\infty);L^{q/2}(E))^3} + \|g_1\|_{L^p_\eta([0,\infty);\mathbb{R}^3)} + \|g_2\|_{L^p_\eta([0,\infty);\mathbb{R}^3)}. \tag{7.8}$$

We have the following result:

Theorem 7.1. *Let $p, q \in (1,\infty)$ and $\eta > 0$ satisfying (7.2). Let $\mathcal{W}_\eta$, $\mathcal{I}$ and $\mathcal{R}_\eta$ be the spaces defined in (7.3), (7.5) and (7.7), respectively. Then for every $(u_0, \ell_0, \omega_0) \in \mathcal{I}$ and every $(f, g_1, g_2) \in \mathcal{R}_\eta$, the system (7.1) admits a unique solution $(u, \pi, \ell, \omega) \in \mathcal{W}_\eta$. Moreover, there exists a positive constant C_L, depending only on p, q, η, E and $\mathcal{O}$ (as in Theorem 6.1, the index L comes from "linear"), such that*

$$\|(u, \pi, \ell, \omega)\|_{\mathcal{W}_\eta} \leqslant C_L \left(\|(u_0, \ell_0, \omega_0)\|_{\mathcal{I}} + \|(f, g_1, g_2)\|_{\mathcal{R}_\eta}\right), \tag{7.9}$$

for every $(u_0, \ell_0, \omega_0) \in \mathcal{I}$ and $(f, g_1, g_2) \in \mathcal{R}_\eta$.

Proof. Let us set

$$U_0 = u_0 \mathbb{1}_E + (\ell_0 + \omega_0 \times y)\mathbb{1}_{\mathcal{O}}, \quad F = \mathbb{P}_q\left(f\mathbb{1}_E + (f_\ell + f_\omega \times y)\mathbb{1}_{\mathcal{O}}\right).$$

Then

$$U_0 \in \mathbb{X}^{q/2} \cap (\mathbb{X}^q, \mathcal{D}(\mathbb{A}_q))_{1-1/p,p},$$

and

$$F \in L^p_\eta([0,\infty); \mathbb{X}^{q/2} \cap \mathbb{X}^q.$$

Moreover, since $p, q \in (1,\infty)$ and $\eta > 0$ satisfy (7.2), it is easy to see that (p, q, η) satisfies the condition (6.2) with $q_0 = q/2$.

Let U be the solution to

$$\frac{\mathrm{d}U}{\mathrm{d}t}(t) = \mathbb{A}_q U(t) + F(t) \quad (t > 0),$$
$$U(0) = U_0.$$

By applying Theorem 6.1, we have $U \in L^p_\eta([0,\infty); \mathcal{D}(\mathbb{A}_q)) \cap W^{1,p}_\eta([0,\infty); \mathbb{X}^q)$. According to Lemma 4.1, there exists π such that (u, π, ℓ, ω) satisfies the system (7.1), where

$$u = U|_E, \quad \ell = \frac{1}{m}\int_{\mathcal{O}} U\ \mathrm{d}y, \quad \omega = -\mathcal{J}^{-1}\int_{\mathcal{O}} U \times y\ \mathrm{d}y,$$

which gives the desired regularity of u, ℓ and ω. Finally, from $(7.1)_1$, we obtain

$$\nabla\pi \in L^p_\eta([0,\infty); L^q(E)).$$

This completes the proof of the result. □

We are now in a position to prove the fixed point argument. We assume that, $p, q \in (1,\infty)$ and $\eta > 0$ satisfy (7.2). Let us also fix $R > 0$ such that

$$\operatorname{diam}(\mathcal{O}) < \frac{R}{2}. \tag{7.10}$$

Recalling the definition of $\mathcal{W}_\eta$ from (7.3), we define, for every $\gamma > 0$, the ball $\mathcal{W}_{\eta,\gamma}$ in $\mathcal{W}_\eta$ by

$$\mathcal{W}_{\eta,\gamma} = \left\{(u,\pi,\ell,\omega) \in \mathcal{W}_\eta \mid \|(u,\pi,\ell,\omega)\|_{\mathcal{W}_\eta} \leqslant \gamma\right\}. \tag{7.11}$$

We are now going to derive various estimates for the nonlinear terms $\mathcal{F}, \mathcal{G}_1$ and $\mathcal{G}_2$ defined in (3.11) - (3.12), beginning with X and Y : (Recall that X, Y, a and b have been defined in Sect. 3 (see (3.4).). (3.7)).

Lemma 7.2. *There exist constants $\gamma_0 \in (0,1)$ and $C > 0$, both depending on p, q_0, q_1, q_2 and η, such that for every $\gamma \in (0,\gamma_0)$ and for every $(u,\pi,\ell,\omega) \in \mathcal{W}_{\eta,\gamma}$, the functions h, a, b, X and Y defined in (3.6), (3.7), (3.4) and Lemma 3.2, satisfy*

$$\begin{aligned}
&\|h\|_{L^\infty((0,\infty);\mathbb{R}^3)} < \frac{R}{2} \quad \text{(in particular (3.2) holds)},\\
&\|\nabla X\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} + \|\nabla Y(X)\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} \leqslant C,\\
&\|a(X)\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} + \|\nabla b\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} \leqslant C,\\
&\|\nabla X - \mathbb{I}_3\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} + \|\nabla Y(X) - \mathbb{I}_3\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} \leqslant C\gamma,\\
&\|a(X) - \mathbb{I}_3\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} + \|b - \mathbb{I}_3\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} \leqslant C\gamma,\\
&\|\partial_t a(X)\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} + \|\partial_t Y(X)\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} \leqslant C\gamma.
\end{aligned}$$

Moreover, for all $i, j, k \in \{1,2,3\}$, we have

$$\begin{aligned}
&\left\|\frac{\partial a_{ik}}{\partial x_j}(X)\right\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} + \left\|\frac{\partial^2 a_{ik}}{\partial x_j^2}(X)\right\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} + \left\|\frac{\partial^2 Y_j}{\partial x_i^2}(X)\right\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} \leqslant C\gamma,\\
&\left\|\frac{\partial Y_j}{\partial x_i}(X)\frac{\partial Y_k}{\partial x_i}(X) - \delta_{j,i}\delta_{k,i}\right\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} \leqslant C\gamma.
\end{aligned}$$

Proof. Recall the definition of h from (3.6). Since $Q(t)$ is an isometry for every $t \geqslant 0$ and $\eta p' > 1$, we deduce that for every $t > 0$ we have

$$|h(t)| \leqslant \int_0^t |\ell(s)| \mathrm{d}s \leqslant \int_0^\infty (1+t^2)^{-\eta/2}(1+t^2)^{\eta/2}|\ell(t)| \ \mathrm{d}t \leqslant C_0 \|\ell\|_{L^p_\eta(0,\infty)} \leqslant C_0\gamma,$$

for some constant $C_0 > 0$ depending only on η and p'. By setting

$$\gamma_0 = \min\left\{1, \frac{R}{2C_0}\right\}, \tag{7.12}$$

we have that

$$\|h\|_{L^\infty([0,\infty);\mathbb{R}^3)} < \frac{R}{2}.$$

The rest of the proof follows easily from the definition of X (see for instance proof of Proposition 6.3 of [13]). □

We next provide estimates of the term $\mathcal{F}$ defined in (3.11).

Lemma 7.3. *Let γ_0 is defined in (7.12). There exists a positive constant $C_{\mathcal{F}}$, depending on p, q_0, q_1, q_2 and η, such that for every $\gamma \in (0, \gamma_0)$, $(u, \pi, \ell, \omega) \in \mathcal{W}_{\eta,\gamma}$, and $(u^i, \pi^i, \ell^i, \omega^i) \in \mathcal{W}_{\eta,\gamma}$, $i = 1, 2$, we have*

$$\begin{aligned}
&\|\mathcal{F}(u,\pi,\ell,\omega)\|_{L^p_\eta([0,\infty);L^{q/2}(E))^3} + \|\mathcal{F}(u,\pi,\ell,\omega)\|_{L^p_\eta(0,\infty;L^q(E))^3} \leqslant C_{\mathcal{F}}\gamma^2,\\
&\left\|\mathcal{F}(u^1,\pi^1,\ell^1,\omega^1) - \mathcal{F}(u^2,\pi^2,\ell^2,\omega^2)\right\|_{L^p_\eta([0,\infty);L^{q/2}(E))^3}\\
&\quad + \left\|\mathcal{F}(u^1,\pi^1,\ell^1,\omega^1) - \mathcal{F}(u^2,\pi^2,\ell^2,\omega^2)\right\|_{L^p_\eta([0,\infty);L^q(E))^3}\\
&\qquad \leqslant C_{\mathcal{F}}\gamma \left\|(u^1,\pi^1,\ell^1,\omega^1) - (u^2,\pi^2,\ell^2,\omega^2)\right\|_{\mathcal{W}_\eta}.
\end{aligned}$$

Proof. Note that, in view of Lemma 3.2, all the terms in the definition of $\mathcal{F}$, with the exception of the seventh one are supported in B_{2R}. Thus for these terms it is enough to estimate $L^p_\eta([0,\infty);L^q(E))$ norm only. Using Lemma 7.2, we estimate the first term of $\mathcal{F}$ by noticing that for $\alpha \in \{1,2,3\}$ we have

$$\begin{aligned}
&\left\|\nu \sum_{i,j,k} b_{\alpha i} \frac{\partial^2 a_{ik}}{\partial x_j^2}(X) u_k\right\|_{L^p_\eta((0,\infty);L^q(E))}\\
&\quad \leqslant C \sum_{i,j,k} \|b_{\alpha i}\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} \left\|\frac{\partial^2 a_{ik}}{\partial x_j^2}(X)\right\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} \|u_k\|_{L^p_\eta(0,\infty;L^q(E))} \leqslant C\gamma^2.
\end{aligned}$$

The other five compactly supported terms can be estimated in a similar way.

Concerning the seventh term of $\mathcal{F}$, we use Hölder's inequality to obtain that

$$\|(u\cdot\nabla)u\|_{L^p_\eta([0,\infty);L^{q/2}(E))} \leqslant \|u\|_{L^\infty_\eta([0,\infty);L^q(E))} \|\nabla u\|_{L^p_\eta([0,\infty);L^q(E))} \leqslant \gamma^2.$$

Since $\dfrac{1}{p} + \dfrac{3}{2q} \leqslant \dfrac{3}{2}$, we have the following embeddings (see for instance [7, Proposition 4.3], [12, Eq. (6.5), (6.7)])

$$\begin{aligned}
&L^p((0,\infty);W^{2,q}(E)) \cap W^{1,p}((0,\infty);L^q(E)) \hookrightarrow L^{3p}(0,\infty;L^{3q}(E)),\\
&L^p((0,\infty);W^{2,q}(E)) \cap W^{1,p}((0,\infty);L^q(E)) \hookrightarrow L^{3p/2}(0,\infty;W^{1,3q/2}(E)).
\end{aligned}$$

Using the above embeddings, we get

$$\|(u\cdot\nabla)u\|_{L^p_\eta([0,\infty);L^q(E))} \leqslant \|u\|_{L^{3p}_\eta([0,\infty);L^{3q}(E))} \|\nabla u\|_{L^{3p/2}_\eta([0,\infty);L^{3q/2}(E))} \leqslant C\gamma^2$$

which ends the proof. □

Concerning $\mathcal{G}_1$ and $\mathcal{G}_2$ defined in (3.12), it can be easily checked that we have:

Lemma 7.4. *Let γ_0 is defined in (7.12). There exists a positive constant $C_{\mathcal{G}}$, depending on p, q_0, q_1, q_2 and η, such that for every $\gamma \in (0, \gamma_0)$, $(u, \pi, \ell, \omega) \in \mathcal{W}_{\eta,\gamma}$, and $(u^i, \pi^i, \ell^i, \omega^i) \in \mathcal{W}_{\eta,\gamma}$, $i = 1, 2$, we have*

$$\begin{aligned}
&\|\mathcal{G}_1(\ell,\omega)\|_{L^p_\eta(0,\infty;\mathbb{R}^3)} + \|\mathcal{G}_2(\ell,\omega)\|_{L^p_\eta(0,\infty;\mathbb{R}^3)} \leqslant C_{\mathcal{G}}\gamma^2,\\
&\left\|\mathcal{G}_1(\ell^1,\omega^1) - \mathcal{G}_1(\ell^2,\omega^2)\right\|_{L^p_\eta(0,\infty;\mathbb{R}^3)} + \left\|\mathcal{G}_2(\ell^1,\omega^1) - \mathcal{G}_2(\ell^2,\omega^2)\right\|_{L^p_\eta(0,\infty;\mathbb{R}^3)}\\
&\quad \leqslant C_{\mathcal{G}}\gamma \left\|(u^1,\pi^1,\ell^1,\omega^1) - (u^2,\pi^2,\ell^2,\omega^2)\right\|_{\mathcal{W}_\eta}.
\end{aligned}$$

We are now in a position to prove Theorems 3.4, 2.3 and Corollary 2.5.

Proof of Theorems 3.4 and 2.3. In view of Proposition 3.5, it is enough to prove Theorem 3.4. We take $(v, \psi, \ell_v, \omega_v) \in \mathcal{W}_{\eta,\gamma}$, and consider the system

$$\begin{cases} \partial_t u - \mu\Delta u + \nabla\pi = \mathcal{F}(v, \psi, \ell_v, \omega_v), \quad \operatorname{div} u = 0 & (t \geqslant 0, x \in E), \\ u = \ell + \omega \times x & (t \geqslant 0, x \in \partial\mathcal{O}), \\ m\dot{\ell} + \displaystyle\int_{\partial\mathcal{O}} \sigma(u, \pi)\nu \, \mathrm{d}s = \mathcal{G}_1(\ell_v, \omega_v) & (t \geqslant 0), \\ \mathcal{J}\dot{\omega} + \displaystyle\int_{\partial\mathcal{O}} x \times \sigma(u, \pi)\nu \, \mathrm{d}s = \mathcal{G}_2(\ell_v, \omega_v) & (t \geqslant 0), \\ u(0) = u_0 & (x \in E), \\ \ell(0) = \ell_0, \quad \omega(0) = \omega_0. \end{cases} \tag{7.13}$$

Let

$$\gamma < \min\left\{\gamma_0, \frac{1}{4C_L C_{\mathcal{F}}}, \frac{1}{4C_L C_{\mathcal{G}}}\right\} \text{ and } \varepsilon_0 = \frac{\gamma}{2C_L}, \tag{7.14}$$

where γ_0 is defined in (7.12), and C_L, $C_{\mathcal{F}}$ and $C_{\mathcal{G}}$ are the constants appearing in Theorem 7.1, Lemmas 7.3 and 7.4 respectively. Assume that, $p, q \in (1, \infty)$ and $\eta > 0$ satisfy (7.2). Moreover, suppose that

$$\|(u_0, \ell_0, \omega_0)\|_{\mathcal{I}} \leqslant \varepsilon_0,$$

where ε_0 satisfies (7.14). Let us define the map $\mathcal{N} : \mathcal{W}_{\eta,\gamma} \to \mathcal{W}_{\eta,\gamma}$ by

$$\mathcal{N}(v, \psi, \ell_v, \omega_v) = (u, \pi, \ell, \omega) \quad ((v, \psi, \ell_v, \omega_v) \in \mathcal{W}_{\eta,\gamma}),$$

where (u, π, ℓ, ω) is the solution to the system (7.13). We are going to show that $\mathcal{N}$ is a strict contraction in $\mathcal{W}_{\eta,\gamma}$.

Since $(v, \psi, \ell_v, \omega_v) \in \mathcal{W}_{\eta,\gamma}$, we can apply Theorem 7.1, Lemmas 7.3 and 7.4 to the system (7.13) and using the choice of γ and ε_0 above, we deduce

$$\begin{aligned} \|(u, \pi, \ell, \omega)\|_{\mathcal{W}_\eta} &\leqslant C_L\left(\|(u_0, \ell_0, \omega_0)\|_{\mathcal{I}} + \|(\mathcal{F}(v, \psi, \ell_v, \omega_v), \mathcal{G}_1(\ell_v, \omega_v), \mathcal{G}_2(\ell_v, \omega_v))\|_{\mathcal{R}_\eta}\right) \\ &\leqslant C_L\varepsilon_0 + C_L C_{\mathcal{F}}\gamma^2 + C_L C_{\mathcal{G}}\gamma^2 \leqslant \gamma. \end{aligned}$$

Therefore $\mathcal{N}$ maps $\mathcal{W}_{\eta,\gamma}$ into itself.

For $j \in \{1, 2\}$, we take $(v^j, \psi^j, \ell_v^j, \omega_v^j) \in \mathcal{W}_{\eta,\gamma}$, and we set

$$\mathcal{N}(v^j, \psi^j, \ell_v^j, \omega_v^j) := (u^j, \pi^j, \ell^j, \omega^j). \tag{7.15}$$

Using Theorem 7.1, Lemmas 7.3, 7.4 and (7.14), we obtain

$$\begin{aligned} &\left\|(u^1, \pi^1, \ell^1, \omega^1) - (u^2, \pi^2, \ell^2, \omega^2)\right\|_{\mathcal{W}_\eta} \\ &\quad\leqslant \gamma C_L(C_{\mathcal{F}} + C_{\mathcal{G}}) \left\|(v^1, \psi^1, \ell_v^1, \omega_v^1) - (v^2, \psi^2, \ell_v^2, \omega_v^2)\right\|_{\mathcal{W}_\eta} \\ &\quad\leqslant \frac{1}{2} \left\|(v^1, \psi^1, \ell_v^1, \omega_v^1) - (v^2, \psi^2, \ell_v^2, \omega_v^2)\right\|_{\mathcal{W}_\eta}. \end{aligned}$$

Thus $\mathcal{N}$ is a strict contraction of $\mathcal{W}_{\eta,\gamma}$. This completes the proof of Theorem 3.4. □

Proof of Corollary 2.5. Since $p > 1$, the estimate (2.10) follows easily from the proof Theorem 2.3. Note that, the condition (2.5) implies that $\eta p' > 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore, for any $t > 0$

$$\begin{aligned} \|h(t)\|_{\mathbb{R}^3} &\leqslant \int_0^t (1 + s^2)^{-\eta/2}(1 + s^2)^{\eta/2} \left\|\dot{h}(s)\right\|_{\mathbb{R}^3} \, \mathrm{d}s \\ &\leqslant \left(\int_0^\infty (1 + t^2)^{-\eta p'/2} \, \mathrm{d}t\right)^{1/p'} \left\|\dot{h}\right\|_{L^p_\eta([0,\infty);\mathbb{R}^3} < \infty. \end{aligned} \tag{7.16}$$

Thus indeed $h \in L^\infty([0, \infty); \mathbb{R}^3)$. □

A few remarks are in order:

Remark 7.5. Let us set $h_f := \lim_{t\to\infty} h(t)$, and recall that we have assumed $h(0) = 0$ in $(1.2)_7$. Then according to the estimate (7.16), $\|h_f - h(0)\|_{\mathbb{R}^3}$ can be made arbitrary small by choosing sufficiently small initial data. This is necessary to extend our result to the case of several rigid bodies.

Remark 7.6. Let us point out that Theorems 2.3 and 3.4 hold for more general class of initial data. Let $p, q_0, q \in (1,\infty)$ and $\eta \geqslant 0$ satisfy the following conditions

$$1 < q_0 < q, \quad \frac{1}{p} + \frac{3}{2q} \leqslant \frac{3}{2}, \quad 1 - \frac{1}{p} < \eta < \frac{3}{2}\left(\frac{1}{q_0} - \frac{1}{q}\right),$$
$$\|fg\|_{L^{q_0}(E)} \lesssim \|f\|_{W^{1,q}(E)} \|g\|_{B^{2(1-1/p)}_{q,p}(E)}.$$

Then Theorem 2.3 is valid for $u_0 \in L^{q_0}(E) \cap B^{2(1-1/p)}_{q,p}(E)$. To simplify the presentation, in Theorem 2.3 we have chosen $q_0 = q/2$.

Another interesting choice is $p = q = 2$ and $q_0 = 1 + \varepsilon$.

8. The Case of Several Rigid Bodies

In this section, we briefly explain why our main result in Theorem 2.3 can be extended to the case of several rigid bodies moving in a viscous incompressible fluid. We insist only on the adaptations needed in the definition of solutions and, most importantly, in the change of variables used to write the governing equations in a fixed spatial domain. With the exception of the above mentioned adaptation of the change of variables, the proofs are very close to those we presented above for a single immersed body, we will not provide their details.

Let $m \in \mathbb{N}$ be the number of rigid bodies. We assume that they are homogeneous and that at instant $t \geqslant 0$ they occupy the smooth bounded domains $\mathcal{S}_i(t)$, with $i \in \{1, 2, \dots m\}$. We assume that the viscous incompressible fluid fills the remaining part of $\mathbb{R}^3$. The domain occupied by the fluid is denoted by

$$\mathcal{F}(t) := \mathbb{R}^3 \setminus \cup_{i=1}^m \overline{\mathcal{S}_i(t)}.$$

We also suppose that initially there is no contact between the rigid bodies, i.e., that

$$\overline{\mathcal{S}_i(0)} \cap \overline{\mathcal{S}_j(0)} = \emptyset \quad (i, j \in \{1, 2 \dots, m\}, \quad i \neq j). \tag{8.1}$$

We denote by $h_i(t), \widetilde{\omega}_i(t)$ and $Q_i(t)$ the position of the centre of mass, the angular velocity, and the orthogonal matrix giving the orientation of the i^{th} rigid body at instant t. We thus have, for $i = 1, 2, \dots, m$,

$$\begin{cases} \mathcal{S}_i(t) = \{h_i(t) + Q_i(t)x \mid x \in \mathcal{S}_i(0)\} & (t \geqslant 0), \\ \dot{Q}_i(t)Q_i(t)^{-1}x = \widetilde{\omega}_i(t) \times x & (t \geqslant 0, \ x \in \mathbb{R}^3), \\ Q_i(0) = \mathbb{I}_3. \end{cases} \tag{8.2}$$

Moreover, the velocity and pressure fields in the fluid are denoted by $\widetilde{u}$ and $\widetilde{\pi}$, respectively. With the above notation, the equations modelling the evolution of several rigid bodies in a viscous incompressible fluid can be written:

$$\begin{cases} \partial_t \widetilde{u} + (\widetilde{u} \cdot \nabla)\widetilde{u} - \mu\Delta\widetilde{u} + \nabla\widetilde{\pi} = 0 & (t > 0, \ x \in \mathcal{F}(t)), \\ \operatorname{div} \widetilde{u} = 0, & (t > 0, \ x \in \mathcal{F}(t)), \\ \widetilde{u}(t,x) = \dot{h}_i(t) + \widetilde{\omega}_i(t) \times (x - h(t)) & (t > 0, \ x \in \partial\mathcal{S}_i(t)), \\ m_i \ddot{h}_i(t) = -\displaystyle\int_{\partial\mathcal{S}_i(t)} \sigma(\widetilde{u}, \widetilde{\pi})\nu_i \, \mathrm{d}s & (t > 0), \\ J_i \dot{\widetilde{\omega}}_i(t) = J_i\widetilde{\omega}_i(t) \times \widetilde{\omega}_i(t) - \displaystyle\int_{\partial\mathcal{S}_i(t)} (x - h(t)) \times \sigma(\widetilde{u}, \widetilde{\pi})\nu_i \, \mathrm{d}s & (t > 0), \\ \widetilde{u}(0,x) = u_0(x) & (y \in \mathcal{F}(0)), \\ h_i(0) = h_{0,i}, \, \dot{h}_i(0) = \ell_{0,i}, \, \widetilde{\omega}_i(0) = \omega_{0,i}. \end{cases} \tag{8.3}$$

In the above equations, ν_i denotes the unit normal to $\partial\mathcal{S}_i(t)$ directed towards the interior of $\mathcal{S}_i(t)$. The constant $m_i > 0$ denotes the mass of the i^{th} rigid body and the matrix $J_i(t)$ stands for the inertia tensor of the i^{th} rigid body at time $t > 0$.

We now introduce the notion of strong solutions to the system (8.3). To this aim, we first define

$$\delta(t) = \min_{i,j\in\{1,2,\ldots,m\}, i\neq j} \operatorname{dist}\left(\overline{\mathcal{S}_i(t)}, \overline{\mathcal{S}_j(t)}\right). \tag{8.4}$$

Notice that, our assumption (8.1), i.e., there is no contact initially between the rigid bodies, imply that $\delta(0) > 0$.

Definition 8.1. We say $(\widetilde{u}, \widetilde{\pi}, (h_i)_{i=1}^m, (Q_i)_{i=1}^m(\widetilde{\omega}_i)_{i=1}^m)$ is a solution of the system (8.2)–(8.3) if

$$h_i \in C([0,\infty);\mathbb{R}^3), \quad Q_i \in W^{1,\infty}(0,\infty;\mathbb{M}_{3\times 3}(\mathbb{R}^3)) \quad \dot{h}_i,\ \widetilde{\omega}_i \in W^{1,p}((0,\infty);\mathbb{R}^3), \quad i \in \{1,2,\ldots,m\},$$
$$\widetilde{u} \in L^p([0,\infty); W^{2,q}(\mathcal{F}(\cdot))) \cap W^{1,p}((0,\infty); L^q(\mathcal{F}(\cdot))), \quad \widetilde{\pi} \in L^p([0,\infty), \widehat{W}^{1,q}(\mathcal{F}(\cdot))),$$

for some $p, q \in (1,\infty)$, (8.2) holds in a classical sense, equations $(8.3)_{1,2}$ in the sense of distribution in $(0,\infty)\times\mathcal{F}(\cdot)$, equations $(8.3)_{4,5}$ are satisfied in the sense of distributions on $(0,\infty)$, equation $(8.3)_3$ holds in the sense of traces, the initial conditions $(8.3)_{6,7}$ hold in a classical sense and $\delta(t) > 0$ for every $t \geqslant 0$.

We now state the main result of this section.

Theorem 8.2. *Let $p, q \in (1,\infty)$ and $\eta > 0$ be such that*

$$q \in (2,\infty), \quad 1 < \frac{1}{p} + \frac{3}{2q} \leqslant \frac{3}{2}, \quad 1 - \frac{1}{p} < \eta < \frac{3}{2q}. \tag{8.5}$$

Let $h_{0,1}, h_{0,2}, \ldots, h_{0,m} \in \mathbb{R}^3$, be such that $\delta(0) > 0$. We assume

$$u_0 \in B_{q,p}^{2(1-1/p)}(\mathcal{F}(0))^3 \cap L^{q/2}(\mathcal{F}(0))^3, \tag{8.6}$$
$$\ell_{0,i} \in \mathbb{R}^3, \quad \omega_{0,i} \in \mathbb{R}^3, \quad i = 1,2,\ldots,m \tag{8.7}$$

satisfying the compatibility conditions

$$\operatorname{div}\ u_0 = 0 \ \text{in}\ \mathcal{F}(0), \quad u_0(x) = \ell_{0,i} + \omega_{0,i} \times x \ \text{for}\ x \in \partial\mathcal{S}_i(0),\ i = 1,2,\ldots,m. \tag{8.8}$$

Then there exists $\varepsilon_0 > 0$ such that for any $(u_0, (\ell_{0,i})_{i=1}^m, (\omega_{0,i})_{i=1}^m)$ satisfying (8.7), (8.7), (8.8) and

$$\|u_0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F}(0))^3} + \|u_0\|_{L^{q/2}(\mathcal{F}(0))^3} + \sum_{i=1}^m \left(\|\ell_{0,i}\|_{\mathbb{R}^3} + \|\omega_{0,i}\|_{\mathbb{R}^3}\right) \leqslant \varepsilon_0, \tag{8.9}$$

the system (8.3) admits a unique solution $(\widetilde{u}, \widetilde{\pi}, (h_i)_{i=1}^m, (Q_i)_{i=1}^m, (\widetilde{\omega}_i)_{i=1}^m)$ in the sense of definition Definition 8.1. Moreover, this solution satisfies

$$\widetilde{u} \in L^p_\eta([0,\infty); W^{2,q}(\mathcal{F}(\cdot))) \cap W^{1,p}_\eta((0,\infty); L^q(\mathcal{F}(\cdot))) \cap C_b([0,\infty); B_{q,p}^{2(1-1/p)}(\mathcal{F}(\cdot))),$$
$$\widetilde{\pi} \in L^p_\eta((0,\infty); \widehat{W}^{1,q}(\mathcal{F}(\cdot))),$$
$$h_i \in C([0,\infty);\mathbb{R}^3), \quad Q_i \in W^{1,\infty}(0,\infty;\mathbb{M}_{3\times 3}(\mathbb{R}^3)),$$
$$\dot{h}_i \in W^{1,p}_\eta(0,\infty;\mathbb{R}^3),\ \widetilde{\omega}_i \in W^{1,p}_\eta(0,\infty;\mathbb{R}^3),\ i = 1,2,\ldots,m.$$

The proof of the above theorem is similar to the proof of Theorem 2.3. The main difference is in the construction of flow X that allows us to reformulate the problem in a fixed cylindrical domain. Thus we limit our presentation to the description of the adaptations needed for the construction of X. In order to be coherent with the notation used for the single rigid body case, we begin by setting $E := \mathcal{F}(0) = \mathbb{R}^3 \setminus \cup_{i=1}^m \mathcal{S}_i(0)$ and $\mathcal{O}_i := \mathcal{S}_i(0)$. For $i \in \{1,2,\ldots,m\}$, let $R_i > 0$ and $h_i : [0,\infty) \to \mathbb{R}^3$ be such that

$$\operatorname{diam}\mathcal{O}_i + \|h_i\|_{L^\infty([0,\infty);\mathbb{R}^3)} < R_i \leqslant \frac{\delta(0)}{3}, \tag{8.10}$$

It is then easy to see that

$$\mathcal{S}(t) \subset B(h_i, R_i) \text{ for all } t \geqslant 0,$$

where $B(h_i, R_i)$ is the open ball of radius R_i and centered at h_i. Let $\psi_i \in [C_0^\infty(B_{2R_i})]^3$ be a cut-off function such that $\psi_i = 1$ on $\overline{B_{R_i}}$. For $i = 1, 2, \ldots, m$, we introduce a function ζ_i defined in $[0, \infty) \times \mathbb{R}^3$ by

$$\zeta_i(t,x) = \dot{h_i}(t) \times (x - h_i(t)) + \frac{|x - h_i(t)|^2}{2}\widetilde{\omega_i}(t) \quad (t \geqslant 0,\ x \in \mathbb{R}^3),$$

For $i \in \{1, 2, \ldots, m\}$ and $t \geqslant 0$ we define $\Lambda_i(t, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Lambda_i(t,x) = \psi_i(x)\left(\dot{h}_i(t) + \widetilde{\omega}_i(t) \times (x - h_i(t))\right) + \begin{bmatrix} \dfrac{\partial \psi_i(x)}{\partial x_2}\zeta_{i3}(t,x) - \dfrac{\partial \psi_i(x)}{\partial x_3}\zeta_{i2}(t,x) \\ \dfrac{\partial \psi_i(x)}{\partial x_3}\zeta_{i1}(t,x) - \dfrac{\partial \psi_i(x)}{\partial x_1}\zeta_{i3}(t,x) \\ \dfrac{\partial \psi_i(x)}{\partial x_1}\zeta_{i2}(t,x) - \dfrac{\partial \psi_i(x)}{\partial x_2}\zeta_{i1}(t,x) \end{bmatrix}. \tag{8.11}$$

for every $x \in \mathbb{R}^3$ and we set

$$\Lambda(t,x) = \sum_{i=1}^{m} \Lambda_i(t,x) \quad (t \geqslant 0,\ x \in \mathbb{R}^3). \tag{8.12}$$

By obvious adaptations of the proof of Lemma 3.1, we can easily check that Λ has the following properties:

Lemma 8.3. *Assume that $\dot{h}_i, \widetilde{\omega}_i \in W_\eta^{1,p}(0,\infty)$ for every $i \in \{1, 2, \ldots, m\}$ and that δ satisfies (8.4). Let Λ be defined by (8.12). Then we have*

1. $\Lambda = 0$ *outside* B_{2R}.
2. $\operatorname{div} \Lambda(t,x) = 0$ *in* $[0,\infty) \times \mathbb{R}^3$.
3. $\Lambda(t,x) = \dot{h}_i(t) + \widetilde{\omega}_i(t) \times (x - h_i(t))$ *for every* $t \in [0,\infty)$ *and* $x \in \mathcal{S}_i(t)$.
4. Λ *is continuous from* $[0,\infty) \times \mathbb{R}^3$ *to* $\mathbb{R}^3$.
5. *For every* $t \geqslant 0$ *the map* $x \mapsto \Lambda(t,x)$ *lies in* $C_0^\infty(\mathbb{R}^3)$.

Let X be the flow associated to the vector Λ, defined by:

$$\begin{cases} \partial_t X(t,y) = \Lambda(t, X(t,y)) & (t > 0), \\ X(0,y) = y \in \mathbb{R}^3. \end{cases} \tag{8.13}$$

Using Lemma 8.3 and following line by line the proof of Lemma 3.2 we obtain

Lemma 8.4. *For every $y \in \mathbb{R}^3$ the initial value problem (3.4) admits a unique solution $X(\cdot, y) : [0,\infty) \mapsto \mathbb{R}^3$, which is a C^1 function in $[0,\infty)$. Furthermore, we have*

1. *For every $t \geqslant 0$, the mapping $y \mapsto X(t,y)$ is a C^∞-diffeomorphism from $\mathbb{R}^3$ onto itself and from E onto $\mathcal{F}(t)$.*
2. *For every $t \geqslant 0$ we denote by $Y(t,\cdot) = [X(t,\cdot)]^{-1}$ the inverse of $X(t,\cdot)$. Then for every $x \in \mathbb{R}^3$ the mapping $t \mapsto Y(t,x)$ is a C^1 function on $[0,\infty)$.*
3. *For every $t > 0$ and $i \in \{1, 2, \ldots, m\}$ we have $X(t, \mathcal{O}_i) = \mathcal{S}_i(t)$ (Thus $Y(t, \mathcal{S}_i(t)) = \mathcal{O}_i$.).*
4. *For every $t \geqslant 0$ and $y \in \mathbb{R}^3$ we have that* $\det(\nabla X(t,y)) = 1$.

The rest of the proof is similar to the proof of Theorem 2.3. We can define a change of co-ordinates similar to (3.5)–(3.6). The system in new variables will satisfy a system similar to (3.9)–(3.12). Then we can define the associated *fluid structure operator*. This new operator has the same properties as the fluid–structure operator associated with the single rigid body. Then one can mimic the steps given in Sects. 4–7, to obtain the proof of Theorem 8.2. In particular, in the spirit of Remark 7.5, we can choose the initial data sufficiently small in order to avoid contacts between the various rigid bodies. Since the calculations are almost identical, lengthy and too much of a repetition, we omit the details here.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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References

[1] Cumsille, P., Takahashi, T.: Wellposedness for the system modelling the motion of a rigid body of arbitrary form in an incompressible viscous fluid. Czechoslov. Math. J. **58**(133), 961–992 (2008)
[2] Dore, G.: L^p regularity for abstract differential equations. In: Functional Analysis and Related Topics, 1991 (Kyoto), Volume 1540 of Lecture Notes in Mathematics, pp. 25–38. Springer, Berlin (1993)
[3] Ervedoza, S., Hillairet, M., Lacave, C.: Long-time behavior for the two-dimensional motion of a disk in a viscous fluid. Commun. Math. Phys. **329**, 325–382 (2014)
[4] Ervedoza, S., Maity, D., Tucsnak, M.: Large time behaviour for the motion of a solid in a viscous incompressible fluid. Math. Ann. **385**, 631–691 (2023)
[5] Ferriere, G., Hillairet, M.: Unbounded-energy solutions to the fluid+disk system and long-time behavior for large initial data. Comptes Rendus Mathématique **361**, 453–485 (2023)
[6] Galdi, G.P.: Large-time behavior of a rigid body of arbitrary shape in a viscous fluid under the action of prescribed forces and torques. arXiv preprint arXiv:2212.11909 (2022)
[7] Geissert, M., Götze, K., Hieber, M.: L^p-theory for strong solutions to fluid-rigid body interaction in Newtonian and generalized Newtonian fluids. Trans. Am. Math. Soc. **365**, 1393–1439 (2013)
[8] Hishida, T.: Large time behavior of a generalized Oseen evolution operator, with applications to the Navier–Stokes flow past a rotating obstacle. Math. Ann. **372**, 915–949 (2018)
[9] Hishida, T., Shibata, Y.: L^p-L^q estimate of the Stokes operator and Navier–Stokes flows in the exterior of a rotating obstacle. Arch. Ration. Mech. Anal. **193**, 339–421 (2009)
[10] Iwashita, H.: L_q-L_r estimates for the solutions of the nonstationary Stokes equations in an exterior domain and the Navier–Stokes initial value problems in L_q spaces. Math. Ann. **285**, 265–288 (1989)
[11] Kato, T.: Strong L^p-solutions of the Navier–Stokes equation in R^m, with applications to weak solutions. Math. Z. **187**, 471–480 (1984)
[12] Maity, D., Takahashi, T.: L^p theory for the interaction between the incompressible Navier-Stokes system and a damped plate. J. Math. Fluid Mech. **23**(103), 23 (2021)
[13] Maity, D., Tucsnak, M.: A Maximal Regularity Approach to the Analysis of Some Particulate Flows, pp. 1–75. Springer, Cham (2017)
[14] Maity, D., Tucsnak, M.: L^p-L^q maximal regularity for some operators associated with linearized incompressible fluid-rigid body problems. In: Mathematical Analysis in Fluid Mechanics—Selected Recent Results, Volume 710 of Contemporary Mathematics, pp. 175–201. American Mathematical Society, Providence (2018)
[15] Murata, M., Shibata, Y.: Global well posedness for a $\mathbb{Q}$-tensor model of nematic liquid crystals. J. Math. Fluid Mech. **24**(34), 30 (2022)
[16] Sabbagh, L.: On the motion of several disks in an unbounded viscous incompressible fluid. Nonlinearity **32**, 2157–2181 (2019)
[17] Serre, D.: Chute libre d'un solide dans un fluide visqueux incompressible. Existence. Jpn. J. Appl. Math. **4**, 99–110 (1987)
[18] Shibata, Y.: New thought on Matsumura–Nishida theory in the L_p-L_q maximal regularity framework. J. Math. Fluid Mech. **24**(66), 23 (2022)
[19] Takahashi, T.: Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain. Adv. Differ. Equ. **8**, 1499–1532 (2003)
[20] Takahashi, T., Tucsnak, M.: Global strong solutions for the two dimensional motion of an infinite cylinder in a viscous fluid. J. Math. Fluid Mech. **6**, 53–77 (2004)

[21] Temam, R.: Problèmes mathématiques en plasticité. Méthodes Mathématiques de l'Informatique [Mathematical Methods of Information Science], vol. 12. Gauthier-Villars, Montrouge (1983)
[22] Triebel, H.: Theory of Function Spaces. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel (2010)
[23] Weis, L.: Operator-valued Fourier multiplier theorems and maximal L_p-regularity. Math. Ann. **319**, 735–758 (2001)

Debayan Maity
TIFR Centre for Applicable Mathematics
Bangalore, Karnataka
560065
India
e-mail: debayan@tifrbng.res.in

Marius Tucsnak
Institut de MathÉmatiques de Bordeaux UMR 5251
UniversitÉ de Bordeaux, CNRS
33400 Talence
France
e-mail: marius.tucsnak@u-bordeaux.fr

(accepted: July 2, 2023; published online: August 7, 2023)